CHAPTER 3

ANALYSIS OF EXPERIMENTAL DATA

3-1 INTRODUCTION

Some form of analysis must be performed on all experimental data. The analysis may be a simple verbal appraisal of the test results, or it may take the form of a complex theoretical analysis of the errors involved in the experiment and matching of the data with fundamental physical principles. Even new principles may be developed in order to explain some unusual phenomenon. Our discussion in this chapter will consider the analysis of data to determine errors, precision, and general validity of experimental measurements. The correspondence of the measurements with physical principles is another matter, quite beyond the scope of our discussion. Some methods of graphical data presentation will also be discussed. The interested reader should consult the monograph by Wilson [4] for many interesting observations concerning correspondence of physical theory and experiment.

The experimentalist should always know the validity of data. The automobile test engineer must know the accuracy of the speedometer and gas gage in order to express the fuel-economy performance with confidence. A nuclear engineer must know the accuracy and precision of many instruments just to make some simple radioactivity measurements with confidence. In order to specify the performance of an amplifier, an electrical engineer must know the accuracy with which the appropriate measurements of voltage, distortion, etc., have been conducted. Many considerations enter into a final determination of the validity of the results of experimental data, and we wish to present some of these considerations in this chapter.

Errors will creep into all experiments regardless of the care which is exerted. Some of these errors are of a random nature, and some will be due to gross blunders on the part of the experimenter. Bad data due to obvious blunders may be discarded immediately. But what of the data points that just "look" bad? We cannot throw out data because they do not conform with our hopes and expectations unless we see something obviously wrong. If such "bad" points fall outside the range of normally expected random deviations, they may be discarded on the basis of some consistent statistical data analysis. The keyword here is "consistent." The elimination of data points must be consistent and should not be dependent on human whims and bias based on what "ought to be." In many instances it is very difficult for the individual to be consistent and unbiased. The pressure of a deadline, disgust with previous experimental failures, and normal impatience all can influence rational thinking processes. However, the competent experimentalist will strive to maintain consistency in the primary data analysis. Our objective in this chapter is to show how one may go about maintaining this consistency.

3-2 CAUSES AND TYPES OF EXPERIMENTAL ERRORS

In this section we present a discussion of some of the types of errors that may be present in experimental data and begin to indicate the way these data may be handled. First, let us distinguish between single-sample and multisample data.

Single-sample data are those in which some uncertainties may not be discovered by repetition. Multisample data are obtained in those instances where enough experiments are performed so that the reliability of the results can be assured by statistics. Frequently, cost will prohibit the collection of multisample data, and the experimenter must be content with single-sample data and prepared to extract as much information as possible from such experiments. The reader should consult Refs. [1] and [4] for further discussions on this subject, but we state a simple example at this time. If one measures pressure with a pressure gage and a single instrument is the only one used for the entire set of observations, then some of the error that is present in the measurement will be sampled only once no matter how many times the reading is repeated. Consequently, such an experiment is a single-sample experiment. On the other hand, if more than one pressure gage is used for the same total set of observations, then we might say that a multisample experiment has been performed. The *number* of observations will then determine the success of this multisample experiment in accordance with accepted statistical principles.

An experimental error is an experimental error. If the experimenter knew what the error was, he or she would correct it and it would no longer be an error. In other words, the real errors in experimental data are those factors that are always vague to some extent and carry some amount of uncertainty. Our task is to determine just how uncertain a particular observation may be and to devise a consistent way of specifying the uncertainty in analytical form. A reasonable definition of experimental uncertainty may be taken as the *possible value* the error may have. This uncertainty may vary a great deal depending upon the circumstances of the experiment. Perhaps it is better to speak of experimental uncertainty instead of experimental error because the magnitude of an error is always uncertain. Both terms are used in practice, however, so the reader should be familiar with the meaning attached to the terms and the ways that they relate to each other.

It is very common for people to speak of experimental errors when the correct terminology should be "uncertainty." Because of this common usage, we ask that the reader accept the faulty semantics when they occur and view each term in its proper context.

At this point we may mention some of the types of errors that may cause uncertainty in an experimental measurement. First, there can always be those gross blunders in apparatus or instrument construction which may invalidate the data. Hopefully, the careful experimenter will be able to eliminate most of these errors. Second, there may be certain *fixed errors* which will cause repeated readings to be in error by roughly the same amount but for some unknown reason. These fixed errors are sometimes called *systematic errors*. Third, there are the *random errors*, which may be caused by personal fluctuations, random electronic fluctuations in the apparatus or instruments, various influences of friction, etc. These random errors usually follow a certain statistical distribution, *but not always*. In many instances it is very difficult to distinguish between fixed errors and random errors.

The experimentalist may sometimes use theoretical methods to estimate the magnitude of a fixed error. For example, consider the measurement of the temperature of a hot gas stream flowing in a duct with a mercury-in-glass thermometer. It is well known that heat may be conducted from the stem of the thermometer, out of the body, and into the surroundings. In other words, the fact that part of the thermometer is exposed to the surroundings at a temperature different from the gas termperature to be measured may influence the temperature of the stem of the thermometer. There is a heat flow from the gas to the stem of the thermometer, and, consequently, the temperature of the stem must be lower than that of the hot gas. Therefore, the temperature we read on the thermometer is not the true temperature of the gas, and it will not make any difference how many readings are taken—we shall always have an error resulting from the heat-transfer condition of the stem of the thermometer. This is a *fixed* error, and its magnitude may be estimated with theoretical calculations based upon known thermal properties of the gas and the glass thermometer.

3-3 ERROR ANALYSIS ON A COMMONSENSE BASIS

We have already noted that it is somewhat more explicit to speak of experimental uncertainty rather than experimental error. Suppose that we have satisfied ourselves with the uncertainty in some basic experimental measurements, taking into consideration such factors as instrument accuracy, competence of the people using the instruments, etc. Eventually, the primary measurements must be combined to calculate a particular result that is desired. We shall be interested in knowing the uncertainty in the final result due to the uncertainties in the primary measurements. This may be done by a commonsense analysis of the data which may take many forms. One rule of thumb that could be used is that the error in the result is equal to the maximum error in any parameter used to calculate the result. Another commonsense analysis would combine all the errors in the most detrimental way in order to determine the maximum error in the final result. Consider the calculation of electric power from

$$P = EI$$

where E and I are measured as

$$E = 100\ \text{V} \pm 2\ \text{V}$$

$$I = 10\ \text{A} \pm 0.2\ \text{A}$$

The nominal value of the power is $100 \times 10 = 1000$ W. By taking the worst possible variations in voltage and current, we could calculate

$$P_{\text{max}} = (100 + 2)(10 + 0.2) = 1040.4\ \text{W}$$

$$P_{\text{min}} = (100 - 2)(10 - 0.2) = 960.4\ \text{W}$$

Thus, using this method of calculation, the uncertainty in the power is +4.04 percent, −3.96 percent. It is quite unlikely that the power would be in error by these amounts because the voltmeter variations would probably not correspond with the ammeter variations. When the voltmeter reads an extreme "high," there is no reason why the ammeter must also read an extreme "high" at that particular instant; indeed, this combination is most unlikely.

The simple calculation applied to the electric-power equation above is a useful way of inspecting experimental data to determine what errors *could* result in a final calculation; however, the test is too severe and should be used only for rough inspections of data. It is significant to note, however, that if the results of the experiments appear to be in error by *more* than the amounts indicated by the above calculation, then the experimenter had better examine the data more closely. In particular, the experimenter should look for certain fixed errors in the instrumentation, which may be eliminated by applying either theoretical or empirical corrections.

As another example we might conduct an experiment where heat is *added* to a container of water. If our temperature instrumentation should indicate a *drop* in temperature of the water, our good sense would tell us that something is wrong and the data point(s) should be thrown out. No sophisticated analysis procedures are necessary to discover this kind error.

The term "common sense" has many connotations and means different things to different people. In the brief example given above, it is intended as a quick and expedient vehicle, which may be used to examine experimental data and results for gross errors and variations. In subsequent sections we shall present methods for determining experimental uncertainties in a more precise manner.

3-4 UNCERTAINTY ANALYSIS

A more precise method of estimating uncertainty in experimental results has been presented by Kline and McClintock [1]. The method is based on a careful specification of the uncertainties in the various primary experimental measurements. For example, a certain pressure reading might be expressed as

$$p = 100\ \text{kN/m}^2 \pm 1\ \text{kN/m}^2$$

When the plus or minus notation is used to designate the uncertainty, the person making this designation is stating the degree of accuracy with which he or she *believes* the measurement has been made. We may note that this specification is in itself uncertain because the experimenter is naturally uncertain about the accuracy of these measurements.

If a very careful calibration of an instrument has been performed recently, with standards of very high precision, then the experimentalist will be justified in assigning a much lower uncertainty to measurements than if they were performed with a gage or instrument of unknown calibration history.

To add a further specification of the uncertainty of a particular measurement, Kline and McClintock propose that the experimenter specify certain odds for the uncertainty. The above equation for pressure might thus be written

$$p = 100\ \text{kN/m}^2 \pm 1\ \text{kN/m}^2\ (20\ \text{to}\ 1)$$

In other words, the experimenter is willing to bet with 20 to 1 odds that the pressure measurement is within $\pm 1\ \text{kN/m}^2$. It is important to note that the specification of such odds can *only* be made by the experimenter based on the total laboratory experience.

Suppose a set of measurements is made and the uncertainty in each measurement may be expressed with the same odds. These measurements are then used to calculate some desired result of the experiments. We wish to estimate the uncertainty in the calculated result on the basis of the uncertainties in the primary measurements. The result R is a given function of the independent variables $x_1, x_2, x_3, \ldots, x_n$. Thus,

$$R = R(x_1, x_2, x_3, \ldots, x_n) \tag{3-1}$$

Let w_R be the uncertainty in the result and $w_1, w_2, \ldots, w_n$ be the uncertainties in the independent variables. If the uncertainties in the independent variables are all given with same odds, then the uncertainty in the result having these odds is given in Ref. [1] as

$$w_R = \left[\left(\frac{\partial R}{\partial x_1} w_1\right)^2 + \left(\frac{\partial R}{\partial x_2} w_2\right)^2 + \cdots + \left(\frac{\partial R}{\partial x_n} w_n\right)^2\right]^{1/2} \qquad (3\text{-}2)$$

If this relation is applied to the electric power relation of the previous section, the expected uncertainty is 2.83 percent instead of 4.04 percent.

Example 3-1. The resistance of a certain size of copper wire is given as

$$R = R_0[1 + \alpha(T - 20)]$$

where $R_0 = 6\ \Omega \pm 0.3$ percent is the resistance at 20°C, $\alpha = 0.004°\text{C}^{-1} \pm 1$ percent is the temperature coefficient of resistance, and the temperature of the wire is $T = 30 \pm 1$°C. Calculate the resistance of the wire and its uncertainty.

Solution. The nominal resistance is

$$R = (6)[1 + (0.004)(30 - 20)] = 6.24\ \Omega$$

The uncertainty in this value is calculated by applying Eq. (3-2). The various terms are:

$$\frac{\partial R}{\partial R_0} = 1 + \alpha(T - 20) = 1 + (0.004)(30 - 20) = 1.04$$

$$\frac{\partial R}{\partial \alpha} = R_0(T - 20) = (6)(30 - 20) = 60$$

$$\frac{\partial R}{\partial T} = R_0\alpha = (6)(0.004) = 0.024$$

$$w_{R_0} = (6)(0.003) = 0.018\ \Omega$$

$$w_\alpha = (0.004)(0.01) = 4 \times 10^{-5}\ °\text{C}^{-1}$$

$$w_T = 1°\text{C}$$

Thus, the uncertainty in the resistance is

$$w_R = [(1.04)^2(0.018)^2 + (60)^2(4 \times 10^{-5})^2 + (0.024)^2(1)^2]^{1/2}$$
$$= 0.0305\ \Omega \text{ or } 0.49\%$$

Particular notice should be given to the fact that the uncertainty propagation in the result w_R predicted by Eq. (3-2) depends on the squares of the uncertainties in the independent variables w_n. This means that if the uncertainty in one variable is significantly larger than the uncertainties in the other variables, say, by a factor of 5 or 10, then it is the largest uncertainty that predominates and the others may probably be neglected.

To illustrate, suppose there are three variables with a product of sensitivity and uncertainty $[(\partial R/\partial x)w_x]$ of magnitude 1, and one variable with a magnitude of 5. The uncertainty in the result would be

$$(5^2 + 1^2 + 1^2 + 1^2)^{1/2} = \sqrt{28} = 5.29$$

The importance of this brief remark concerning the relative magnitude of uncertainties is evident when one considers the design of an experiment, procurement of instrumentation, etc. Very little is gained by trying to reduce the "small" uncertainties. Because of the square propagation it is the "large" ones that predominate, and any improvement in the overall experimental result must be achieved by improving the instrumentation or technique connected with these relatively large uncertainties. In the examples and problems that follow, both in this chapter and throughout the book, the reader should always note the relative effect of uncertainties in primary measurements on the final result.

In Sec. 2-11 (Table 2-7) the reader was cautioned to examine possible experimental errors *before* the experiment is conducted. Equation (3-2) may be used very effectively for such analysis, as we shall see in the sections and chapters that follow. A further word of caution may be added here. It is equally as unfortunate to overestimate uncertainty as to underestimate it. An underestimate gives false security, while an overestimate may make one discard important results, miss a real effect, or buy much too expensive instruments. The purpose of this chapter is to indicate some of the methods for obtaining reasonable estimates of experimental uncertainty.

In the previous discussion of experimental planning we noted that an uncertainty analysis may aid the investigator in selecting alternative methods to measure a particular experimental variable. It may also indicate how one may improve the overall accuracy of a measurement by attacking certain critical variables in the measurement process. The next three examples illustrate these points.

Example 3-2 Selection of measurement method. A resistor has a nominal stated value of $10\ \Omega \pm 1$ percent. A voltage is impressed on the resistor, and the power dissipation is to be calculated in two different ways: (1) from $P = E^2/R$ and (2) from $P = EI$. In (1) only a voltage measurement will be made, while both current and voltage will be measured in (2). Calculate the uncertainty in the power determination in each case when the measured values of E and I are:

$$E = 100\ \text{V} \pm 1\% \qquad \text{(for both cases)}$$

$$I = 10\ \text{A} \pm 1\%$$

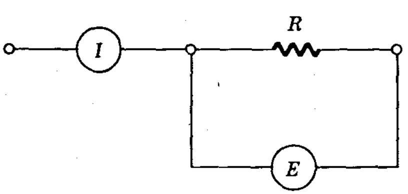


FIGURE EXAMPLE 3-2
Power measurement across a resistor.

