ME 4251: Calibration and Curve-Fitting

Fall 2019

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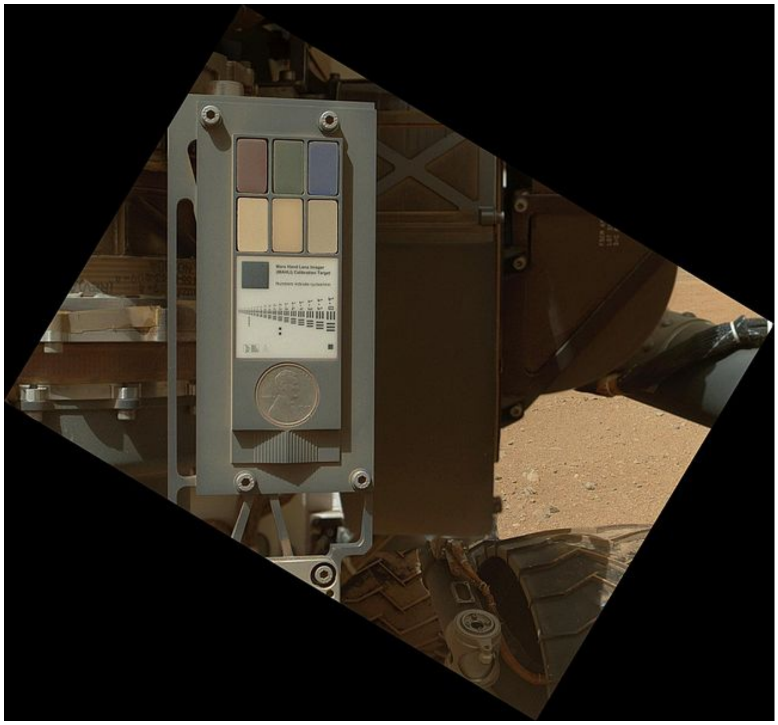
From here, it's possible.

Definition

Measurement is a process of comparison: the measured quantity is compared to a known standard.

Before we measure, we must establish the relationship between the readout values of our instrument and known input values of the measurand. This process is **calibration**.

Calibration Target for Camera on Mars Curiosity Rover



NASA/JPL-Caltech/Malin Space Science Systems

Linear Calibration

In many cases, we may know in advance that there is a linear relationship between the measurand and the readout of our instrument.

For example,

$$\boxed{p_{\text{true}} = a_1 \; v_{\text{indicated}} + b_1} \tag{1a}$$

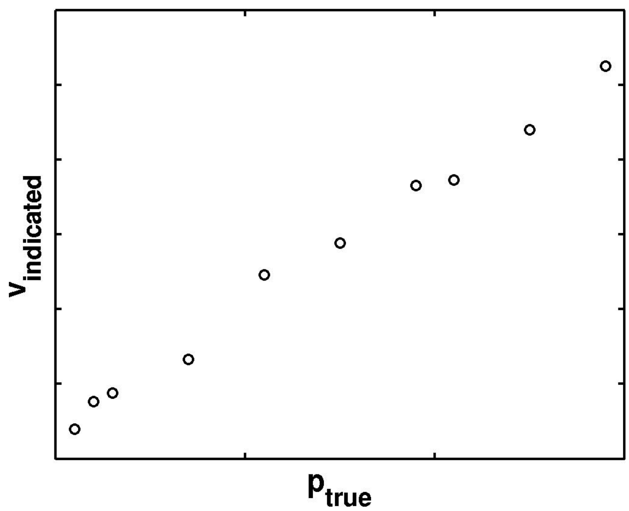
or

$$\boxed{p_{\text{true}} = a_2 \; p_{\text{indicated}} + b_2} \tag{1b}$$

where “indicated” refers to the readout values and “true” refers to the true values of the measurand.

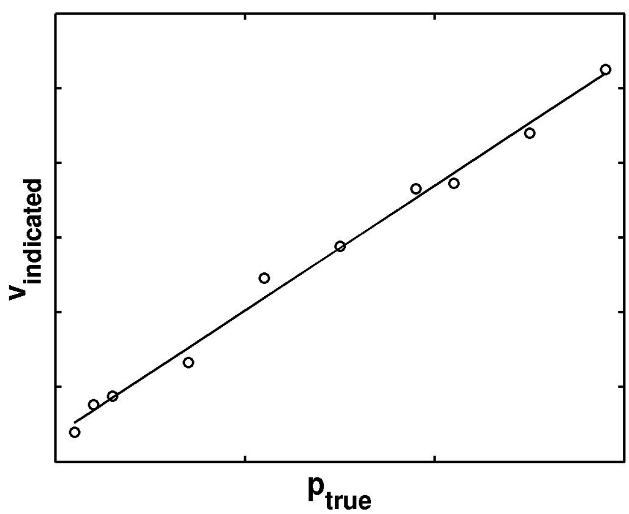
These relationships are our **goal**.

