CE 5333 – Special Studies in Water Resources
Essay 1.2 Analysis and Propogation of Experimental Error

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1 Introduction

This essay is a continuation of the previous essay on uncertainty analysis and related topics. The last essay concluded with the first step **Estimate the uncertainty interval for each measured quantity**.

This essay covers the next two steps **State the confidence limit on each measurement**, and **Propagate the uncertainty into results calculated from experimental data.**

2 Estimate the uncertainty interval for each measured quantity

This step was covered in the previous essay.

3 State the confidence limit on each measurement.

The uncertainty interval should be stated at specified odds (specified probability). For instance, the experimenter may write $h = 752.6 \pm 0.5$ mm (20 to 1)[1]. This expression implies that the experimenter is "betting" with 20:1 odds that the height of the mercury column is actually within ± 0.5 mm of the stated value. The experimenter must understand that:

> ... the specification of such odds can only be made by the experimenter based on ... total laboratory experience. There is no substitute for sound judgement in estimating the uncertainty of a measured variable. (Doebelin, 1990)

The importance of this quote cannot be overstated — there is certainly gut feeling involved, but the results should make sense to the experimenter, they should be consistent with similar

[1] This expression is probably archaic — a more current way would be to replace the odds with a confidence limit. For normal distributed measurements 20 to 1 corresponds to ± 2 standard deviations or a 95% confidence interval or a 5% risk level. The thought that goes into making some guess as to the chance of the actual value being within a measured range is as important as the numbers themselves.

prior work, and they should agree with some theory. At the very least, the values should be within physical boundaries known to the researcher.

As a simple example, consider a velocity measurement in a natural system (a river). As of 1994 from 2950 measured values by the U.S. Geological Survey for a set of representative large rivers in the United States, the median maximum velocity (by a price-current meter), was 4.11 feet per second. For the Mississippi river only, its median maximum velocity was 8.0 feet per second. The 90-th percentile (90% of values are less than or equal to this value) for the United States was 13 feet per second. the largest recorded value in a natural river was 22.4 feet per second. The point is that in the absence of any other information, a velocity value in excess of 10 feet per second is unlikely and the experimenter should check the instrument and retake the reading if at all possible. The reading could be correct, but a measurement error is far more likely. Knowledge of such physical or observational boundaries is invaluable for the experimenter. Again in the absence of any other knowledge, these bounds establish a quasi six-sigma range[2]

The confidence interval interpretation is based on the concept of standard deviations for a normal distribution. Odds of 100:1 correspond to $\pm 3\sigma$; 99 of 100 observations of all future readings for **identical experimental conditions** are expected to fall within this interval. Odds of 20:1 correspond to $\pm 2\sigma$; 95 of 100 observations of all future readings for **identical experimental conditions** are expected to fall within this interval. Odds of 2:1 correspond to $\pm 1\sigma$; 50 of 100 observations of all future readings for **identical experimental conditions** are expected to fall within this interval.

The 95% interval is typical for engineering work (medical research usually adheres to this interval although the 99% is common in medical research)[3].

To conclude this section — make a statement of the confidence in each measurement; not how confident you are in making the measurement, but the confidence in the value.

[2] This is the author's concept and not standard practice. One sixth of this range can be used to approximate the variance in Chebychev's Inequality which states: For X a discrete or continuous random variable with $\mu = E(X)$ and $V(X) = \sigma^2$ that

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}$$

This inequality provides a probabilistic bound on the distance from the mean value of any observation provided the variance is known. Assuming ϵ is constant, then repeated measurements can reduce the variance and thus the probability of being far from the desired measurement.

[3] As a side comment "rare" in medicine is 1 in 100 — thus medicines that have a rare side effect of "death" are expected to kill 1 of every 100 patients who take the medicine. My advice — read the product literature and proceed accordingly.

4 Propagate the uncertainty into results calculated from experimental data.

This step is important, usually we do not directly measure a useable quantity, but instead the measurement is used in a calculation of a useful quantity. For example, we generally do not measure discharge, but instead a set of velocities and depths and then calculate the discharge[4].