Solution. The schematic is shown in the accompanying figure. For the first case we have

$$\frac{\partial P}{\partial E} = \frac{2E}{R} \qquad \frac{\partial P}{\partial R} = -\frac{E^2}{R^2}$$

and we apply Eq. (3-2) to give

$$w_P = \left[\left(\frac{2E}{R}\right)^2 w_E^2 + \left(-\frac{E^2}{R^2}\right)^2 w_R^2\right]^{1/2} \tag{a}$$

Dividing by $P = E^2/R$ gives

$$\frac{w_P}{P} = \left[4\left(\frac{w_E}{E}\right)^2 + \left(\frac{w_R}{R}\right)^2\right]^{1/2} \tag{b}$$

Inserting the numerical values for uncertainty,

$$\frac{w_P}{P} = [4(0.01)^2 + (0.01)^2]^{1/2} = 2.236\%$$

For the second case we have

$$\frac{\partial P}{\partial E} = I \qquad \frac{\partial P}{\partial I} = E$$

and after similar algebraic manipulation, we obtain

$$\frac{w_P}{P} = \left[\left(\frac{w_E}{E}\right)^2 + \left(\frac{w_I}{I}\right)^2\right]^{1/2} \tag{c}$$

Inserting the numerical values of uncertainty,

$$\frac{w_P}{P} = [(0.01)^2 + (0.01)^2]^{1/2} = 1.414\%$$

Thus, the second method of power determination provides considerably less uncertainty than the first method, even though the primary uncertainties in each quantity are the same. In this example the utility of the uncertainty analysis is that it affords the individual a basis for *selection of a measurement method* to produce a result with less uncertainty.

Example 3-3 Instrument selection. The power measurement in Example 3-2 is to be conducted by measuring voltage and current across the resistor with the circuit shown in the accompanying figure. The voltmeter has an internal resistance R_m, and the value of R is known only approximately. Calculate the nominal value of the power dissipated in R and the uncertainty for the following conditions:

$$R = 100\ \Omega \qquad \text{(not known exactly)}$$

$$R_m = 1000\ \Omega \pm 5\%$$

$$I = 5\ \text{A} \pm 1\%$$

$$E = 500\ \text{V} \pm 1\%$$

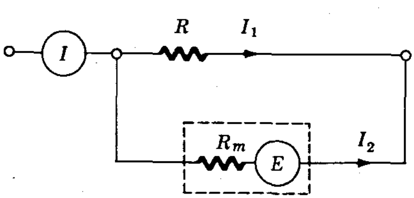


FIGURE EXAMPLE 3-3
Effect of meter impedance on measurement.

Solution. A current balance on the circuit yields

$$I_1 + I_2 = I$$

$$\frac{E}{R} + \frac{E}{R_m} = I$$

and

$$I_1 = I - \frac{E}{R_m} \qquad (a)$$

The power dissipated in the resistor is

$$P = EI_1 = EI - \frac{E^2}{R_m} \qquad (b)$$

The nominal value of the power is thus calculated as

$$P = (500)(5) - \frac{500^2}{1000} = 2250 \text{ W}$$

In terms of known quantities the power has the functional form $P = f(E, I, R_m)$, and so we form the derivatives

$$\frac{\partial P}{\partial E} = I - \frac{2E}{R_m} \qquad \frac{\partial P}{\partial I} = E$$

$$\frac{\partial P}{\partial R_m} = \frac{E^2}{R_m^2}$$

The uncertainty for the power is now written as

$$w_P = \left[\left(I - \frac{2E}{R_m}\right)^2 w_E^2 + E^2 w_I^2 + \left(\frac{E^2}{R_m^2}\right)^2 w_{R_m}^2\right]^{1/2} \qquad (c)$$

Inserting the appropriate numerical values gives

$$w_P = \left[\left(5 - \frac{1000}{1000}\right)^2 5^2 + (25 \times 10^4)(25 \times 10^{-4}) + \left(25 \times \frac{10^4}{10^6}\right)^2 (2500)\right]^{1/2}$$

$$= [16 + 25 + 6.25]^{1/2}(5)$$

$$= 34.4 \text{ W}$$

or

$$\frac{w_P}{P} = \frac{34.4}{2250} = 1.53\%$$

In order of influence on the final uncertainty in the power we have

1. Uncertainty of current determination
2. Uncertainty of voltage measurement
3. Uncertainty of knowledge of internal resistance of voltmeter

Comment. There are other conclusions we can draw from this example. The relative influence of the experimental quantities on the overall power determination is noted above. But this listing may be a bit misleading in that it implies that the uncertainty of the meter impedance does not have a large effect on the final uncertainty in the power determination. This results from the fact that $R_m \gg R$ ($R_m = 10R$). If the meter impedance were lower, say, 200 Ω, we would find that it was a dominant factor in the overall uncertainty. For a very *high* meter impedance there would be little influence, even with a very inaccurate knowledge of the exact value of R_m. Thus, we are led to the simple conclusion that we need not worry too much about the precise value of the internal impedance of the meter as long as it is very large compared with the resistance we are measuring the voltage across. This fact should influence *instrument selection* for a particular application.

Example 3-4 Ways to reduce uncertainties. A certain obstruction-type flowmeter (orifice, venturi, nozzle), shown in the accompanying figure, is used to measure the flow of air at low velocities. The relation describing the flow rate is

$$\dot{m} = CA\left[\frac{2g_c p_1}{RT_1}(p_1 - p_2)\right]^{1/2} \qquad (a)$$

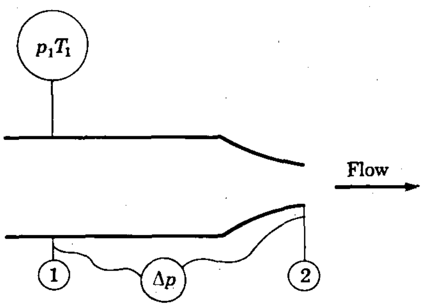


FIGURE EXAMPLE 3-4
Uncertainty in a flowmeter.

where C is an empirical-discharge coefficient, A is the flow area, p_1 and p_2 are the upstream and downstream pressures, T_1 is the upstream temperature, and R is the gas constant for air. Calculate the percent uncertainty in the mass flow rate for the following conditions:

$C = 0.92 \pm 0.005$ (from calibration data)

$p_1 = 25$ psia ± 0.5 psia

$T_1 = 70°\text{F} \pm 2°\text{F} \qquad T_1 = 530°\text{R}$

$\Delta p = p_1 - p_2 = 1.4$ psia ± 0.005 psia (measured directly)

$A = 1.0\ \text{in}^2 \pm 0.001\ \text{in}^2$

Solution. In this example the flow rate is a function of several variables, each subject to an uncertainty.

$$\dot{m} = f(C, A, p_1, \Delta p, T_1) \qquad (b)$$

Thus, we form the derivatives:

$$\frac{\partial \dot{m}}{\partial C} = A\left(\frac{2g_c p_1}{RT_1}\Delta p\right)^{1/2}$$

$$\frac{\partial \dot{m}}{\partial A} = C\left(\frac{2g_c p_1}{RT_1}\Delta p\right)^{1/2}$$

$$\frac{\partial \dot{m}}{\partial p_1} = 0.5CA\left(\frac{2g_c}{RT_1}\Delta p\right)^{1/2} p_1^{-1/2} \qquad (c)$$

$$\frac{\partial \dot{m}}{\partial \Delta p} = 0.5CA\left(\frac{2g_c p_1}{RT_1}\right)^{1/2} \Delta p^{-1/2}$$

$$\frac{\partial \dot{m}}{\partial T_1} = -0.5CA\left(\frac{2g_c p_1}{R}\Delta p\right)^{1/2} T_1^{-3/2}$$

The uncertainty in the mass flow rate may now be calculated by assembling these derivatives in accordance with Eq. (3-2). Designating this assembly as Eq. (*c*) and then dividing by Eq. (*a*) gives

$$\frac{w_{\dot{m}}}{\dot{m}} = \left[\left(\frac{w_C}{C}\right)^2 + \left(\frac{w_A}{A}\right)^2 + \frac{1}{4}\left(\frac{w_{p_1}}{p_1}\right)^2 + \frac{1}{4}\left(\frac{w_{\Delta p}}{\Delta p}\right)^2 + \frac{1}{4}\left(\frac{w_{T_1}}{T_1}\right)^2\right]^{1/2} \qquad (d)$$

We may now insert the numerical values for the quantities to obtain the percent uncertainty in the mass flow rate.

$$\frac{w_{\dot{m}}}{\dot{m}} = \left[\left(\frac{0.005}{0.92}\right)^2 + \left(\frac{0.001}{1.0}\right)^2 + \frac{1}{4}\left(\frac{0.5}{25}\right)^2 + \frac{1}{4}\left(\frac{0.005}{1.4}\right)^2 + \frac{1}{4}\left(\frac{2}{530}\right)^2\right]^{1/2}$$

$$= [29.5 \times 10^{-6} + 1.0 \times 10^{-6} + 1.0 \times 10^{-4} + 3.19 \times 10^{-6} + 3.57 \times 10^{-6}]^{1/2}$$

$$= [1.373 \times 10^{-4}]^{1/2} = 1.172\% \qquad (e)$$

The main contribution to uncertainty is the p_1 measurement with its basic uncertainty of 2 percent. Thus, to improve the overall situation the accuracy of this measurement should be attacked first. In order of influence on the flow-rate uncertainty, we have

1. Uncertainty in p_1 measurement (± 2 percent)
2. Uncertainty in value of C
3. Uncertainty in determination of T_1
4. Uncertainty in determination of Δp
5. Uncertainty in determination of A

By inspecting Eq. (*e*) we see that the first two items make practically the whole contribution to uncertainty. The value of the uncertainty analysis in this example is that it shows the investigator how to improve the overall measurement accuracy of this technique. First, obtain a more precise measurement of p_1. Then try to obtain a better calibration of the device, i.e., a better value of C. In Chap. 7 we shall see how values of the discharge coefficient C are obtained.

3-5 EVALUATION OF UNCERTAINTIES FOR COMPLICATED DATA REDUCTION

We have seen in the preceding discussion and examples how uncertainty analysis can be a useful tool to examine experimental data. In many cases data reduction is a rather complicated affair and is often performed with a computer routine written specifically for the task. A small adaptation of the routine can provide for direct calculation of uncertainties without resorting to an analytical determination of the partial derivatives in Eq. (3-2). We still assume that this equation applies, although it could involve several computational steps. We also assume that we are able to obtain estimates by some means of the uncertainties in the primary measurements, i.e., w_1, w_2, etc.

Suppose a set of data is collected in the variables $x_1, x_2, \ldots, x_n$ and a result calculated. At the same time one may perturb the variables by Δx_1, Δx_2, and so on, and calculate new results. We would have

$$R(x_1) = R(x_1, x_2, \ldots, x_n)$$

$$R(x_1 + \Delta x_1) = R(x_1 + \Delta x_1, x_2, \ldots, x_n)$$

$$R(x_2) = R(x_1, x_2, \ldots, x_n)$$

$$R(x_2 + \Delta x_2) = R(x_1, x_2 + \Delta x_2, \ldots, x_n)$$

For small enough values of Δx the partial derivatives can be well approximated by

$$\frac{\partial R}{\partial x_1} \simeq \frac{R(x_1 + \Delta x_1) - R(x_1)}{\Delta x_1}$$

$$\frac{\partial R}{\partial x_2} \simeq \frac{R(x_2 + \Delta x_2) - R(x_2)}{\Delta x_2}$$

and these values could be inserted in Eq. (3-2) to calculate the uncertainty in the result.

At this point we must again alert the reader to the ways uncertainties or errors of instruments are normally specified. Suppose a pressure gage is available and the manufacturer states that it is accurate within ± 1.0 percent. This statement normally refers to *percent of full scale*. So a gage with a range of 0 to 100 kPa would have an uncertainty of ± 10 percent when reading a pressure of only 10 kPa. Of course, this means that the uncertainty in the calculated result, either as an absolute value or percentage, can vary widely depending on the range of operation of instruments used to make the primary measurements. The above procedure can be used to advantage in complicated data-reduction schemes.

A very full description of this technique and many other considerations of uncertainty analysis are given by Moffat [7]. An example of an industry standard on uncertainty analysis is given in Ref. [8].

Example 3-5. Calculate the uncertainty of the wire resistance in Example 3-1 using the technique of this section.

Solution. In Example 3-1 we have already calculated the nominal resistance as 6.24 Ω. We now perturb the three variables R_0, α, and T by small amounts to evaluate the partial derivatives. We shall take

$$\Delta R_0 = 0.01 \qquad \Delta\alpha = 1 \times 10^{-5} \qquad \Delta T = 0.1$$

Then

$$R(R_0 + \Delta R_0) = (6.01)[1 + (0.004)(30 - 20)] = 6.2504$$

and the derivative is approximated as

$$\frac{\partial R}{\partial R_0} \approx \frac{R(R_0 + \Delta R_0) - R}{\Delta R_0} = \frac{6.2504 - 6.24}{0.01} = 1.04$$

or the same result as in Example 3-1. Similarly,

$$R(\alpha + \Delta\alpha) = (6.0)[1 + (0.00401)(30 - 20)] = 6.2406$$

$$\frac{\partial R}{\partial \alpha} \approx \frac{R(\alpha + \Delta\alpha) - R}{\Delta\alpha} = \frac{6.2406 - 6.24}{1 \times 10^{-5}} = 60$$

$$R(T + \Delta T) = (6)[1 + (0.004)(30.1 - 20)] = 6.2424$$

$$\frac{\partial R}{\partial T} \approx \frac{R(T + \Delta T) - R}{\Delta T} = \frac{6.2424 - 6.24}{0.1} = 0.24$$

All the derivatives are the same as in Example 3-1 so the uncertainty in R would be the same, or 0.0305 Ω.

3-6 STATISTICAL ANALYSIS OF EXPERIMENTAL DATA

We shall not be able to give an extensive presentation of the methods of statistical analysis of experimental data; we may only indicate some of the more important methods currently employed. First, it is important to define some pertinent terms.

When a set of readings of an instrument is taken, the individual readings will vary somewhat from each other, and the experimenter is usually concerned with the *mean* of all the readings. If each reading is denoted by x_i and there are n readings, the *arithmetic mean* is given by

$$x_m = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{3-3}$$

The *deviation* d_i for each reading is defined by

$$d_i = x_i - x_m \tag{3-4}$$

We may note that the average of the deviations of all the readings is zero since

$$\bar{d}_i = \frac{1}{n} \sum_{i=1}^{n} d_i = \frac{1}{n} \sum_{i=1}^{n} (x_i - x_m)$$

$$= x_m - \frac{1}{n}(nx_m) = 0 \qquad (3\text{-}5)$$

The average of the absolute values of the deviations is given by

$$|\bar{d}_i| = \frac{1}{n} \sum_{i=1}^{n} |d_i| = \frac{1}{n} \sum_{i=1}^{n} |x_i - x_m| \qquad (3\text{-}6)$$

Note that this quantity is not necessarily zero.