Plot of Calibration Data



Use the more precisely known data for the **abscissa**.
Use the data we are attempting to calibrate for the **ordinate**.
Can a linear function be drawn through this data? No!

Linear Curve-fit to Data

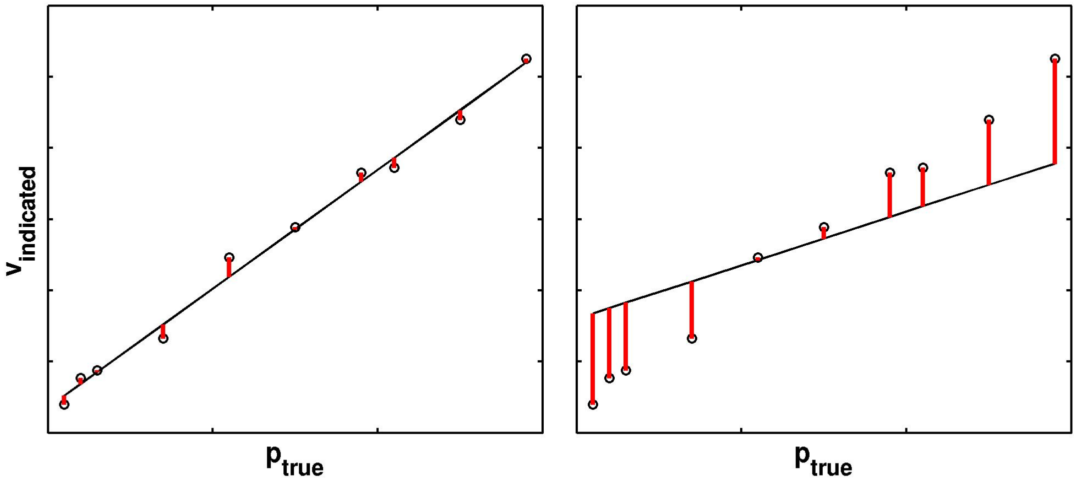


$$v_{\text{indicated}} = c_1 \; p_{\text{true}} + c_2 \tag{2a}$$

For generality, we will express as:

$$y = c_1 \; x + c_2 \tag{2b}$$

Method of Least-Squares



The Method of Least-Squares finds the coefficients, c_1 and c_2, that minimize the sum of the squares of the deviations between the data, y_i, and the fit, $c_1 x_i + c_2$. As in the left figure.

Mathematical Development of Method of Least Squares

The **deviation** between the data and the fit:

$$y_i - (c_1 x_i + c_2)$$

The **square of the deviation**:

$$[y_i - (c_1 x_i + c_2)]^2$$

The **sum of the squares of the deviations**:

$$E = \sum [y_i - (c_1 x_i + c_2)]^2$$

We minimize by the methods of calculus:

$$\frac{\partial E}{\partial c_1} = 0 \tag{3a}$$

$$\frac{\partial E}{\partial c_2} = 0 \tag{3b}$$

Equations 3a and 3b are **two** Equations in **two** Unknowns

Solve them for c_1 and c_2. **Our result**:

$$c_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} \tag{4a}$$

$$c_2 = \frac{\sum y_i \sum x_i^2 - \sum x_i y_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2} \tag{4b}$$

where

n is the number of data pairs in the curve-fit.

the y_i's are the ordinate values (the **less** precisely known).

the x_i's are the abscissa values (the **more** precisely known).

Note: This assumes uniform uncertainty in each y_i.

Compare Equations 1a and 2a

Equation 1a:

$$p_{\text{true}} = a_1 \; v_{\text{indicated}} + b_1$$

Equation 2a:

$$v_{\text{indicated}} = c_1 \; p_{\text{true}} + c_2$$

To obtain our final result, we must find a_1 and b_1.
We can rearrange 2a, like so:

$$\boxed{p_{\text{true}} = \frac{1}{c_1} v_{\text{indicated}} - \frac{c_2}{c_1}} \tag{5}$$

We have calibrated the instrument!

