CIVE 3331 - ENVIRONMENTAL ENGINEERING

Spring 2003

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Purpose: Lecture #17 CIVE3331

<u>Population Growth Mathematics</u>

Population growth mathematics is important in civil engineering for design and operation of engineered systems. It is used to predict the size of a system to serve a population and to project revenue from the project, taxation, and other funding sources used to pay for the project. Grossly incorrect predictions usually leads to cost overruns and inadequate service.

Typically money people over-predict revenue and engineers under-predict costs. This lecture will not prevent this situation from changing, but should give you an idea of how projections are made. Examples of population model uses

1. Determine the growth of a bio-engineered compound (vaccine, contaminant eating microorganism, food, etc.). How fast can it be produced in quantity? How fast can it be brought to market? If there is an upset (failure) how fast can it be replaced? Will the development money run out before ti generates revenue?
2. Determine the population of Harris County in 2020. Will the transportation network be adequate? Will there be enough drinking water? Will there be enough air? Will current projects be obsolete even before completion? Can the current projects be paid for even if the population does not grow as projected?

Discrete Interval Growth

Discrete interval growth is the growth of a population (people, cells, dogs, dollars, etc.) that occurs over some fixed or specified time interval. The growth of money in a bank is a good example.

N_0 - initial population (beginning of interval)

N_1 - final population (end of interval)

$N_1 - N_0 = \Delta N$ - change in population.

Now if we divide the change by the initial population we obtain a value called the growth rate (intrinsic growth rate).

$\frac{\Delta N}{N_0} = r$ - intrinsic growth rate.

If we know the rate we can predict future populations from

$$N_1 = N_0 + rN_0 = N_0(1+r)$$

If we are interested in a series of intervals then the future population is

$$N_n = N_0(1+r)^n$$

Continuous Growth

Often we are interested in the population value at some instant in time that may not coincide with the discrete interval time. We can rewrite the discrete model as

$$N_{t+\Delta t} = N_t + rN_t \frac{\Delta t}{\Delta \bar{t}}$$

If the time step length is the same at the interval length (t_bar) then the model is identical to the discrete model. But lets assume it is not. Rearrange as

$$N_{t+\Delta t} = N_t + \frac{r}{\Delta \bar{t}} N_t \Delta t$$

Now examine the finite-difference form

$$\frac{N_{t+\Delta t} - N_t}{\Delta t} = \frac{r}{\Delta \bar{t}} N_t$$

Take the limit as Δt vanishes

$$\lim_{\Delta t \to 0} \frac{N_{t+\Delta t} - N_t}{\Delta t} = \frac{dN}{dt} = \frac{r}{\Delta \bar{t}} N$$

The solution to the last expression is

$$N(t) = N_0 e^{(\frac{r}{\Delta \bar{t}} t)}$$

Usually the intrinsic growth rate and the characteristic time are taken as a single value called the growth rate.

$$N(t) = N_0 e^{(rt)}$$

This last model is called continuous compounding.

For exponential growth (or decay) the doubling time (or half-life) is an important concept. It relates how fast a growth (decay) process proceeds.

Doubling time for continuous growth

$$2N_o = N_0 e^{(rt)}$$
$$2 = e^{rt}$$
$$\ln 2 = rt$$
$$t = \frac{\ln 2}{r}$$

This the doubling time in continuous growth is 0.693/r.

Half-Life in Continuous Growth

In the case of decay (negative growth) the half-life is obtained from

$$
\begin{aligned}
0.5N_o &= N_0 e^{(-rt)} \\
0.5 &= e^{-rt} \\
\ln 0.5 &= -rt \\
t &= -\frac{\ln 0.5}{r}
\end{aligned}
$$

So the half-life is –(-0.693/r) = 0.693/r (How convenient!)

Doubling time for discrete growth

Repeating the same analysis for discrete growth produces similar results.

$$
\begin{aligned}
2N_0 &= N_0(1+r)^n \\
2 &= (1+r)^n \\
\ln 2 &= n\ln(1+r) \\
n &= \frac{\ln 2}{\ln(1+r)}
\end{aligned}
$$

Thus the number of intervals required to double is 0.693/ln(1+r)

Suppose you have money invested at 7% per year (simple interest). What is its doubling time?