The *standard deviation* or *root-mean-square deviation* is defined by

$$\sigma = \left[\frac{1}{n} \sum_{i=1}^{n} (x_i - x_m)^2 \right]^{1/2} \qquad (3\text{-}7)$$

and the square of the standard deviation σ^2 is called the *variance*. This is sometimes called the *population* or *biased* standard deviation because it strictly applies only when a large number of samples is taken to describe the population.

In many circumstances the engineer will not be able to collect as many data points as necessary to describe the underlying population. Generally speaking, it is desirable to have at least 20 measurements in order to obtain reliable estimates of standard deviation and general validity of the data. For small sets of data an *unbiased* or *sample standard deviation* is defined by

$$\sigma = \left[\frac{\sum_{i=1}^{n} (x_i - x_m)^2}{n - 1} \right]^{1/2} \qquad (3\text{-}8)$$

Note that the factor $n - 1$ is used instead of n as in Eq. (3-7). The sample or unbiased standard deviation should be used when the underlying population is not known. However, when comparisons are made against a known population or standard, Eq. (3-7) is the proper one to use for standard deviation. An example would be the calibration of a voltmeter against a known voltage source.

There are other kinds of mean values of interest from time to time in statistical analysis. The *median* is the value that divides the data points in half. For example, if measurements made on five production resistors give 10, 12, 13, 14, and 15 kΩ, the median value would be 13 kΩ. The *arithmetic* mean, however, would be

$$R_m = \frac{10 + 12 + 13 + 14 + 15}{5} = 12.8 \text{ k}\Omega$$

In some instances it may be appropriate to divide data into quartiles and deciles also. So, when we say that a student is in the upper quartile of the class we mean that that student's grade is among the top 25 percent of all students in the class.

Sometimes it is appropriate to use a *geometric mean* when studying phenomena which grow in proportion to their size. This would apply to certain biological processes and to growth rates in financial resources. The geometric mean is defined by

$$x_g = [x_1 \cdot x_2 \cdot x_3 \cdots x_n]^{1/n} \tag{3-9}$$

As an example of the use of this concept, consider the 5-year record of a mutual fund investment:

Year	Asset value	Rate of increase over previous year
1	1000	
2	890	0.89
3	990	1.1124
4	1100	1.1111
5	1250	1.1364

The average growth rate is therefore

$$\begin{aligned}\text{Average growth } &= [(0.89)(1.1124)(1.1111)(1.1364)]^{1/4} \\ &= 1.0574\end{aligned}$$

To see that this is indeed a valid average growth rate we can observe that

$$(1000)(1.0574)^4 = 1250$$

Example 3-6. The following readings are taken of a certain physical length. Compute the mean reading, standard deviation, variance, and average of the absolute value of the deviation, using the "biased" basis.

Reading	x, cm
1	5.30
2	5.73
3	6.77
4	5.26
5	4.33
6	5.45
7	6.09
8	5.64
9	5.81
10	5.75

Solution. The mean value is given by

$$x_m = \frac{1}{n}\sum_{i=1}^{n} x_i = \tfrac{1}{10}(56.13) = 5.613 \text{ cm}$$

The other quantities are computed with the aid of the following table:

Reading	$d_i = x_i - x_m$	$(x_i - x_m)^2 \times 10^2$
1	−0.313	9.797
2	0.117	1.369
3	1.157	133.865
4	−0.353	12.461
5	−1.283	164.609
6	−0.163	2.657
7	0.477	22.753
8	0.027	0.0729
9	0.197	3.881
10	0.137	1.877

$$\sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - x_m)^2\right]^{1/2} = [\tfrac{1}{10}(3.533)]^{1/2} = 0.5944\ \text{cm}$$

$$\sigma^2 = 0.3533\ \text{cm}^2$$

$$|\bar{d}_i| = \frac{1}{n}\sum_{i=1}^{n}|d_i| = \frac{1}{n}\sum_{i=1}^{n}|x_i - x_m|$$

$$= \tfrac{1}{10}(4.224) = 0.4224\ \text{cm}$$

Example 3-7. Calculate the best estimate of standard deviation for the data of Example 3-6 based on the "sample" or unbiased basis.

Solution. The calculation gives

$$\sigma = \left[\frac{1}{10-1}(3.536)\right]^{1/2} = (0.3929)^{1/2} = 0.627\ \text{cm}$$

Suppose an "honest" coin is flipped a large number of times. It will be noted that after a large number of tosses heads will be observed about the same number of times as tails. If one were to consistently bet on either heads or tails the best one could hope for would be a break-even proposition over a long period of time. In other words, the *frequency of occurrence* is the same for both heads or tails for a very large number of tosses. It is common knowledge that a few tosses of a coin, say 5 or 10, may not be a break-even proposition, as a large number of tosses would be. This observation illustrates the fact that frequency of occurrence of an event may be dependent on the total number of events which are observed.

The *probability* that one will get a head when flipping an unweighted coin is $\frac{1}{2}$, regardless of the number of times the coin is tossed. The probability that a tail will occur is also $\frac{1}{2}$. The probability that either a head or a tail will occur is $\frac{1}{2} + \frac{1}{2}$ or unity. (We ignore the possibility that the coin will stand on edge.) *Probability* is a mathematical quantity that is linked to the frequency with which a certain phenomenon occurs after a large number of tries. In the case of the coin, it is the number of times heads would be expected to result in a large number of tosses divided by the total number of tosses. Similarly, the toss of an unloaded die results in the occurrence of each side one-sixth of the time. Probabilities are expressed in values of less than one, and a probability of unity corresponds to certainty. In other words, if the probabilities for all possible events are added, the result must be unity. For separate events, the probability that one of the events will occur is the sum of the individual probabilities for the events. For a die, the probability that any one side will occur is $\frac{1}{6}$. The probability for one of three given sides is $\frac{1}{6} + \frac{1}{6} + \frac{1}{6}$, or $\frac{1}{2}$, and so on.

Suppose two dice are thrown and we wish to know the probability that both will display a 6. The probability for a 6 on a single die is $\frac{1}{6}$. By a short listing of the possible arrangements that the dice may have, it can be seen that there can be 36 possibilities and that the desired result of two 6's represents only one of these possibilities. Thus, the probability is $\frac{1}{36}$. For a throw of 7 or 11, there are 6 possible ways of getting a 7; thus the probability of getting a 7 is $\frac{6}{36}$ or $\frac{1}{6}$. There are only 2 ways of getting an 11; thus the probability is $\frac{2}{36}$ or $\frac{1}{18}$. The probability of getting *either* a 7 or an 11 is $\frac{6}{36} + \frac{2}{36}$.

If several *independent* events occur at the same time such that each event has a probability p_i, the probability that all events will occur is given as the product of the probabilities of the individual events. Thus, $p = \Pi\ p_i$, where the Π designates a product. This rule could be applied to the problem of determining the probability of a double 6 in the throw of two dice. The probability of getting a 6 on each die is $\frac{1}{6}$, and the total probability is therefore $(\frac{1}{6})(\frac{1}{6})$, or $\frac{1}{36}$. This reasoning could not be applied to the problem of obtaining a 7 on the two dice because the number on each die is not *independent* of the number on the other die, since a 7 can be obtained in more than one way.

As a final example we ask what the chances are of getting a royal flush in the first five cards drawn off the top of the deck. There are 20 suitable possibilities for the first draw (4 suits, 5 possible cards per suit) out of a total of 52 cards. On the second draw we have fixed the suit so that there are only 4 suitable cards out of the 51 remaining. There are three suitable cards on the third draw, two on the fouth, and only one on the fifth draw. The total probability of drawing the royal flush is thus the product of the probabilities of each draw, or

$$\frac{20}{52} \times \frac{4}{51} \times \frac{3}{50} \times \frac{2}{49} \times \frac{1}{48} = \frac{1}{649{,}740}$$

In the above discussion we have seen that the probability is related to the number of ways a certain event may occur. In this case we are assuming that all events are equally likely, and hence the probability that an event will occur is the number of ways the event may occur divided by the number of possible events. Our primary concern is the application of probability and statistics to the analysis of experimental data. For this purpose we need to discuss next the meaning and use of *probability distributions*. We shall be concerned with a few particular distributions that are directly applicable to experimental data analysis.

3-7 PROBABILITY DISTRIBUTIONS

Suppose we toss a horseshoe some distance x. Even though we make an effort to toss the horseshoe the same distance each time, we would not always meet with success. On the first toss the horseshoe might travel a distance x_1, on the second toss a distance of x_2, and so on. If one is a good player of the game, there would be more tosses which have an x distance equal to that of the objective. Also, we would expect fewer and fewer tosses for those x distances which are further and further away from the target. For a large number of tosses, the probability that it will travel a distance is obtained by dividing the number traveling this distance by the total number of tosses. Since each x distance will vary somewhat from other x distances, we might find it advantageous to calculate the probability of a toss landing in a certain increment of x between x and $x + \Delta x$. When this calculation is made, we might get something like the situation shown in Fig. 3-1. For a good player, the maximum probability is expected to surround the distance x_m designating the position of the target.

The curve shown in Fig. 3-1 is called a *probability distribution*. It shows how the probability of success in a certain event is distributed over the distance x. Each value of the ordinate $p(x)$ gives the probability that the horseshoe will land between x and $x + \Delta x$, where Δx is allowed to approach zero. We might

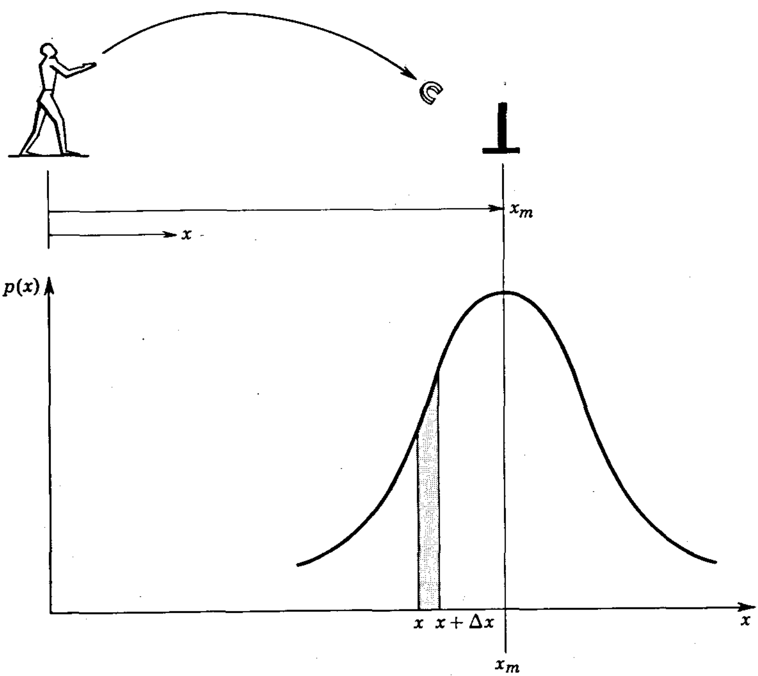


FIGURE 3-1
Distribution of throws for a "good" horseshoes player.

consider the deviation from x_m as the error in the throw. If the horseshoe player has good aim, large errors are less likely than small errors. The area under the curve is unity since it is certain that the horseshoe will land somewhere.

We should note that more than one variable may be present in a probability distribution. In the case of the horseshoes player, a person might throw the object an exact distance of x_m and yet to one side of the target. The sideways distance is another variable, and a large number of throws would have some distribution in this variable as well.

A particular probability distribution is the *binomial distribution*. This distribution gives the number of successes n out of N possible independent events when each event has a probability of success p. The probability that n events will succeed is given in Ref. [2] as

$$p(n) = \frac{N!}{(N-n)!n!} p^n(1-p)^{N-n} \tag{3-10}$$

It will be noted that the quantity $(1-p)$ is the probability of failure of each independent event.

Example 3-8 Binomial distribution. An unweighted coin is flipped three times. Calculate the probability of getting zero, one, two, or three heads in these tosses.

Solution. The binomial distribution applies in this case. The probability of getting a head on each throw is $p = \frac{1}{2}$ and $N = 3$, while n takes on the values 0, 1, 2, and 3. The probabilities are calculated as

$$p(0) = \frac{3!}{(3!)(0!)}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$p(1) = \frac{3!}{(2!)(1!)}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

$$p(2) = \frac{3!}{(1!)(2!)}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = \frac{3}{8}$$

$$p(3) = \frac{3!}{(0!)(3!)}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{1}{8}$$

Now suppose that the number of possible independent events N is very large and the probability of occurrence of each p is very small. The calculation of the probability of n successes out of the N possible events using Eq. (3-10) would be most cumbersome because of the size of the numbers. The limit of the binomial distribution as $N \to \infty$ and $p \to 0$, such that

$$Np = a = \text{const}$$

is called the *Poisson distribution* and is given by

$$p_a(n) = \frac{a^n e^{-a}}{n!} \tag{3-11}$$

The Poisson distribution is applicable to the calculation of the decay of radioactive nuclei, as we shall see in a subsequent chapter. It may be shown that the standard deviation of the Poisson distribution is

$$\sigma = \sqrt{a} \tag{3-12}$$

We have noted that a probability distribution like Fig. 3-1 is obtained when we observe frequency of occurrence over a large number of observations. When a limited number of observations is made and the raw data plotted, we call the plot a *histogram*. For example, the following distribution of throws might be observed for a horseshoes player:

Distance from target, cm	Number of throws
0–10	5
10–20	15
20–30	13
30–40	11
40–50	9
50–60	8
60–70	10
70–80	6
80–90	7
90–100	5
100–110	5
110–120	3
Over 120	2
Total	99

These data are plotted in Fig. 3-2 using increments of 10 cm in Δx. The same data are plotted in Fig. 3-3 using a Δx of 20 cm. The *relative frequency*, or

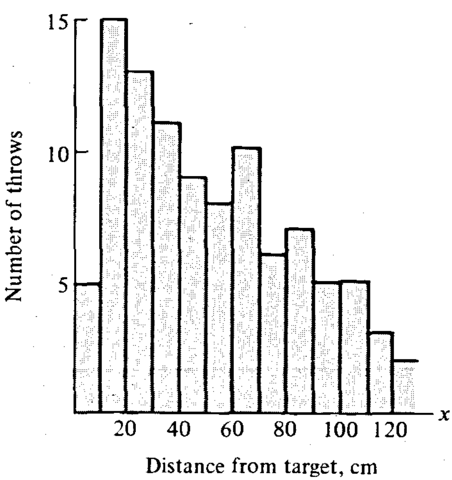


FIGURE 3-2
Histogram with $\Delta x = 10$ cm.