Suppose that measurements of independent variables $x_1, x_2, \ldots, x_n$ are made in the laboratory Suppose the corresponding relative uncertainty of each quantity is estimated as u_i. The measurements are then used to calculate some result, R, for the experiment. As experimenters we need to know how errors in the x_i's *propogate* into the calculation of R from measured values.

The quantity R can be expressed as a function of the measured values as $R = R(x_1, x_2, \ldots, x_n)$ The sensitivity of R to an error in an individual value x_i is the partial derivative of R with respect to that variable.

$$\frac{\delta R}{\delta x_i} = \frac{\partial R}{\partial x_i}$$

Typically the sensitivity is normalized by the magnitude of the result[5] as

$$\frac{\delta R}{R} = \frac{1}{R}\frac{\partial R}{\partial x_i}\delta x_i$$

Generally the variation in the x_i is also normalized so the relationship is

$$\frac{\delta R}{R} = \frac{x_i}{R}\frac{\partial R}{\partial x_i}\frac{\delta x_i}{x_i}$$

Now examine the rightmost fraction — this is the uncertainty, either known or estimated, so we substitute that value as

$$\frac{\delta R}{R} = \frac{x_i}{R}\frac{\partial R}{\partial x_i}u_i$$

This result is really the relative error in R from uncertainty in x_i and typically the notation is

$$u_{R_i} = \frac{x_i}{R}\frac{\partial R}{\partial x_i}u_{x_i}$$

[4]Direct measurement of discharge is difficult in a laboratory and nearly impossible in most field situations. Weighing a quantity of water over time is considered the most accurate direct measurement of discharge, while a volume catch is second most accurate. Weighing 100 cubic feet per second would require a scale that has a frequency response well in excess of a few milliseconds, and an ability to hold 6400 lbs of water without failing and then release the water — feasible in a laboratory, not exactly field portable.

[5]Kind of a relative error concept.

Generally we are interested in average uncertainty and Kline and McClintock (1953) argue that the most likely average uncertainty is the root-mean-uncertainty which is expressed as

$$u_R = \pm \sqrt{\sum_{i=1}^{n} (\frac{x_i}{R}\frac{\partial R}{\partial x_i} u_i)^2}$$

Now there are two tricks worth mentioning about evaluating the partial derivatives. The first is that they can be approximated by finite difference computations so we don't necessarily need to do the calculus for complex result models. The second is a variant of the first — we can sample from an assumed distribution of each contributing variable, compute the results and compute their mean and variance. This is called a Monte-Carlo simulation of error propagation. Neither trick is particularly elegant, but they are brute force techniques that will work. They generally require the experimenter to be able to program a computer to "roll the dice." There are some programs that can do the work for the researcher[6]

For pedagogical reasons (and the preferable way if you can do the calculus) an example in the old-fashioned way is presented.

4.1 Example 1: Uncertainty in the Volume of a Cylinder

The volume of a cylinder in terms of radius and height is

$$V = \pi r^2 h$$

The partial derivatives for the quantity r is,

$$\frac{\partial V}{\partial r} = 2\pi r h$$

The partial derivative for the quantity h is

$$\frac{\partial V}{\partial h} = \pi r^2$$

[6]The Excel VBA program called @Risk is a business program for financial computations that can perform perfectly adequate monte-carlo simulations. It [the program] is available from `http://www.palisade.com/RISK/`. It is not too hard to write your own code but if you don't know how the program can help.

The fractional uncertainty from the radius is

$$u_{V_r} = \frac{r}{V}\frac{\partial V}{\partial r}u_r = \frac{r}{\pi r^2 h}(2\pi r h)\ u_r = 2u_r$$

The fractional uncertainty from the height is

$$u_{V_h} = \frac{h}{V}\frac{\partial V}{\partial h}u_h = \frac{h}{\pi r^2 h}(\pi r^2)\ u_h = u_h$$

The uncertainty caused by the radius contribution is twice that of the height, hence more effort is required for the radius measurement as opposed to the height measurement to achieve the same absolute uncertainty in the two measurements.