Determining Uncertainty **in the General Case**

Any **measured result** using our calibration curve must include an **uncertainty estimate**.

We calculate the **standard error** (analogous to σ) as:

$$S = u_y = \sqrt{\frac{\sum [y_i - (c_1 x_i + c_2)]^2}{n-2}} \tag{6}$$

We propagate the error as in the previous lecture:

$$u_x = \frac{\partial x}{\partial y} u_y \tag{7}$$

Determining Uncertainty **for the Current Example**

Any **measured result** using our calibration curve must include an **uncertainty estimate**.

We calculate the **standard error** (analogous to σ) as:

$$S = u_v = \sqrt{\frac{\sum [v_i - (c_1 p_i + c_2)]^2}{n-2}} \tag{8}$$

We propagate the error as in the previous lecture:

$$u_p = \frac{\partial p}{\partial v} u_v \tag{9a}$$

$$= \frac{1}{c_1} u_v \tag{9b}$$

Using the Calibrated Instrument

Suppose we use the instrument to measure a pressure, p.

The instrument outputs a voltage, v.

We would calculate and report our measured pressure as

$$\boxed{p = \left(\frac{1}{c_1}v - \frac{c_2}{c_1}\right) \pm z_{c/2}\frac{S}{c_1} \qquad \text{at a confidence level } c} \tag{10}$$

Common Curve Fits

Linear: $y = ax + b$ (as in our example)

Exponential: $y = ae^{bx}$

Power Law: $y = ax^b$

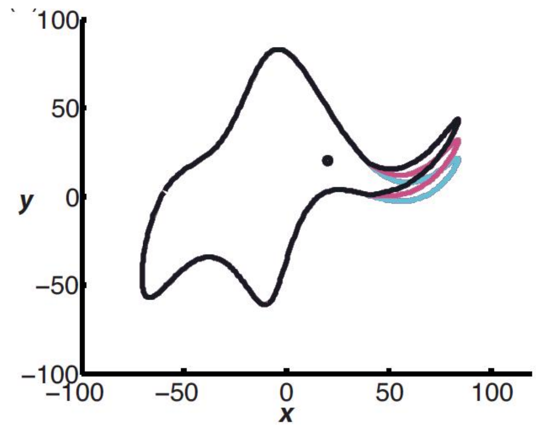
These functional forms, and others, are candidates for fitting to data. Often, there is a strong theoretical reason to prefer one or another. If there is no theoretical preference, the appearance of the data should guide our choice.

The latter two can be manipulated so that all our preceding work can be applied.

John von Neumann's caution against high-order curve fits:

"With four parameters I can fit an elephant, and with five I can make him wiggle his trunk."

Nature 427 (22 January 2004) p. 297.



American Journal of Physics – June 2010 – Volume 78, Issue 6, pp. 648.

Exponential Curve Fit: $y = ae^{bx}$

Can you picture what this plot looks like? It's not linear.

But…Take logarithm of both sides:

$$\ln(y) = \ln(a) + bx$$

Let $Y = \ln(y)$. Then, we obtain

$$Y = \ln(a) + bx$$

which does plot as a straight line. We can do a linear curve fit to obtain a and b for the original relation.

Power Law Curve Fit: $y = ax^b$

Can you picture what this plot looks like? It's not linear.

But...Take logarithm of both sides:

$$\ln(y) = \ln(a) + b\ln(x)$$

Let $Y = \ln(y)$ and $X = \ln(x)$. Then, we obtain

$$Y = \ln(a) + bX$$

which does plot as a straight line. We can do a linear curve fit to obtain a and b for the original relation.

References

I have based these slides on those of my predecessor, Darryl James.

I have also used the following sources (in order of ease of comprehension and in reverse order of rigor):

1. Beckwith, Marangoni, and Lienhard, Mechanical Measurements, Fifth Edition, Addison-Wesley Publishing Co., Reading, Massachusetts, 1993.
2. Bevington and Robinson, Data Reduction and Error Analysis for the Physical Sciences, Second Edition, WCB/McGraw Hill, Boston, Massachusetts, 1992.
3. Meyer, Paul L., Introductory Probability and Statistical Applications, 2nd Edition, Addison-Wesley Publishing Co., Reading, Massachusetts, 1970.