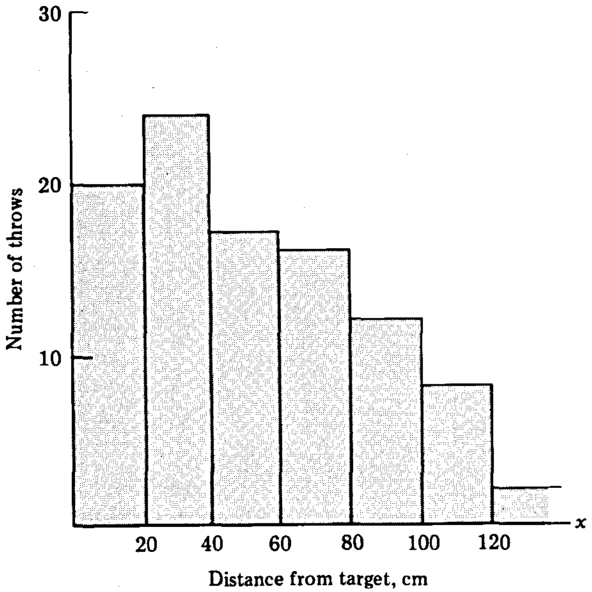


FIGURE 3-3
Histogram with $\Delta x = 20$ cm.

fraction of throws in each Δx increment, could also be used to convey the same information. A *cumulative frequency* diagram could be employed for these data, as shown in Fig. 3-4. If this figure had been constructed on the basis of a very large number of throws, then we could appropriately refer to the ordinate as the probability that the horseshoe will land within a distance x of the target.

3-8 THE GAUSSIAN OR NORMAL ERROR DISTRIBUTION

Suppose an experimental observation is made and some particular result recorded. We know (or would strongly suspect) that the observation has been subjected to many random errors. These random errors may make the final reading either too large or too small, depending on many circumstances which are unknown to us. Assuming that there are many small errors that contribute to the final error and that each small error is of equal magnitude and equally likely to be positive or negative, the *gaussian* or *normal error distribution* may be derived. If the measurement is designated by x, the gaussian distribution gives the probability that the measurement will lie between x and $x + dx$ and is written

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-x_m)^2/2\sigma^2} \tag{3-13}$$

In this expression, x_m is the mean reading and σ is the standard deviation. Some may prefer to call $P(x)$ the *probability density*. The units of $P(x)$ are

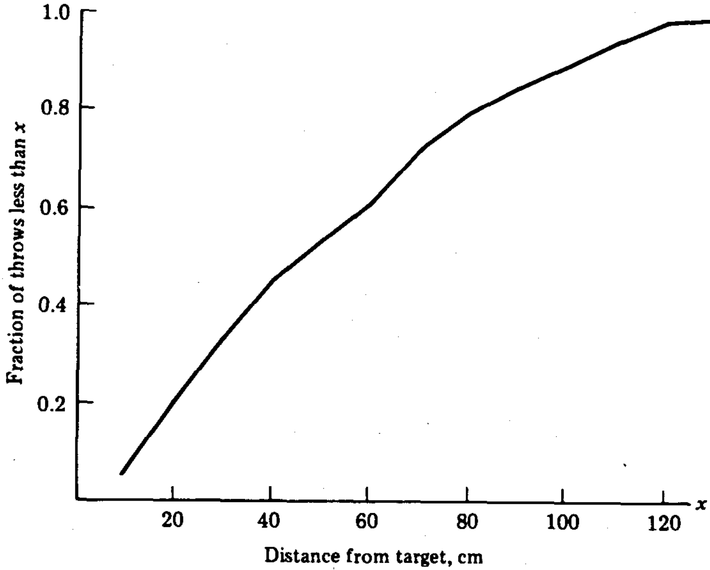


FIGURE 3-4
Cumulative frequency diagram.

those of $1/x$ since these are the units of $1/\sigma$. A plot of Eq. (3-13) is given in Fig. 3-5. Note that the most probable reading is x_m. The standard deviation is a measure of the width of the distribution curve; the larger the value of σ, the flatter the curve and hence the larger the expected error of all the measurements. Equation (3-13) is normalized so that the total area under the curve is unity. Thus,

$$\int_{-\infty}^{+\infty} P(x)\,dx = 1.0 \qquad (3\text{-}14)$$

At this point we may note the similarity between the shape of the normal error curve and the expected experimental distribution for tossing horseshoes as shown in Fig. 3-1. This is what we would expect, because the good horseshoes player's throws will be bunched around the target. The better the player is at the game, the more closely the throws will be grouped around the mean and the more probable will be the mean distance x_m. Thus, in the case of the horseshoes player, a smaller standard deviation would mean a larger percentage of "ringers."

We may quickly anticipate the next step in the analysis as one of trying to determine the precision of a set of experimental measurements through an application of the normal error distribution. One may ask: But how do you know that the assumptions pertaining to the derivation of the normal error distribution apply to experimental data? The answer is that for sets of data

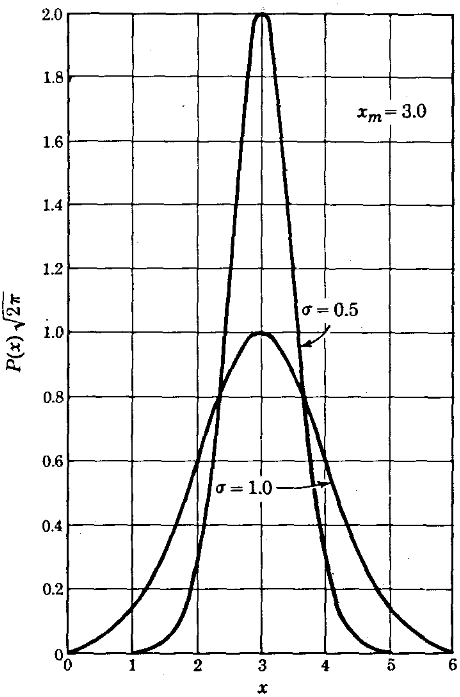


FIGURE 3-5
The gaussian or normal error distribution for two values of the standard deviation.

where a large number of measurements are taken, experiments indicate that the measurements do indeed follow a distribution like that shown in Fig. 3-5 when the experiment is under control. If an important parameter is not controlled, one gets just scatter, i.e., no sensible distribution at all. Thus, as a matter of experimental verification the gaussian distribution is believed to represent the *random* errors in an adequate manner for a properly controlled experiment.

By inspection of the gaussian distribution function of Eq. (3-13) we see that the maximum probability occurs at $x = x_m$, and the value of this probability is

$$P(x_m) = \frac{1}{\sigma\sqrt{2\pi}} \tag{3-15}$$

It is seen from Eq. (3-15) that smaller values of the standard deviation produce larger values of the maximum probability, as would be expected in an intuitive sense. $P(x_m)$ is sometimes called a *measure of precision* of the data because it has a larger value for smaller values of the standard deviation.

We next wish to examine the gaussian distribution to determine the likelihood that certain data points will fall within a specified deviation from the mean of all the data points. The probability that a measurement will fall within a certain range x_1 of the mean reading is

$$P = \int_{x_m - x_1}^{x_m + x_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-x_m)^2/2\sigma^2} \, dx \tag{3-16}$$

Making the variable substitution,

$$\eta = \frac{x - x_m}{\sigma}$$

Equation (3-16) becomes

$$P = \frac{1}{\sqrt{2\pi}} \int_{-\eta_1}^{+\eta_1} e^{-\eta^2/2} \, d\eta \tag{3-17}$$

where

$$\eta_1 = \frac{x_1}{\sigma} \tag{3-18}$$

Values of the gaussian normal error function

$$\frac{1}{\sqrt{2\pi}} e^{-\eta^2/2}$$

and integrals of the gaussian function corresponding to Eq. (3-17) are given in Tables 3-1 and 3-2.

If we have a sufficiently large number of data points, the error for each point should follow the gaussian distribution and we can determine the probability that certain data fall within a specified deviation from the mean value. Example 3-9 illustrates the method of computing the chances of finding data points within one or two standard deviations from the mean. Table 3-3 gives the chances for certain deviations from the mean value of the normal-distribution curve.

Example 3-9. Calculate the probabilities that a measurement will fall within one, two, and three standard deviations of the mean value, and compare them with the values in Table 3-3.

Solution. We perform the calculation using Eq. (3-17) with $\eta_1 = 1, 2,$ and 3. The values of the integral may be obtained from Table 3-2. We observe that

$$\int_{-\eta_1}^{+\eta_1} e^{-\eta^2/2} \, d\eta = 2 \int_0^{\eta_1} e^{-\eta^2/2} \, d\eta$$

so that

$$P(1) = (2)(0.34134) = 0.6827$$

$$P(2) = (2)(0.47725) = 0.9545$$

$$P(3) = (2)(0.49865) = 0.9973$$

Using the odds given in Table 3-3, we would calculate the probabilities as

$$P(1) = \frac{2.15}{2.15 + 1} = 0.6827$$

$$P(2) = \frac{21}{21 + 1} = 0.9545$$

$$P(3) = \frac{369}{369 + 1} = 0.9973$$

It is a rare circumstance indeed when an experimenter does not find that some of the data points look bad and out of place in comparison with the bulk of the data. The experimenter is therefore faced with the task of deciding if

TABLE 3-1

Values of the gaussian normal error distribution

Values of the function $(1/\sqrt{2\pi})e^{-\eta^2/2}$ for different values of the argument η. (Each figure in the body of the table is preceded by a decimal point.)

η	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	39894	39892	39886	39876	39862	39844	39822	39797	39767	39733
0.1	39695	39654	39608	39559	39505	39448	39387	39322	39253	39181
0.2	39104	39024	38940	38853	38762	38667	38568	38466	38361	38251
0.3	38139	38023	37903	37780	37654	37524	37391	37255	37115	36973
0.4	36827	36678	36526	36371	36213	36053	35889	35723	35553	35381
0.5	35207	35029	34849	34667	34482	34294	34105	33912	33718	33521
0.6	33322	33121	32918	32713	32506	32297	32086	31874	31659	31443
0.7	31225	31006	30785	30563	30339	30114	29887	29658	29430	29200
0.8	28969	28737	28504	28269	28034	27798	27562	27324	27086	26848
0.9	26609	26369	26129	25888	25647	25406	25164	24923	24681	24439
1.0	24197	23955	23713	23471	23230	22988	22747	22506	22265	22025
1.1	21785	21546	21307	21069	20831	20594	20357	20121	19886	19652
1.2	19419	19186	18954	18724	18494	18265	18037	17810	17585	17360
1.3	17137	16915	16694	16474	16256	16038	15822	15608	15395	15183
1.4	14973	14764	14556	14350	14146	13943	13742	13542	13344	13147
1.5	12952	12758	12566	12376	12188	12001	11816	11632	11450	11270
1.6	11092	10915	10741	10567	10396	10226	10059	09893	09728	09566
1.7	09405	09246	09089	08933	08780	08628	08478	08329	08183	08038
1.8	07895	07754	07614	07477	07341	07206	07074	06943	06814	06687
1.9	06562	06438	06316	06195	06077	05959	05844	05730	05618	05508
2.0	05399	05292	05186	05082	04980	04879	04780	04682	04586	04491
2.1	04398	04307	04217	04128	04041	03955	03871	03788	03706	03626
2.2	03547	03470	03394	03319	03246	03174	03103	03034	02965	02898
2.3	02833	02768	02705	02643	02582	02522	02463	02406	02349	02294
2.4	02239	02186	02134	02083	02033	01984	01936	01888	01842	01797
2.5	01753	01709	01667	01625	01585	01545	01506	01468	01431	01394
2.6	01358	01323	01289	01256	01223	01191	01160	01130	01100	01071
2.7	01042	01014	00987	00961	00935	00909	00885	00861	00837	00814
2.8	00792	00770	00748	00727	00707	00687	00668	00649	00631	00613
2.9	00595	00578	00562	00545	00530	00514	00499	00485	00470	00457
3.0	00443									
3.5	008727									
4.0	0001338									
4.5	0000160									
5.0	000001487									

TABLE 3-2

Integrals of the gaussian normal error function

Values of the integral $(1/\sqrt{2\pi})\int_0^{\eta_1} e^{-\eta^2/2}\,d\eta$ are given for different values of the argument η_1. It may be observed that

$$\frac{1}{\sqrt{2\pi}}\int_{-\eta_1}^{+\eta_1} e^{-\eta^2/2}\,d\eta = 2\,\frac{1}{\sqrt{2\pi}}\int_0^{\eta_1} e^{-\eta^2/2}\,d\eta$$

The values are related to the error function since

$$\operatorname{erf}\eta_1 = \frac{1}{\sqrt{\pi}}\int_{-\eta_1}^{+\eta_1} e^{-\eta^2}\,d\eta$$

so that the tabular values are equal to $\frac{1}{2}\operatorname{erf}(\eta_1/\sqrt{2})$. (Each figure in the body of the table is preceded by a decimal point.)

η_1	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	00000	00399	00798	01197	01595	01994	02392	02790	03188	03586
0.1	03983	04380	04776	05172	05567	05962	06356	06749	07142	07355
0.2	07926	08317	08706	09095	09483	09871	10257	10642	11026	11409
0.3	11791	12172	12552	12930	13307	13683	14058	14431	14803	15173
0.4	15554	15910	16276	16640	17003	17364	17724	18082	18439	18793
0.5	19146	19497	19847	20194	20450	20884	21226	21566	21904	22240
0.6	22575	22907	23237	23565	23891	24215	24537	24857	25175	25490
0.7	25804	26115	26424	26730	27035	27337	27637	27935	28230	28524
0.8	28814	29103	29389	29673	29955	30234	30511	30785	31057	31327
0.9	31594	31859	32121	32381	32639	32894	33147	33398	33646	33891
1.0	34134	34375	34614	34850	35083	35313	35543	35769	35993	36214
1.1	36433	36650	36864	37076	37286	37493	37698	37900	38100	38298
1.2	38493	38686	38877	39065	39251	39435	39617	39796	39973	40147
1.3	40320	40490	40658	40824	40988	41198	41308	41466	41621	41774
1.4	41924	42073	42220	42364	42507	42647	42786	42922	43056	43189
1.5	43319	43448	43574	43699	43822	43943	44062	44179	44295	44408
1.6	44520	44630	44738	44845	44950	45053	45154	45254	45352	45449
1.7	45543	45637	45728	45818	45907	45994	46080	46164	46246	46327
1.8	46407	46485	46562	46638	46712	46784	46856	46926	46995	47062
1.9	47128	47193	47257	47320	47381	47441	47500	47558	47615	47670
2.0	47725	47778	47831	47882	47932	47962	48030	48077	48124	48169
2.1	48214	48257	48300	48341	48382	48422	48461	48500	48537	48574
2.2	48610	48645	48679	48713	48745	48778	48809	48840	48870	48899
2.3	48928	48956	48983	49010	49036	49061	49086	49111	49134	49158
2.4	49180	49202	49224	49245	49266	49286	49305	49324	49343	49361
2.5	49379	49296	49413	49430	49446	49461	49477	49492	49506	49520
2.6	49534	49547	49560	49573	49585	49598	49609	49621	49632	49643
2.7	49653	49664	49674	49683	49693	49702	49711	49720	49728	49736
2.8	49744	49752	49760	49767	49774	49781	49788	49795	49801	49807
2.9	49813	49819	49825	49831	49836	49841	49846	49851	49856	49861
3.0	49865									
3.5	4997674									
4.0	4999683									
4.5	4999966									
5.0	4999997133									

TABLE 3-3
Chances for deviations from mean value of normal-distribution curve

Deviation	Chance of results falling within specified deviation
$\pm 0.6745\sigma$	1–1
σ	2.15–1
2σ	21–1
3σ	369–1

these points are the result of some gross experimental blunder and hence may be neglected or if they represent some new type of physical phenomenon that is peculiar to a certain operating condition. The engineer cannot just throw out those points that do not fit with expectations—there must be some consistent basis for elimination.