The root-mean-uncertainty is

$$u_V = \pm\sqrt{(2u_r)^2 + (u_h)^2}$$

4.2 Example 2: Uncertainty in Mass Flow Rate

The mass flow of water is to be determined by collecting water in a vessel (bucket). The mass flow rate is calculated from the net mass of water collected dived by the time interval.

$$\dot{m} = \frac{\Delta m}{\Delta t}$$

where $\Delta m = m_{full} - m_{empty}$. Error estimates for the measured quantities are

$$m_{full} = m_f = 400 \pm 2 \text{ g (95\% confidence interval)}$$
$$m_{empty} = m_e = 200 \pm 2 \text{ g (95\% confidence interval)}$$
$$\Delta t = 10 \pm 0.2 \text{ sec (95\% confidence interval)}$$

The partial derivatives for the quantities are

$$\frac{\partial \dot{m}}{\partial m_f} = \frac{1}{\Delta t}$$
$$\frac{\partial \dot{m}}{\partial m_e} = -\frac{1}{\Delta t}$$
$$\frac{\partial \dot{m}}{\partial \Delta t} = -\frac{m_f - m_e}{\Delta t^2}$$

The fractional uncertainties for the quantities are

$$u_{m_f} = \frac{m_f \Delta t}{m_f - m_e} \frac{1}{\Delta t} \frac{\pm}{m_f} = \pm \frac{2g}{200g} = \pm 0.01$$

$$u_{m_e} = \frac{m_e \Delta t}{m_f - m_e} \frac{-1}{\Delta t} \frac{\pm}{m_e} = \pm \frac{2g}{200g} = \pm 0.01$$

$$u_{\Delta t} = \frac{\Delta t^2}{m_f - m_e} (-\frac{m_f - m_e}{\Delta t^2}) \frac{\pm}{\Delta t} = \pm \frac{0.2s}{10s} = \pm 0.02$$

The root-mean-uncertainty is

$$u_{\dot{m}} = \pm \sqrt{(0.01)^2 + (0.01)^2 + (0.02)^2} = \pm 0.0245$$

Thus the 95% confidence mass flow rate by this method using the tools implied in this problem would be specified as

$$\dot{m} = 20g/s \pm 0.49g/s$$

The 95% confidence interval for this measurement means the mass flow reading 95 in 100 readings will fall in the range $[19.51, 20.49]$ g/s. The uncertainty in the time measurement is the larger contributor to the overall uncertainty, and if lower uncertainty is required, then a more precise timer is of greater benefit than a more precise scale.

5 Summary

This essay illustrated how, using partial derivatives to propagate measurement uncertainty into a calculated quantity, and report that quantity using the 95% confidence interval as the default standard. In class similar examples for discharge are presented. The next essay illustrates how to use numerical methods (finite-difference approximations) to estimate the uncertainty.

Statistical methods are not discussed in this class but are an alternate tool to accomplish the same kind of analysis. Of note, the readings do illustrate a principle to reject what appear to be poor readings. These rejections are always done after the fact (assuming no gross errors, in which case the experimenter would re-make the reading on the spot).

Finally uncertainty analysis is a bit of an art, its real purpose is to point out to the experimenter and engineer when things are different and when they are not.

6 Readings

1. Holman (1989) pp. 37-49.
2. Kline and McClintock (1953)

7 Exercises

1. (1) Problem 3-3, Holman (1989) p. 84
2. (2) Problem 3-4, Holman (1989) p. 84
3. (3) Problem 3-20, Holman (1989) p. 87
4. (4) (Extra Credit) Problem 3-21, Holman (1989) p. 87

References

Doebelin, E. O. (1990). *Measurement Systems* (4th ed.). New York, NY: McGraw-Hill.

Holman, J. P. (1989). *Experimental Methods for Engineers, 5th Edition.* New York, NY: McGraw-Hill.

Kline, S. J. and F. A. McClintock (1953). Describing uncertainties in single-sample experiments. *Mechanical Engineering* (No. 75), 3–9.