Compute (1+r) as 1.07. Insert into the formula and discover that it takes 10.24 intervals (years) for the money to double.

$$n = \frac{\ln 2}{\ln(1.07)} = 10.24\,\text{years}$$

In the continuous case, it only takes 9.9 years (not much difference at low rates, but compare at high rates!)

$$t = \frac{\ln 2}{0.07} = \frac{0.693}{0.07} = 9.9\,\text{years.}$$

The rule of 72 is a short-cut to compute doubling times when comparing financial produces. It states that the doubling time in years is 72 divided by the interest rate in %/year. Thus for our example

72/7 = 10.28 (pretty close)

For continuous compounding (which is not offered to consumers) the rule is 70 divided by the interest rate in %/year. Thus for our example 70/7 = 10 (pretty close).

Half-Life in Discrete Growth

$$0.5N_0 = N_0(1-r)^n$$
$$0.5 = (1-r)^n$$
$$\ln 0.5 = n\ln(1-r)$$
$$n = \frac{\ln 0.5}{\ln(1-r)}$$

Thus the number of intervals is given by –0.693/ln(1-r).

It is relatively uncommon to calculate discrete decay half-lives, but the concept is clear in the context of growth arithmetic.

<u>Disaggregated Growth Rates</u>

Many times a quantity can be expressed as the product of several contributing factors. In the case of some population we might represent this concept as

$$N(t) = P_1(t)P_2(t)...P_n(t)$$

If each of the influencing factors is governed by some exponential type of growth/decay model then the aggregate model can be expressed as

$$N(t) = P_1e^{r_1t}P_2e^{r_2t}...P_ne^{r_nt}$$

By collecting terms we can write the aggregate model as

$$N(t) = P_1P_2...P_ne^{(r_1+r_2+...+r_n)t}$$

In the last case the initial term is the product of all the individual "P"s and the rate is the sum of the individual rates.

Resource Consumption

Projection of resource consumption and resource reserves is critical in engineering planning and project management. As an example we will consider energy resources, but the ideas can be extended to any finite resource.

Suppose P is a production rate (or consumption rate). The total amount of resource produced (consumed) in a time interval is

$$Q(t_2 - t_1) = \int_{t_1}^{t_2} P(t)dt$$

And since the beginning of production the amount to date is

$$Q(t) = \int_0^t P(t)dt$$

If the production function P happens to follow an exponential model we can write

$$Q(t) = \int_0^t P_0 e^{rt} dt = \frac{P_0}{r}(e^{rt} - 1)$$

Symmetrical Production Curve

Most depletible resources do not experience exponential growth for long. Typically there is a discovery, rapid growth, peak, then decline.

The Gaussian growth curve is an example

$$P(t) = P_m e^{-\frac{1}{2}(\frac{t-t_m}{s})^2}$$

When integrated over a lifetime of production we have

$$Q(t) = \int_{-\infty}^{t} P_m e^{-\frac{1}{2}(\frac{t-t_m}{s})^2} dt = P_m s \sqrt{2p} (1 + erf(\frac{t-t_m}{s\sqrt{2}}))$$

erf(z) is the error function.

Error function – approximations (useful in statistics – note how the integral is identical to the normal density)

Rational Approximation

$$erf(z) \approx \frac{1}{[1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4]^4}$$

The constants are:

$$a_1 = 0.278393$$
$$a_2 = 0.230389$$
$$a_3 = 0.000972$$
$$a_4 = 0.078108$$

Symmetry relationships for negative values of "z"

$$erf(0) = 0$$
$$erf(\infty) = 1$$
$$erf(-\infty) = -1$$
$$erf(-z) = -erf(z)$$

[Future revision – insert graphs of Production curve illustrate effect of variance and Pm – also show use of Normal density]

Population Models

Exponential growth and Gaussian production models are not realistic for many population modeling issues.

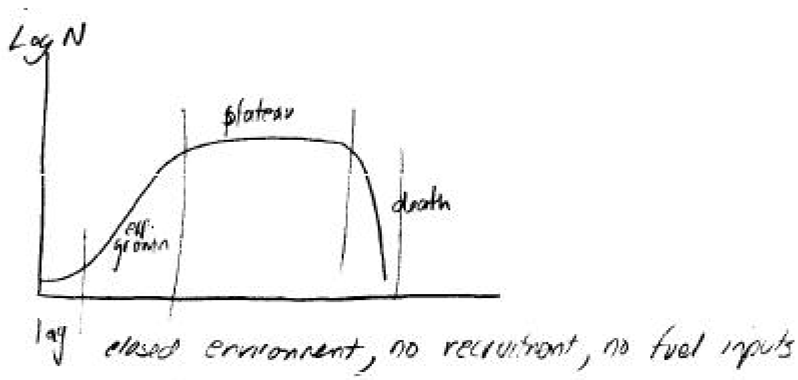

Figure 1. Bacterial Growth Curve - Closed System

[Future revision – insert logistic growth curve – open system approach to equilibrium]

Logistic Model is one population model used to explain bacterial growth in a system where a constant amount of food is maintained. It can be obtained from the original discrete growth model by adding a "resource limitation" process.