Suppose n measurements of a quantity are taken and n is large enough that we may expect the results to follow the gaussian error distribution. This distribution may be used to compute the probability that a given reading will deviate a certain amount from the mean. We would not expect a probability much smaller than $1/n$ because this would be unlikely to occur in the set of n measurements. Thus, if the probability for the observed deviation of a certain point is less than $1/n$, a suspicious eye would be cast at that point with an idea toward eliminating it from the data. Actually, a more restrictive test is usually applied to eliminate data points. It is known as *Chauvenet's criterion* and specifies that a reading may be rejected if the probability of obtaining the particular deviation from the mean is less than $1/2n$. Table 3-4 lists values of the ratio of deviation to standard deviation for various values of n according to this criterion.

TABLE 3-4
Chauvenet's criterion for rejecting a reading

Number of readings, n	Ratio of maximum acceptable deviation to standard deviation, d_{max}/σ
3	1.38
4	1.54
5	1.65
6	1.73
7	1.80
10	1.96
15	2.13
25	2.33
50	2.57
100	2.81
300	3.14
500	3.29
1,000	3.48

In applying Chauvenet's criterion to eliminate dubious data points, one first calculates the mean value and standard deviation using all data points. The deviations of the individual points are then compared with the standard deviation in accordance with the information in Table 3-4 (or by a direct application of the criterion), and the dubious points are eliminated. For the final data presentation a new mean value and standard deviation are computed with the dubious points eliminated from the calculation. Note that Chauvenet's criterion might be applied a second or third time to eliminate additional points; but this practice is unacceptable, and only the first application may be used.

Example 3-10. Using Chauvenet's criterion, test the data points of Example 3-6 for possible inconsistency. Eliminate the questionable points and calculate a new standard deviation for the adjusted data.

Solution. The best estimate of the standard deviation is given in Example 3-7 as 0.627 cm. We first calculate the ratio d_i/σ and eliminate data points in accordance with Table 3-4.

Reading	d_i/σ
1	0.499
2	0.187
3	1.845
4	0.563
5	2.046
6	0.260
7	0.761
8	0.043
9	0.314
10	0.219

In accordance with Table 3-4, we may eliminate only point number 5. When this point is eliminated, the new mean value is

$$x_m = \tfrac{1}{9}(51.80) = 5.756 \text{ cm}$$

The new value of the standard deviation is now calculated with the following table:

Reading	$d_i = x_i - x_m$	$(x_i - x_m)^2 \times 10^2$
1	−0.456	20.7936
2	−0.026	0.0676
3	1.014	102.8196
4	−0.496	24.602
6	−0.306	9.364
7	0.334	11.156
8	−0.116	1.346
9	0.054	0.292
10	−0.006	0.0036

$$\sigma = \left[\frac{1}{n-1} \sum_{i=1}^{n} (x_i - x_m)^2 \right]^{1/2} = [\tfrac{1}{8}(1.7044)]^{1/2}$$

$$= (0.213)^{1/2} = 0.4615 \text{ cm}$$

Thus, by the elimination of the one point, the standard deviation has been reduced from 0.627 to 0.462 cm. This is a 26.5 percent reduction.

3-9 PROBABILITY GRAPH PAPER

We have seen that the normal error distribution offers a means for examining experimental data for statistical consistency. In particular, it enables us to eliminate questionable readings with the Chauvenet criterion and thus obtain a better estimate of the standard deviation and mean reading. If the distribution of random errors is not *normal*, then this elimination technique will not apply. It is to our advantage, therefore, to determine if the data are following a normal distribution before making too many conclusions about the mean value, variances, etc. Specially constructed probability graph paper is available for this purpose and may be purchased from a technical drawing shop. The paper uses the coordinate system shown in Fig. 3-6. The ordinate has the percent of readings at or below the value of the abscissa, and the abscissa is the value of a particular reading. The ordinate spacings are arranged so that the gaussian-distribution curve will plot as a straight line on the graph. In addition, this straight line will intersect the 50 percent ordinate at an abscissa equal to the arithmetic mean of the data.

Thus, to determine if a set of data points is distributed normally, we plot the data on probability paper and see how well they match with the theoretical straight line. It is to be noted that the largest reading cannot be plotted on the graph because the ordinate does not extend to 100 percent. In assessing the validity of the data we should not place as much reliance on the points near the upper and lower ends of the curve since they are closer to the "tails" of the probability distribution and are thus less likely to be valid.

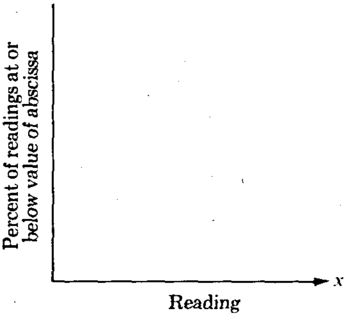


FIGURE 3-6
Probability graph paper.

Example 3-11. The following data are collected for a certain measurement. Plot the data on probability paper and comment on the normality of the distribution.

Reading	x_i, cm
1	4.62
2	4.69
3	4.86
4	4.53
5	4.60
6	4.65
7	4.59
8	4.70
9	4.58
10	4.63
	$\sum x_i = 46.45$

From these data the mean value is calculated as

$$x_m = \tfrac{1}{10} \sum x_i = \tfrac{1}{10}(46.45) = 4.645 \text{ cm}$$

The data are plotted in the accompanying figure indicating a reasonably normal distribution. It should be noted that the straight line crosses the 50 percent ordinate at a value of approximately $x = 4.62$, which is not in agreement with the calculated value of x_m. Note that point 3, $x = 4.86$, does not appear on the plot since it would represent the 100 percent ordinate.

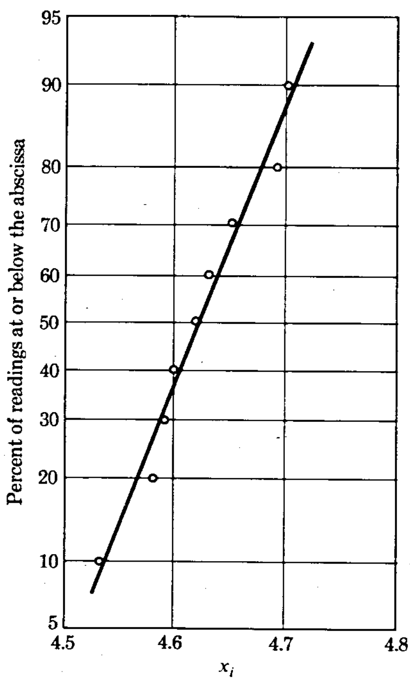


FIGURE EXAMPLE 3-10

3-10 THE CHI-SQUARE TEST OF GOODNESS OF FIT

In the previous discussion we have noted that random experimental errors would be expected to follow the gaussian distribution, and the examples illustrated the method of calculating the probability of occurrence of a particular experimental determination. We might ask how it is known that the random errors or deviations do approximate a gaussian distribution. In general, we may ask how we can determine if experimental observations match some particular expected distribution for the data. As a simple example, consider the tossing of a coin. We would like to know if a certain coin is "honest," i.e., unweighted toward either heads or tails. If the coin is unweighted, then heads should occur half the time and tails should occur half the time. But suppose we do not want to take the time to make thousands of tosses to get a frequency distribution of heads and tails for a large number of tosses. Instead, we toss the coin a few times and wish to infer from these few tosses whether the coin is unweighted or weighted. Common sense tells us not to expect exactly six heads and six tails out of, say, twelve tosses. But how much deviation from this arrangement could we tolerate and still expect the coin to be unweighted? The chi-square test of goodness of fit is a suitable way of answering this question. It is based on a calculation of the quantity chi squared, defined by

$$\chi^2 = \sum_{i=1}^{n} \frac{[(\text{observed value})_i - (\text{expected value})_i]^2}{(\text{expected value})_i} \tag{3-19}$$

where n is the number of cells or groups of observations. The expected value is the value which would be obtained if the measurements matched the expected distribution perfectly.

The chi-square test may be applied to check the validity of various distributions. Calculations have been made [2] of the probability that the actual measurements match the expected distribution, and these probabilities are given in Table 3-5. In this table, F represents the number of degrees of freedom in the measurements and is given by

$$F = n - k \tag{3-20}$$

where n is the number of cells and k is the number of imposed conditions on the expected distribution. A plot of the chi-square function is given in Fig. 3-7.

While we initiated the discussion on the chi-square test in terms of random errors following the gaussian distribution, the test is an important tool for testing any expected experimental distribution. In other words, we might use the test to analyze random errors or to check the adherence of certain data to an expected distribution. We interpret the test by calculating the number of degrees of freedom and χ^2 from the experimental data. Then, consulting Table 3-5, we obtain the probability P that this value of χ^2 or higher value could occur by chance. If $\chi^2 = 0$, then the assumed or expected distribution and measured distribution match exactly. The larger the value of χ^2, the larger is the disagreement between the assumed distribution and the observed values, or

TABLE 3-5

Chi-squared. *P* is the probability that the value in the table will be exceeded for a given number of degrees of freedom *F*†

F = \ P	0.995	0.990	0.975	0.950	0.900	0.750	0.500	0.250	0.100	0.050	0.025	0.010	0.005
1	0.0^4393	0.0^3157	0.0^3982	0.0^2393	0.0158	0.102	0.455	1.32	2.71	3.84	5.02	6.63	7.88
2	0.0100	0.0201	0.0506	0.103	0.211	0.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6
3	0.0717	0.115	0.216	0.352	0.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	0.207	0.297	0.484	0.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9
5	0.412	0.554	0.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	0.676	0.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	0.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.35	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.3	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

† From C. M. Thompson: *Biometrika*, vol. 32, 1941, as abridged by A. M. Mood and F. A. Graybill, "Introduction to the Theory of Statistics," 2d ed., McGraw-Hill Book Company, New York, 1963.

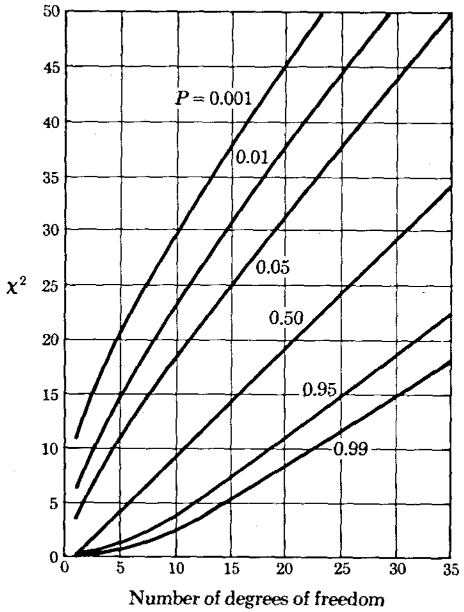


FIGURE 3-7
The chi-square function.

the smaller the probability that the observed distribution matches the expected distribution. The reader should consult Refs. [2] and [4] for more specific information on the chi-square test and the derivation of the probabilities associated with it.

One may note that the heading of this section includes the term "goodness of fit." We see that the chi-square test may be used to determine how well a set of experimental observations fits an assumed distribution. In connection with this test we may remark that data may sometimes be "too good" or "too consistent." For example, we would be quite surprised if in the conduct of an experimental test, the results were found to check with theory *exactly* or to follow some well-defined relationship exactly. We might find, for instance, that a temperature controller maintained a set point temperature *exactly*, with no measurable deviation whatsoever. Experienced laboratory people know that controllers usually do not operate this way and would immediately suspect that the temperature recorder might be stuck or otherwise defective. The point of this brief remark is that one must be suspicious of high values of P as well as low values. A good rule of thumb is that if P lies between 0.1 and 0.9, the observed distribution may be considered to follow the assumed distribution. If P is either less than 0.02 or greater than 0.98, the assumed distribution may be considered unlikely.

Let us return for a moment to the tossing of a coin. Suppose a coin is tossed twice, resulting in one head and one tail. This observation certainly matches exactly with what would be expected for an unweighted coin; however, our common sense tells us not to believe the coin is unweighted on the basis of only two tosses. In other words, we must have a certain minimum number of samples for statistics to apply. For the chi-square test the generally accepted minimum number of expected values for each ith cell is 5. If some frequencies fall below 5 it is recommended that the cells or groups be redefined to alleviate the problem.

For example, a plastics company produces two types of styrofoam cups (call them A and B) which can experience eight kinds of defects. One hundred defective samples of each cup are collected and the number of each type of defect determined. The following table results:

Type defect	Cup A	Cup B
1	1	5
2	2	3
3	3	3
4	25	23
5	10	12
6	15	16
7	38	30
8	6	8
Total	100	100

We would like to know if the two cups have the same pattern of defects. To do this we could compute chi squared for cup B assuming cup A has the expected distribution. But we encounter a problem. Defects 1, 2, and 3 do not meet our criterion of a minimum of 5 expected values in each cell. So we must reconstruct the cells by combining 1, 2, and 3 to obtain:

Type defect	Cup A	Cup B
1, 2, 3	6	11
4	25	23
5	10	12
6	15	16
7	38	30
8	6	8
Total	100	100

For the former case we had eight cells or groups and one imposed condition (total observations = 100), so $F = 8 - 1 = 7$. After grouping defects 1, 2, and 3 we have $F = 6 - 1 = 5$. Using this new tabulation the value of chi-squared is calculated as 7.145. Consulting Table 3-5, we obtain the value of P as 0.43. Thus, we might expect that the two cups have approximately the same pattern of defects.

