CE 5333 – Special Studies in Water Resources
Essay 1.1 Analysis and Propogation of Experimental Error

Theodore G. Cleveland, Ph.D., P.E.

1 Introduction

Experimental tests (measurements) are vital in engineering analysis and design. The quality of the results is variable; the validity of data should be documented before results are used in engineering design. Analysis of uncertainty and analysis of computational propagation of errors are tools used to quantify data validity and precision[1]. Uncertainty analysis is also useful in experimental design, a careful examination of procedures and computations may indicate potential sources of unacceptable error and suggest improved measurement methods.

2 Types of Error

Measurements always contain error. Aside from gross blunders by the analyst[2], measurement error comes in two flavors: systematic error, and random error.

2.1 Systematic Error

Systematic error is error that causes repeated measurements to be in error by the same magnitude for each measurement (of the same conditions). Such error is also called bias. If the bias is constant for a given set of conditions (ideally for a range of conditions) then the error can be removed by a correction. If an instrument is reporting known bias, then calibration is in order to adjust the instrument back to its best performance level, although one could use an un-calibrated instrument as long as the correction terms are known.

Un-calibrated instruments (if precise) can always be used for comparative measurements to

[1] Precision is a concept related to repeatability, accuracy is how close to the “truth” the measurement comes. Both words are incorrectly used interchangeably — they are not interchangeable. An inaccurate device might well be very precise — if we discover the inaccuracy and the inaccuracy is some constant, then the bias can be corrected and the device becomes both accurate and precise.

[2] Data entry off by a decimal point for instance — sometimes hard to detect during the measurement. These errors are often uncorrectable; there is a reason for repeat measurements and spatial distributed measurements of the “same” variable in experimental and field work.

detect changes in variable magnitude (even though the absolute magnitudes will be incorrect), however as a general rule this use is not recommended.

Systematic error is important to quantify, but generally is not the error of greatest concern in many experimental studies[3]

2.2 Random Error

Random error is an error in a measurement that is not repeatable (the error, not the measurement!). This error is different in magnitude for every reading and cannot be removed by a simple correction term, if at all. The causes of random error are uncertain by definition. The objective of uncertainty analysis is to estimate the probable random error in measurements, then propagate these errors in values calculated from such measurements.

The result is a value along with an uncertainty range — the value and the range are important in design. Two different values that lie within the uncertainty range of each other are for all practical purposes the same value. Most designers ignore this fact and can get into meaningless arguments over numerical differences that are, in fact, nonexistent.

2.3 Measurement Assumptions

In uncertainty analysis we assume the following [4]:

1. The equipment has been constructed correctly and calibrated to eliminate fixed errors.
2. The instrumentation has adequate resolution and that fluctuations in readings are not excessive.
3. Care is used in making measurements and recording observations so that only random errors remain.

[3]This statement is not meant to discount systematic error — it is important, but as long as there is known systematic error, it is generally not advisable to change measurement techniques during a series of experiments.

[4]Sometimes these assumptions are violated out of necessity, schedule, or forces beyond the experimenters control; This list should be interpreted in the context that there is a conscious effort to make the best measurements possible with the tools avaliable.

3 Estimation of Uncertainty

The goal is to estimate the uncertainty in experimental measurements and calculated results caused by random errors. The overall recipe has three steps:

1. Estimate the uncertainty interval for each measured quantity.
2. State the confidence limit on each measurement.
3. Propagate the uncertainty into results calculated from experimental data.

These steps are discussed individually in the next three subsections.

3.1 Estimate the uncertainty interval for each measured quantity

Suppose the measured variables in an experiment are $x_1, x_2, \ldots, x_n$. One way to find the uncertainty would be to repeat the measurements many times. The result would be a distribution of data for each variable — while desirable, it is usually impractical.

Random errors in measurements usually approximate a *normal* frequency distribution of measured values. The dispersion (scatter) of measurements is quantified by the variance σ^2, whose square root is called the standard deviation σ. The uncertainty interval for each measured variable x_i is typically reported as $\pm n\sigma_i$, where $n = 1, 2$, or 3.

For approximately normal data, 99 % of the measured values for x_i will lie within $\pm 3\sigma_i$ of the mean value, 95 % of the measured values for x_i will lie within $\pm 2\sigma_i$ of the mean value, and 50 % of the measured values for x_i will lie within $\pm 1\sigma_i$ of the mean value. Thus in theory it would be possible to quantify expected errors within any desired confidence limit if a statistically significant set of data were collected.

The required number of measurements for a statistically significant set of data is usually quite impractical. However the normal distribution as an error model suggests several important concepts:

1. Small errors are more frequent that large errors.
2. Positive and negative error are equally likely.
3. There is no finite bound to maximum error [5].

An added complication in most engineering work is a "single-sample" experiment, where only one measurement is made for each quantity and the situation during the measurement

[5] As a practical matter the instrument itself will censor errors beyond its measurement range.

is unrepeatable. Field flow measurements are generally unrepeatable — the flow in a river changes from day to day, and experimental conditions are not repeatable.

In the single-sample situation, the conventional estimate of measurement uncertainty is $\pm$ one-half for the smallest scale division of the instrument (or the last digit in a digital readout). This convention needs to be used with some care, as is illustrated in Example 1.

3.1.1 Example 1. Uncertainty in reading a barometer

Suppose the observed height of oil in a barometer column is $h = 752.6\ mm\ Hg$. The smallest division on the scale is $0.1\ mm$, so the conventional measurement error would be reported as $\pm 0.05\ mm$.

Suppose such a measurement could be made to this precision. Then the uncertainty expressed as a fraction of the reported value would be

$$u_h = \pm \frac{0.05mm}{752.6mm} = \pm 0.0000664. \tag{1}$$

This result is about 6 parts in one hundred thousand, pretty good indeed!

Such a measurement cannot not be made this precisely, the barometer slider and meniscus must be aligned by the analyst's eyes. The slider has a minimum count of 1 mm. As a conservative estimate, a measurement could probably be made to the nearest millimeter. Using this practical knowledge the uncertainty is at least one order of magnitude larger, 6 parts in ten thousand, still pretty good.

Now one last bit of information, the barometer is calibrated at standard temperature and pressure, but typical room temperature is about 15 degrees Fahrenheit above the standard temperature and a temperature correction factor of nearly 3 mm must be subtracted from the measurement. The correction factor itself has uncertainty, plus we have to now measure temperature, and probably interpolate in a table of values.

The message is that a seemingly simple measurement has many other components that we routinely don't think about (instrument reading, analyst interaction, temperature reading, correction factor table look-up, interpolation, and the final calculation to report the actual pressure).

The microprocessor age has certainly simplified some of these steps (automatic temperature compensation is typical of most instruments) but even these machines have limited precision.

4 Readings

1. **?** — This is a short article on uncertainty analysis. The article describes all three steps, these notes only discuss the first step, the other two are discussed in subsequent lectures.
2. **?** — Chapter 3 (about 50 pages), copy waiting to be scanned??.

5 Exercises

1. (1) Assume a manual barometer reports a value of $h = 752.6\ mm\ Hg$, and the measurement can probably be made to only the nearest millimeter. What is the relative uncertainty of this measurement?
2. (2) Assume a temperature corrected digital barometer reports a value of $h = 752.6\ mm\ Hg$. What is the relative uncertainty of this measurement?
3. (3) Which of the two instruments above has less uncertainty? By how much (express as a percentage)?