Example 3-12. Two dice are rolled 300 times and the following results are noted:

Number	Number of occurrences
2	6
3	9
4	27
5	36
6	39
7	57
8	45
9	39
10	24
11	12
12	6

Calculate the probability that the dice are unloaded.

Solution. Eleven cells have been observed with only one restriction: the number of rolls of the dice is fixed. Thus, $F = 11 - 1 = 10$. If the dice are unloaded, a short listing of the combinations of the dice will give the probability of occurrence for each number. The expected value of each number is then the probability multiplied by 300, the total number of throws. The values of interest are tabulated below.

Number	Observed	Probability	Expected
2	6	1/36	8.333
3	9	1/18	16.667
4	27	1/12	25.0
5	36	1/9	33.333
6	39	5/36	41.667
7	57	1/6	50.0
8	45	5/36	41.667
9	39	1/9	33.333
10	24	1/12	25.0
11	12	1/18	16.667
12	6	1/36	8.333

From these data the value of chi squared is calculated as 8.034. If Table 3-5 is consulted the probability is given as $P = 0.626$.

Example 3-13. A coin is tossed 20 times, resulting in 6 heads and 14 tails. Using the chi-square test, estimate the probability that the coin is unweighted. Suppose another set of tosses of the same coin is made and 8 heads and 12 tails are obtained. What is the probability of having an unweighted coin based on the information from both sets of data?

Solution. For each set of data we may make only two observations: the number of heads and the number of tails. Thus, $n = 2$. Furthermore, we impose one restriction on the data: the number of tosses is fixed. Thus, $k = 1$ and the number of degrees of freedom is

$$F = n - k = 2 - 1 = 1$$

The values of interest are

	Observed	Expected
Heads	6	10
Tails	14	10

For these values χ^2 is calculated as

$$\chi^2 = \frac{(6-10)^2}{10} + \frac{(14-10)^2}{10} = 3.20$$

Consulting Table 3-5 we find $P = 0.078$; that is, there is an 8 percent chance that this distribution is just the result of random fluctuations and that the coin may be unweighted.

Now consider the additional information we gain about the coin from the second set of observations. We now have four observations: the number of heads and tails in each set. There are only two restrictions on the data: the total number of tosses is fixed in each set. Thus, the number of degrees of freedom is

$$F = n - k = 4 - 2 = 2$$

For the second set of data the values of interest are

	Observed	Expected
Heads	8	10
Tails	12	10

Chi squared is now calculated on the basis of all four observations.

$$\chi^2 = \frac{(6-10)^2}{10} + \frac{(14-10)^2}{10} + \frac{(8-10)^2}{10} + \frac{(12-10)^2}{10} = 4.0$$

Consulting Table 3-5 again, we find $P = 0.15$. So, with the additional information we find a stronger likelihood that the tosses are following a random variation and that the coin is unweighted.

Example 3-14. A test is conducted to determine the effect of cigarette smoke on the eating habits and weight of mice. One group is fed a certain diet while being exposed to a controlled atmosphere containing cigarette smoke. A control group is fed the same diet but in the presence of clean air. The observations are given below. Does the presence of smoke cause a loss in weight?

	Gained weight	Lost weight	Total
Exposed to smoke	61	89	150
Exposed to clean air	65	77	142
Total	126	166	292

Solution. Clearly, there are four observations in this experiment, but we are faced with the problem of deciding on the expected values. We cannot just take the "clean-air" data as the expected values because some of the behavior might be a result of the special diet that is fed to both groups of mice. Consequently, about the best estimate we can make is one based on the total sample of mice. Thus, the expected frequencies would be

$$\text{Expected fraction to gain weight} = \tfrac{126}{292}$$

$$\text{Expected fraction to lose weight} = \tfrac{166}{292}$$

The expected values for the groups would thus be

	Gained weight	Lost weight
Exposed to smoke	$\frac{126}{292}\,150 = 64.7$	$\frac{166}{292}\,150 = 85.3$
Exposed to clean air	$\frac{126}{292}\,142 = 61.3$	$\frac{166}{292}\,142 = 80.7$

We observe that there are three restrictions on the data: (1) the number exposed to smoke, (2) the number exposed to clean air, and (3) the additional restriction involved in the calculation of the expected fractions which gain and lose weight. The number of degrees of freedom is thus

$$F = 4 - 3 = 1$$

The value of chi squared is calculated from

$$\chi^2 = \frac{(61-64.7)^2}{64.7} + \frac{(89-85.3)^2}{85.3} + \frac{(65-61.3)^2}{61.3} + \frac{(77-80.7)^2}{80.7} = 0.767$$

From Table 3-5 we find $P = 0.41$, or there is a 41 percent chance that the difference in the observations for the two groups is just the result of random fluctuations. One may not conclude from this information that the presence of cigarette smoke causes a loss in weight for the mice.

3-11 METHOD OF LEAST SQUARES

Suppose we have a set of observations $x_1, x_2, \ldots, x_n$. The sum of the squares of their deviations from some mean value is

$$S = \sum_{i=1}^{n} (x_i - x_m)^2 \tag{3-21}$$

Now suppose we wish to minimize S with respect to the mean value x_m. We set

$$\frac{\partial S}{\partial x_m} = 0 = \sum_{i=1}^{n} -2(x_i - x_m) = -2\left(\sum_{i=1}^{n} x_i - n x_m\right) \tag{3-22}$$

where n is the number of observations. We find that

$$x_m = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{3-23}$$

or the mean value which minimizes the sum of the squares of the deviations is the arithmetic mean. This example might be called the simplest application of the method of least squares. We shall be able to give only two other applications of the method, but it is of great utility in analyzing experimental data.

Suppose that the two variables x and y are measured over a range of values. Suppose further that we wish to obtain a simple analytical expression for y as a function of x. The simplest type of function is a linear one; hence we might try to establish y as a linear function of x. (Both x and y may be complicated functions of other parameters so arranged that x and y vary approximately in a linear manner. This matter will be discussed later.) The problem is one of finding the *best* linear function, for the data may scatter a considerable amount. We could solve the problem rather quickly by plotting the data points on graph paper and drawing a straight line through them by eye. Indeed this is common practice, but the method of least squares gives a more reliable way to obtain a better functional relationship than the guesswork of plotting. We seek an equation of the form

$$y = ax + b \tag{3-24}$$

We therefore wish to minimize the quantity

$$S = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2 \tag{3-25}$$

This is accomplished by setting the derivatives with respect to a and b equal to zero. Performing these operations, there results

$$nb + a \sum x_i = \sum y_i \tag{3-26}$$

$$b \sum x_i + a \sum x_i^2 = \sum x_i y_i \tag{3-27}$$

Solving Eqs. (3-26) and (3-27) simultaneously gives

$$a = \frac{n \sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{n \sum x_i^2 - \left(\sum x_i\right)^2} \tag{3-28}$$

$$b = \frac{\left(\sum y_i\right)\left(\sum x_i^2\right) - \left(\sum x_i y_i\right)\left(\sum x_i\right)}{n \sum x_i^2 - \left(\sum x_i\right)^2} \tag{3-29}$$

Designating the computed value of y as $\hat{y}$, we have

$$\hat{y} = ax + b$$

and the standard error of estimate of y for the data is

$$\text{Standard error} = \left[\frac{\sum (y_i - \hat{y}_i)^2}{n-2} \right]^{1/2} \tag{3-30}$$

$$= \left[\frac{\sum (y_i - ax_i - b)^2}{n-2} \right]^{1/2} \tag{3-31}$$

The method of least squares may also be used for determining higher-order polynomials for fitting data. One only needs to perform additional differentiations to determine additional constants. For example, if it were desired to obtain a least-squares fit according to the quadratic function

$$y = ax^2 + bx + c$$

the quantity

$$S = \sum_{i=1}^{n} [y_i - (ax_i^2 + bx_i + c)]^2$$

would be minimized by setting the following deivatives equal to zero:

$$\frac{\partial S}{\partial a} = 0 = \sum 2[y_i - (ax_i^2 + bx_i + x)](-x_i^2)$$

$$\frac{\partial S}{\partial b} = 0 = \sum 2[y_i - (ax_i^2 + bx_i + c)](-x_i)$$

$$\frac{\partial S}{\partial c} = 0 = \sum 2[y_i - (ax_i^2 + bx_i + c)](-1)$$

Expanding and collecting terms,

$$a \sum x_i^4 + b \sum x_i^3 + c \sum x_i^2 = \sum x_i^2 y_i \tag{3-32}$$

$$a \sum x_i^3 + b \sum x_i^2 + c \sum x_i = \sum x_i y_i \tag{3-33}$$

$$a \sum x_i^2 + b \sum x_i + cn = \sum y_i \tag{3-34}$$

These equations may then be solved for the constants a, b, and c.

Regression analysis

In the above discussion of the method of least squares no mention has been made of the influence of experimental uncertainty on the calculation. We are considering the method primarily for its utility in fitting an algebraic relationship to a set of data points. Clearly, the various x_i and y_i could have different experimental uncertainties. To take all these into account requires a rather tedious calculation procedure which we shall not present here; however, we may state the following rules:

1. If the values of x_i and y_i are taken as the data value in y and the value of x *on the fitted curve for the same value of* y, then there is a presumption that the uncertainty in x is large compared with that in y.

2. If the values of x_i and y_i are taken as the data value in y and the value *on the fitted curve for the same value of* x, the presumption is that the uncertainty in y dominates.
3. If the uncertainties in x_i and y_i are believed to be of approximately equal magnitude, a special averaging technique must be used.

In rule 1 we say we are taking a *regression* of x on y, and in rule 2 there is a regression of y on x. In the second case we are minimizing the sum of the squares of the deviations of the actual points from the assumed curve and also assuming *that x does not vary appreciably at each point.* If we obtained

$$y = a + bx$$

and then solved to get

$$x = \frac{1}{b} y - \frac{a}{b}$$

this second relation would *not* necessarily give a good calculation for x, since the minimization was carried out in the y direction, and not the x direction. In Example 3-15, rule 2 is assumed to apply.

Example 3-15. From the following data obtain y as a linear function of x using the method of least squares:

y_i	x_i
1.2	1.0
2.0	1.6
2.4	3.4
3.5	4.0
3.5	5.2
$\sum y_i = 12.6$	$\sum x_i = 15.2$

Solution. We seek an equation of the form

$$y = ax + b$$

We first calculate the quantities indicated in the following table:

$x_i y_i$	x_i^2
1.2	1.0
3.2	2.56
8.16	11.56
14.0	16.0
18.2	27.04
$\sum x_i y_i = 44.76$	$\sum x_i^2 = 58.16$

We calculate the value of a and b using Eqs. (3-28) and (3-29) with $n = 5$:

$$a = \frac{(5)(44.76) - (15.2)(12.6)}{(5)(58.16) - (15.2)^2} = 0.540$$

$$b = \frac{(12.6)(58.16) - (44.76)(15.2)}{(5)(58.16) - (15.2)^2} = 0.879$$

Thus, the desired relation is

$$y = 0.540x + 0.879$$

A plot of this relation and the data points from which it was derived is shown in the accompanying figure.

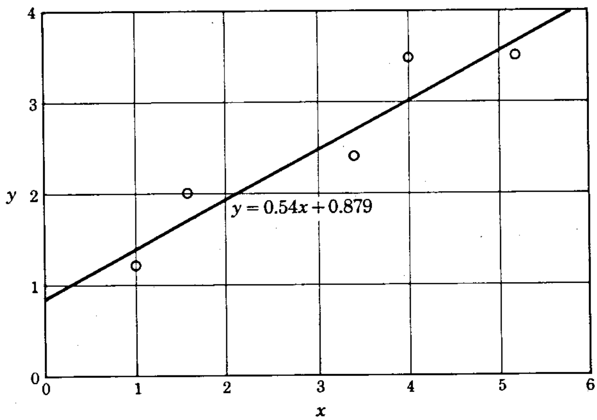


FIGURE EXAMPLE 3-15

3-12 THE CORRELATION COEFFICIENT

Let us assume that a suitable correlation between y and x has been obtained, either by least-squares analysis or graphical curve fitting. We want to know how good this fit is, and the parameter which conveys this information is the *correlation coefficient* r defined by

$$r = \left[1 - \frac{\sigma_{y,x}^2}{\sigma_y^2}\right]^{1/2} \tag{3-35}$$

where σ_y is the standard deviation of y given as

$$\sigma_y = \left[\frac{\sum_{i=1}^{n} (y_i - y_m)^2}{n-1}\right]^{1/2} \tag{3-36}$$

and

$$\sigma_{y,x} = \left[\frac{\sum_{i=1}^{n} (y_i - y_{ic})^2}{n-2}\right]^{1/2} \tag{3-37}$$

The y_i are the actual values of y, and the y_{ic} are the values computed from the correlation equation for the same value of x. It may be noted that many calculators have built-in routines which calculate the correlation coefficient as well as other statistical functions. In addition, there are many computer software packages which accomplish these calculations, for example, those of Ref. [9-13].

Example 3-16. Calculate the correlation coefficient for the least-square correlation of Example 3-15.

Solution. From Example 3-15

$$y_m = \frac{\sum y_i}{n} = \frac{12.6}{5} = 2.52$$

and from the correlating equation $y_{ic} = 0.540x + 0.879$

i	y_i	y_{ic}	$(y_i - y_{ic})^2$
1	1.2	1.419	0.048
2	2.0	1.743	0.066
3	2.4	2.715	0.0992
4	3.5	3.039	0.0212
5	3.5	3.687	0.035
			$\sum = 0.2694$

so that

$$\sigma_{y,x} = \left(\frac{0.2694}{3}\right)^{1/2} = 0.2997$$

Also, $\sigma_y = 0.987$, so that the correlation coefficient is

$$r = \left[1 - \left(\frac{0.2997}{0.987}\right)^2\right]^{1/2} = 0.9528$$

3-13 STANDARD DEVIATION OF THE MEAN

We have taken the arithmetic mean value as the best estimate of the true value of a set of experimental measurements. Considerable discussion has been devoted to the gaussian normal error distribution and to an examination of the various types of errors and deviations that may occur in an experimental measurement. But one very important question has not yet been answered; i.e., how *good* (or precise) is this arithmetic mean value which is taken as the best estimate of the true value of a set of readings? To obtain an experimental answer to this question it would be necessary to repeat the set of measurements and find a new arithmetic mean. In general, we would find that this new arithmetic mean would differ from the previous value, and thus we would not be able to resolve the problem until a large number of *sets of data* were collected. We would then know how well the mean of a single set approximated the mean which would be obtained with a large number of sets. The mean value of a large number of sets is presumably the true value. Consequently, we wish to know the standard deviation of the mean of a single set of data from this true value.

It turns out that the problem may be resolved with a statistical analysis which we shall not present here. The result is

$$\sigma_m = \frac{\sigma}{\sqrt{n}} \tag{3-38}$$

where σ_m = standard deviation of the mean value
σ = standard deviation of the set of measurements
n = number of measurements in the set

The following example illustrates the use of Eq. (3-38).

Example 3-17. For the data of Example 3-6, estimate the uncertainty in the calculated mean value of the readings.

Solution. We shall make this estimate for the original data and for the reduced data of Example 3-10. For the original data the standard deviation of the mean is

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{0.627}{\sqrt{10}} = 0.198 \text{ cm}$$

The arithmetic mean value calculated in Example 3-6 was $x_m = 5.613$ cm. We could now specify the uncertainty of this value by using the odds of Table 3-3:

$$\begin{aligned} x_m &= 5.613 \pm 0.198 \text{ cm} \qquad (2.15 \text{ to } 1) \\ &= 5.756 \pm 0.396 \text{ cm} \qquad (21 \text{ to } 1) \\ &= 5.613 \pm 0.594 \text{ cm} \qquad (369 \text{ to } 1) \end{aligned}$$

Using the data of Example 3-10, where one point has been eliminated by Chauvenet's criterion, we may make a better estimate of the mean value with less uncertainty. The standard deviation of the mean is calculated as

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{0.465}{\sqrt{9}} = 0.155 \text{ cm}$$

for the mean value of 5.756 cm. Thus, we would estimate the uncertainty as

$$\begin{aligned} x_m &= 5.756 \pm 0.155 \text{ cm} \qquad (2.15 \text{ to } 1) \\ &= 5.756 \pm 0.310 \text{ cm} \qquad (21 \text{ to } 1) \\ &= 5.756 \pm 0.465 \text{ cm} \qquad (369 \text{ to } 1) \end{aligned}$$

We should note that the calculation of statistical parameters like standard deviation and least-square fits to data is easily performed with standard computer programs which are available on even small hand calculators. Rapid expansion of the availability of compact programs for data analysis is to be expected.

3-14 GRAPHICAL ANALYSIS AND CURVE FITTING

Engineers are well known for their ability to plot many curves of experimental data and to extract all sorts of significant facts from these curves. The better one understands the physical phenomena involved in a certain experiment, the better is one able to extract a wide variety of information from graphical displays of experimental data. Because these physical phenomena may encompass all engineering science, we cannot discuss them here except to emphasize that the person who is usually most successful in analyzing experimental data is the one who understands the physical processes behind the data. Blind curve-plotting and cross-plotting usually generate an excess of displays, which are confusing not only to the management or supervisory personnel who must pass on the experiments, but sometimes even to the experimenter. To be blunt, the engineer should give considerable thought to the kind of information being looked for before even taking the graph paper out of the package.

Assuming that the engineer knows what is to be examined with graphical presentations, the plots may be carefully prepared and checked against appropriate theories. Frequently, a *correlation* of the experimental data is desired in terms of an analytical expression between variables that were measured in the experiment. When the data may be approximated by a straight line, the analytical relation is easy to obtain; but when almost any other functional variation is present, difficulties are usually encountered. This fact is easy to understand since a straight line is easily recognizable on graph paper, whereas the functional form of a curve is rather doubtful. The curve could be a polynomial, exponential, or complicated logarithmic function and still present roughly the same appearance to the eye. It is most convenient, then, to try to plot the data in such a form that a straight line will be obtained for certain types of functional relationships. If the experimenter has a good idea of the type of function that will represent the data, then the type of plot is easily selected. It is frequently possible to estimate the functional form that the data will take on the basis of theoretical considerations and the results of previous experiments of a similar nature.

Table 3-6 summarizes several different types of functions and plotting methods that may be used to produce straight lines on graph paper. The graphical measurements, which may be made to determine the various constants, are also shown. It may be remarked that the method of least squares may be applied to all these relations to obtain the best straight line to fit the experimental data. A number of personal computer software packages are available to accomplish the functional plots illustrated in Table 3-6. See, for example, Refs. [9], [11], and [12].

Please note that when using logarithmic or semilog graph paper it is unnecessary to make log calculations; the scaling of the paper automatically accomplishes this.

Incorporation of graphics in reports and presentations is discussed in Chapter 15.

3-15 GENERAL CONSIDERATIONS IN DATA ANALYSIS

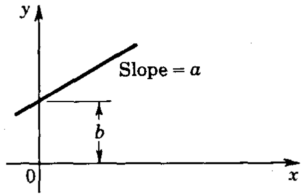
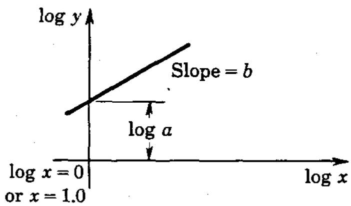
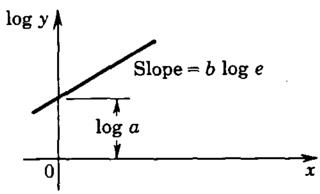
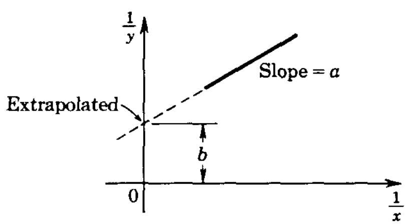
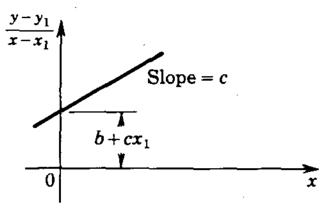
Our discussions in this chapter have considered a variety of topics: statistical analysis, uncertainty analysis, curve plotting, least squares, etc. With these tools the reader is equipped to handle a variety of circumstances that may occur in experimental investigations. As a summary to this chapter let us now give an approximate outline of the manner in which one would go about analyzing a set of experimental data.

1. *Examine the data for consistency.* No matter how hard one tries, there will always be some data points that appear to be grossly in error. If we add heat to a container of water, the temperature must rise, and so if a particular data point indicates a *drop* in temperature for a heat *input*, that point might be eliminated. In other words, the data should follow commonsense consistency, and points that do not appear proper should be eliminated. If very many data points fall in the category of "inconsistent," perhaps the entire experimental procedure should be investigated for gross mistakes or miscalculation.
2. *Perform a statistical analysis of data where appropriate.* A statistical analysis is only appropriate when measurements are repeated several times. If this is the case, make estimates of such parameters as standard deviation, etc.
3. *Estimate the uncertainties in the results.* We have discussed uncertainties at length. Hopefully, these calculations will have been performed in advance, and the investigator will already know the influence of different variables by the time the final results are obtained.
4. *Anticipate the results from theory.* Before trying to obtain correlations of the experimental data, the investigator should carefully review the theory appropriate to the subject and try to glean some information that will indicate the trends the results may take. Important dimensionless groups, pertinent functional relations, and other information may lead to a fruitful interpretation of the data.
5. *Correlate the data.* The word "correlate" is subject to misinterpretation. In the context here we mean that the experimental investigator should make sense of the data in terms of physical theories or on the basis of previous experimental work in the field. Certainly, the results of the experiments should be analyzed to show how they conform to or differ from previous investigations or standards that may be employed for such measurements.

3-16 SUMMARY

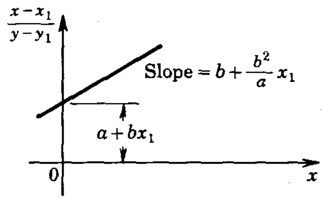
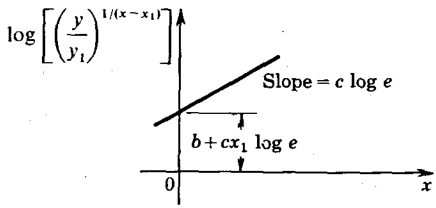
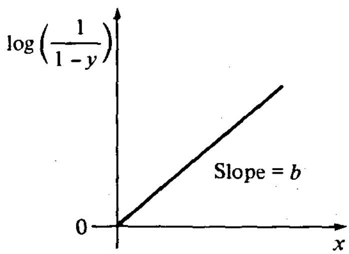
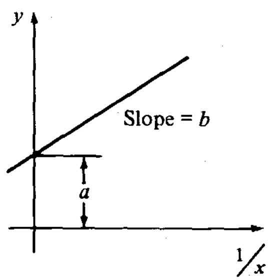
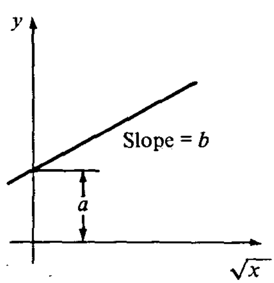
By now the reader will have sensed the central theme of this chapter as that of uncertainty analysis and the use of this analysis to influence experiment design, instrument selection, and evaluation of the results of experiments. At this point we must reiterate statements we have made before. We still must recognize that uncertainty is *not* the same as error, even though some people interchange the terms. As we saw in Chap. 2, the determination of "error" is eventually related to a comparison with a standard. Even then, there is still "uncertainty" in the error because the "standard" has its own uncertainty.

TABLE 3-6
Methods of plotting various functions to obtain straight lines

Functional relationship	Method of plot	Graphical determination of parameters
$y = ax + b$	y versus x on linear paper	
$y = ax^b$	$\log y$ versus $\log x$ on log-log paper	
$y = ae^{bx}$	$\log y$ versus x on semilog paper	
$y = \frac{x}{a + bx}$	$\frac{1}{y}$ versus $\frac{1}{x}$ on linear paper	
$y = a + bx + cx^2$	$\frac{y - y_1}{x - x_1}$ versus x on linear paper	

(*Continued*)

TABLE 3-6
Methods of plotting various functions to obtain straight lines (*Continued*)

Functional relationship	Method of plot	Graphical determination of parameters
$y = \dfrac{x}{a + bx} + c$	$\dfrac{x - x_1}{y - y_1}$ versus x on linear paper	
$y = ae^{bx + cx^2}$	$\log\left[\left(\dfrac{y}{y_1}\right)^{1/(x-x_1)}\right]$ versus x on semilog paper	
$y = 1 - e^{-bx}$	$\log\left(\dfrac{1}{1-y}\right)$ versus x on semilog paper	
$y = a + \dfrac{b}{x}$	y versus $\dfrac{1}{x}$ on linear paper	
$y = a + b\sqrt{x}$	y versus $\sqrt{x}$ on linear paper	

In the chapters which follow we shall examine a large number of instruments and measurement devices and will see how the concepts of error, uncertainty, and calibration apply to each.

REVIEW QUESTIONS

3-1. How does an error differ from an uncertainty?
3-2. What is a fixed error; random error?
3-3. Define standard deviation and variance.
3-4. In the normal error distribution, what does $P(x)$ represent?
3-5. What is meant by measure of precision?
3-6. What is Chauvenet's criterion and how is it applied?
3-7. What are some purposes of uncertainty analyses?
3-8. Why is an uncertainty analysis important in the preliminary stages of experiment planning?
3-9. How can an uncertainty analysis help to reduce overall experimental uncertainty?
3-10. What is meant by standard deviation of the mean?
3-11. What is a least-squares analysis?
3-12. What is the correlation coefficient?
3-13. What is meant by a regression analysis?

PROBLEMS

3-1. The resistance of a resistor is measured 10 times, and the values determined are 100.0, 100.9, 99.3, 99.9, 100.1, 100.2, 99.9, 100.1, 100.0, and 100.5. Calculate the uncertainty in the resistance.

3-2. A certain resistor draws 110.2 V and 5.3 A. The uncertainties in the measurements are ±0.2 V and ±0.06 A, respectively. Calculate the power dissipated in the resistor and the uncertainty in the power.

3-3. A small plot of land has measured dimensions of 50.0 by 150.0 ft. The uncertainty in the 50-ft dimension is ±0.01 ft. Calculate the uncertainty with which the 150-ft dimension must be measured to ensure that the total uncertainty in the area is not greater than 150 percent of that value it would have if the 150-ft dimension were exact.

3-4. Two resistors R_1 and R_2 are connected in series and parallel. The values of the resistances are

$$R_1 = 100.0 \pm 0.1\ \Omega$$

$$R_2 = 50.0 \pm 0.03\ \Omega$$

Calculate the uncertainty in the combined resistance for both the series and the parallel arrangements.

3-5. A resistance arrangement of 50 Ω is desired. Two resistances of 100.0 ± 0.1 Ω and two resistances of 25.0 ± 0.02 Ω are available. Which should be used, a series arrangement with the 25-Ω resistors or a parallel arrangement with the 100-Ω resistors? Calculate the uncertainty for each arrangement.

3-6. The following data are taken from a certain heat-transfer test. The expected correlation equation is $y = ax^b$. Plot the data in an appropriate manner, and use the method of least squares to obtain the best correlation.

x	2040	2580	2980	3220	3870	1690	2130	2420	2900	3310	1020	1240	1360	1710	2070
y	33.2	32.0	42.7	57.8	126.0	17.4	21.4	27.8	52.1	43.1	18.8	19.2	15.1	12.9	78.5

Calculate the mean deviation of these data from the best correlation.

3-7 A horseshoes player stands 30 ft from the target. The results of the tosses are

Toss	Deviation from target, ft	Toss	Deviation from target, ft
1	0	6	+2.4
2	+3	7	−2.6
3	−4.2	8	+3.5
4	0	9	+2.7
5	+1.5	10	0

On the basis of these data would you say that this is a good player or a poor player? What advice would you give this player in regard to improving at the game?

3-8. Calculate the probability of drawing a full house (three of a kind and two of a kind) in the first 5 cards from a 52-card deck.

3-9. Calculate the probability of filling an inside straight with one draw from the remaining 48 cards of a 52-card deck.

3-10. A voltmeter is used to measure a known voltage of 100 V. Forty percent of the readings are within 0.5 V of the true value. Estimate the standard deviation for the meter. What is the probability of an error of 0.75 V?

3-11. In a certain mathematics course the instructor informs the class that grades will be distributed according to the following scale provided that the average class score is 75:

Grade	A	B	C	D	F
Score	90–100	80–90	70–80	60–70	Below 60

Estimate the percentage distribution of grades for 5, 10, and 15 percent failing. Assume that there are just as many A's as F's.

3-12. For the following data points y is expected to be a quadratic function of x. Obtain this quadratic function by means of a graphical plot and also by the method of least squares.

x	1	2	3	4	5
y	1.9	9.3	21.5	42.0	115.7

3-13. It is suspected that the rejection rate for a plastic-cup-molding machine is dependent on the temperature at which the cups are molded. A series of short tests is conducted to examine this hypothesis with the following results:

Temperature	Total production	Number rejected
T_1	150	12
T_2	75	8
T_3	120	10
T_4	200	13

On the basis of these data do you agree with the hypothesis?

3-14. A capacitor discharges through a resistor according to the relation $E/E_0 = e^{-t/RC}$ where E_0 = voltage at time zero; R = resistance; C = capacitance. The value of the capacitance is to be measured by recording the time necessary for the voltage to drop to a value E_1. Assuming that the resistance is known accurately, derive an expression for the percent uncertainty in the capacitance as a function of the uncertainty in the measurements of E_1 and t.

3-15. In heat-exchanger applications, a log mean temperature is defined by

$$\Delta T_m = \frac{(T_{h_1} - T_{c_1}) - (T_{h_2} - T_{c_2})}{\ln[(T_{h_1} - T_{c_1})/(T_{h_2} - T_{c_2})]}$$

where the four temperatures are measured at appropriate inlet and outlet conditions for the heat-exchanger fluids. Assuming that all four temperatures are measured with the same absolute uncertainty w_T, derive an expression for the percentage uncertainty in ΔT_m in terms of the four temperatures and the value of w_T. Recall that the percentage uncertainty is

$$\frac{w_{\Delta T_m}}{\Delta T_m} \times 100$$

3-16. A certain length measurement is made with the following results:

Reading	1	2	3	4	5	6	7	8	9	10
x, in	49.36	50.12	48.98	49.24	49.26	50.56	49.18	49.89	49.33	49.39

Calculate the standard deviation, the mean reading, and the uncertainty. Apply Chauvenet's criterion as needed.

3-17. Devise a method for plotting the gaussian normal error distribution such that a straight line will result. (*Ans.* $(1/\eta)\ln[\sqrt{2\pi}P(\eta)]$ versus η.) Show how such a plot may be labeled so that it can be used to estimate the fraction of points which lie below a certain value of η. Subsequently show that this plot may be used to investigate the normality of a set of data points. Apply this reasoning to the data points of Example 3-6 and Probs. 3-6 and 3-7.

3-18. A citizens' traffic committee decides to conduct its own survey and analysis of the influence of drinking on car accidents. By some judicious estimates the committee determines that in their community 30 percent of the drivers on a Saturday evening between 10 P.M. and 2 A.M. have consumed some alcohol. During this same period there were 50 accidents, varying from minor scratched fenders to fatalities. In these 50 accidents 50 of the drivers had had something to drink (there are 100 drivers for 50 accidents). From these data what conclusions do you draw about the influence of drinking on car accidents? Can you devise a better way to perform this analysis?

3-19. The grades for a certain class fall in the following ranges:

Number	10	30	50	40	10	8
Score	90–100	80–90	70–80	60–70	50–60	Below 50

The arithmetic mean grade is 68. Devise your own grade distribution for this class. Be sure to establish the criteria for the distribution.

3-20. A certain length measurement is performed 100 times. The arithmetic mean reading is 6.823 ft, and the standard deviation is 0.01 ft. How many readings fall within (*a*) ±0.005 ft, (*b*) ±0.02 ft, (*c*) ±0.05 ft, and (*d*) 0.001 ft of the mean value?

3-21. A series of calibration tests is conducted on a pressure gage. At a known pressure of 1000 psia, it is found that 30 percent of the readings are within 1 psia of the true value. At a known pressure of 500 psia, 40 percent of the readings are within 1 psia. At a pressure of 200 psia, 45 percent of the readings are within 1 psia. What conclusions do you draw from these readings? Can you estimate a standard deviation for the pressure gage?

3-22. Two resistors are connected in series and have the following values:

$$R_1 = 10{,}000\ \Omega \pm 5\% \qquad R_2 = 1\ \text{M}\Omega \pm 10\%$$

Calculate the percent uncertainty for the series total resistance.

3-23. Apply Chauvenet's criterion to the data of Example 3-11 and then replot the data on probability paper, omitting any excluded points.

3-24. Plot the data of Example 3-6 on probability paper. Replot the data, taking into account the point eliminated in Example 3-10. Comment on the normality of these two sets of data.

3-25. Two groups of secretaries operate under the same manager. Both groups have the same number of people, use the same equipment, and turn out about the same amount of work. During one maintenance period, group *A* had 10 service calls on the equipment while group *B* had only 6 calls. From these data would you conclude that group *A* was harder on the equipment?

3-26. A laboratory experiment is conducted to measure the viscosity of a certain oil. A series of tests gives the values as 0.040, 0.041, 0.041, 0.042, 0.039, 0.040, 0.043, 0.041, and 0.039 ft^2/s. Calculate the mean reading, the variance, and the standard deviation. Eliminate any data points as necessary.

3-27. The following data are expected to follow a linear relation of the form $y = ax + b$. Obtain the best linear relation in accordance with a least-squares analysis. Calculate the standard deviation of the data from the predicted straight-line relation.

x	0.9	2.3	3.3	4.5	5.7	6.7
y	1.1	1.6	2.6	3.2	4.0	5.0

3-28. The following data points are expected to follow a functional variation of $y = ax^b$. Obtain the values of *a* and *b* from a graphical analysis.

x	1.21	1.35	2.40	2.75	4.50	5.1	7.1	8.1
y	1.20	1.82	5.0	8.80	19.5	32.5	55.0	80.0

3-29. The following data points are expected to follow a functional variation of $y = ae^{bx}$. Obtain the values of a and b from a graphical analysis.

x	0	0.43	1.25	1.40	2.60	2.9	4.3
y	9.4	7.1	5.35	4.20	2.60	1.95	1.15

3-30. The following heat-transfer data points are expected to follow a functional form of $N = aR^b$. Obtain the values of a and b from a graphical analysis and also by the method of least squares.

R	12	20	30	40	100	300	400	1000	3000
N	2	2.5	3	3.3	5.3	10	11	17	30

What is the average deviation of the points from the correlating relationship?

3-31. In a student laboratory experiment a measurement is made of a certain resistance by different students. The values obtained were

Reading	1	2	3	4	5	6	7	8	9	10	11	12
Resistance, kΩ	12.0	12.1	12.5	11.8	13.6	11.9	12.2	11.9	12.0	12.3	12.1	11.85

Calculate the standard deviation, the mean reading, and the uncertainty.

3-32. In a certain decade resistance box resistors are arranged so that four resistances may be connected in series to obtain a desired result. The first selector uses 10 resistances of 1000, 2000, . . . , 9000, the second uses 10 of 100, 200, . . . , 900, the third uses 10 of 10, 20, . . . , 90, and the fourth, 1, 2, . . . , 9 Ω. Thus, the overall range is 0 to 9999 Ω. If all the resistors have an uncertainty of ±1.0 percent, calculate the percent uncertainties for total resistances of 9, 56, 148, 1252, and 9999 Ω.

3-33. Calculate the chances and probabilities that data following a normal-distribution curve will fall within 0.2, 1.2, and 2.2 standard deviations of the mean value.

3-34. Suggest improvements in the measurement uncertainties for Example 3-4 which will result in reduction in the overall uncertainty of flow measurement to ±1.0 percent.

3-35. What uncertainty in the resistance for the first part of Example 3-2 is necessary to produce the same uncertainty in power determination as results from the current and voltage measurements?

3-36. Use the technique of Sec. 3-5 with Example 3-4.

3-37. Use the technique of Sec. 3-5 with Examples 3-3 and 3-2.

3-38. Obtain the correlation coefficient for Prob. 3-27.

3-39. Obtain the correlation coefficient for Prob. 3-28.

3-40. Obtain the correlation coefficient for Probs. 3-29 and 3-30.

3-41. Obtain the correlation coefficient for Probs. 3-6 and 3-12.

3-42. For the heat exchanger of Prob. 3-15 the temperatures are measured as $T_{h_1} = 100°C$, $T_{h_2} = 80°C$, $T_{c_1} = 75°C$, and $T_{c_2} = 55°C$. All temperatures have an uncertainty of ±1°C. Calculate the uncertainty in ΔT_m using the technique of Sec. 3-5.

3-43. Repeat Prob. 3-41 but with $T_{c_1} = 90°C$ and $T_{c_2} = 70°C$.

3-44. Four resistors having nominal values of 1, 1.5, 3, and 2.5 kΩ are connected in parallel. The uncertainties are ±10 percent. A voltage of 100 V ± 1.0 V is impressed on the combination. Calculate the power drawn and its uncertainty. Use Sec. 3-5.

3-45. A radar speed-measurement device for state police is said to have an uncertainty of ±4 percent when directed straight at an oncoming vehicle. When directed at some angle θ from the straight-on position, the device measures a component of the vehicle speed. The police officer can only obtain a value for the angle θ through a visual observation having an uncertainty of ±10°. Calculate the uncertainty of the speed measurement for θ values of 0, 10, 20, 30, and 45°. Use the techniques of both Secs. 3-4 and 3-5.

3-46. An automobile is to be tested for its acceleration performance and fuel economy. Plan this project taking into account the measurements which must be performed and expected uncertainties in these measurements. Assume that three different drivers will be used for the tests. Make plans for the number of runs which will be used to reduce the data. Also prepare a detailed outline with regard to form and content of the report which will be used to present the results.

3-47. A thermocouple is used to measure the temperature of a known standard maintained at 100°C. After converting the electrical signal to temperature the readings are: 101.1, 99.8, 99.9, 100.2, 100.5, 99.6, 100.9, 99.7, 100.1, and 100.3. Using whatever criteria seem appropriate, make some statements about the calibration of the thermocouple.

3-48. Seven students are asked to make a measurement of the thickness of a steel block with a micrometer. The actual thickness of the block is known very accurately as 2.000 cm. The seven measurements are: 2.002, 2.001, 1.999, 1.997, 1.998, 2.003, and 2.003 cm. Comment on these measurements using whatever criteria you think appropriate.

3-49. A collection of 120 rock aggregate samples is taken and the volumes measured for each. The mean volume is 6.8 cm^3 and the standard deviation is 0.7 cm^3. How many rocks would you expect to have volumes ranging from 6.5 to 7.2 cm^3?

3-50. Plot the equation $y = 5e^{1.2x}$ on semilog paper. Arbitrarily assign fictitious data points on both sides of the line so that the line appears by eye as a reasonable representation. Then, using these points, perform a least-squares analysis to obtain the best fit to the points. What do you conclude from this comparison?

3-51. The following data are presumed to follow the relation $y = ax^b$. Plot the values of x and y on log-log graph paper and draw a straight line through the points. Subsequently, obtain the values of a and b. Then determine the values of a and b by the method of least squares. Compute the standard deviation for both cases. If a packaged computer routine for the least-squares analysis is available, use it.

x	y
4	105
5.3	155
11	320
21	580
30	1050
50	1900

3-52. The variables x and y are related by the quadratic equation

$$y = 2 - 0.3x + 0.01x^2$$

for $0 < x < 2$. Compute the percentage uncertainty in y for uncertainties in x of ± 1, 2, and 3 percent. Use both an analytical technique and the numerical technique discussed in Sec. 3-5.

3-53. For the relation given in Prob. 3-52 consider y as the primary variable with uncertainties of ± 1, 2, and 3 percent. On this basis, compute the resulting uncertainties in x. Use both the analytical and numerical techniques.

3-54. Reynolds numbers for pipe flow may be expressed as

$$\text{Re} = \frac{4\dot{m}d}{\mu}$$

where $\dot{m}$ is mass flow in kg/s, d is pipe diameter in m, and μ is viscosity in kg/m·s. In a certain system the flow rate is 12 lbm/min, ± 0.5 percent, through a 0.5-in diameter (± 0.005-in) pipe. The viscosity is 4.64×10^{-4} lbm/hr·ft, ± 1 percent. Calculate the value of the Reynolds number and its uncertainty. Use both the analytical and numerical techniques.

3-55. The specific heat of a gas at constant volume is measured by determining the temperature rise resulting from a known electrical heat input to a fixed mass and volume. Then

$$P = EI = mc_v \, \Delta T = mc_v(T_2 - T_1)$$

where the mass is calculated from the ideal gas law and the volume, that is,

$$m = \frac{p_1 V}{RT_1}$$

Suppose the gas is air with $R = 287$ J/kg·K and $c_v = 0.714$ kJ/kg·°C, and the measurements are to be performed on a 1-liter volume (known accurately) starting at $p_1 = 150$ kPa and $T_1 = 30$°C. Determine suitable power and temperature requirements, assign some uncertainties to the measured variables, and estimate the uncertainty in the value of specific heat determined.

3-56. A model race car is placed on a tethered circular track having a diameter of 10 m ± 1 cm. The speed of the car is determined by measuring the time required for traveling each lap. A hand-held stopwatch is used for the measurement, and the estimated uncertainty in *both* starting and stopping the watch is ± 0.2 sec. For a nominal speed of 100 mi/hr calculate the uncertainty in the speed measurement when made over 1, 2, 3, and 4 laps.

REFERENCES

1. Kline, S. J., and F. A. McClintock: Describing Uncertainties in Single-Sample Experiments, *Mech. Eng.*, p. 3, January 1953.
2. Mood, A. M., and F. A. Graybill: "Introduction to the Theory of Statistics," 2d ed., McGraw-Hill Book Company, New York, 1963.
3. Schenck, H.: "Theories of Engineering Experimentation," McGraw-Hill Book Company, New York, 1961.
4. Wilson, E. B.: "An Introduction to Scientific Research," McGraw-Hill Book Company, New York, 1952.

5. Young, H. D.: "Statistical Treatment of Experimental Data," McGraw-Hill Book Company, New York, 1962.
6. Hicks, C. R.: "Fundamental Concepts in the Design of Experiments," Holt, Rinehart, and Winston, Inc., New York, 1964.
7. Moffat, R. J.: Contributions to the Theory of Single-Sample Uncertainty Analysis, *J. Fluids Engr.*, vol. 104, p. 250, June 1982.
8. "Measurement Uncertainty," ANSI/ASME Power Test Code 19.1-1985, American Society of Mechanical Engineers, New York, 1986.
9. Johnson, A. T.: Microcomputer Programs for Instrumentation and Data Analysis, *Int. J. Appl. Engineering Education*, vol. 3, no. 2, p. 149, 1987.
10. Press, W. H., B. P. Flannery, S. A. Teukolsky, W. T. Vettering: "Numerical Recipes, the Art of Scientific Computing," Chap. 12, pp. 381–449, Cambridge University Press, 1986. Also FORTRAN Example Book and diskette.
11. Canale, R. P., and S. C. Chapra: "Electronic Tool Kit," McGraw-Hill Book Company, 1988.
12. Chapra, S. C., and R. P. Canale: "Numerical Methods for Engineers," 2d Ed., McGraw-Hill Book Company, 1988.
13. Constantinides, A.: "Applied Numerical Methods with Personal Computers," McGraw-Hill Book Company, 1987.