Start with the discrete model (birth process)

$$N(t+\Delta t) = N(t) + \frac{r}{\Delta \bar{t}} N(t)\Delta t$$

Add a resource limitation process

$$\text{resource limit} = \frac{N(t)}{K}$$

In this process if the population number is small relative to the constant "K" (carrying capacity) then the resource limit is negligible. If the population number is large relative to "K" (the number of individuals that can be sustained by the environment) then the growth is negative (death). We incorporate into the model as

$$N(t+\Delta t) = N(t) + \frac{r}{\Delta \bar{t}} N(t)\Delta t - \frac{r}{\Delta \bar{t}} \frac{N(t)}{K} N(t)\Delta t$$

Rearrange the model somewhat into the finite-difference form as

$$\frac{N(t+\Delta t)-N(t)}{\Delta t}=\frac{r}{\Delta\bar{t}}N(t)[1-\frac{N(t)}{K}]$$

In the limit as the time step length vanishes

$$\frac{dN}{dt}=\frac{r}{\Delta\bar{t}}N(t)[1-\frac{N(t)}{K}]$$

This last expression is a logistic equation.

The solution (separate and integrate) is

$$N(t)=\frac{K}{1+e^{\frac{r}{\Delta\bar{t}}(t-t^*)}}$$

t* is the time when the population number is ½ the carrying capacity constant value.

Maximum Sustained Yield

A useful population concept is called sustained yield. It represents the amount of a resource that can be harvested without adversely impacting the population size. It is used in agriculture, wildlife management, and in engineering biological processes (drug manufacture, vaccines, sewage treatment plants, etc.).

In the case of any population it is the solution to

$$\frac{d}{dt}(\frac{dN}{dt})=0; N^*\neq 0$$

That is it is the target population number (non-zero) that occurs at maximum growth rate.

In the logistic case

$$\frac{d}{dt}(\frac{dN}{dt})=\frac{d}{dt}[\frac{r}{\Delta\bar{t}}N(1-\frac{N}{K})]=\frac{r}{\Delta\bar{t}}\frac{dN}{dt}-\frac{r}{K\Delta\bar{t}}(2N\frac{dN}{dt})=0$$

$$\frac{r}{\Delta\bar{t}}=1-\frac{2N^*}{K}\rightarrow N^*=\frac{K}{2}$$

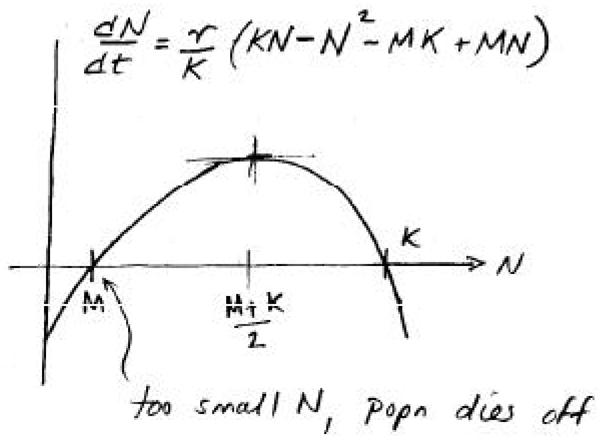

Figure 2. Population Growth Curve with Capacity Limit and Minimum Sustainable Limit

Human Population Growth

Logistic-type equations are not too useful for human populations and other species that have relatively long fertility cycles. The logistic models assume very short fertility cycles (like bacteria or rabbits), unlimited family sizes. In the case of humans, time-lagged age group models are used.

Age groups are called cohorts – represent a group of people born at the same time.

Crude birth rate – number born/1000 in any year.

Total fertility rate (TFR) – number of children/woman/lifetime (the woman's lifetime).

Replacement fertility level (RFL) – number of children/woman to produce one daughter who survives to fertility.

Population momentum – growth beyond replacement.

Crude death rate – deaths/1000 in any year.

Infant mortality rate- deaths/1000 births.

Intrinsic growth rate = crude birth – crude death.

Migration rate = net recruitment rate (from outside population).

Age structure

Age structure diagrams are used along with birth, death, and fertility rates to predict future population structures. It is important in planning for how to pay for civil infrastructure, especially if the infrastructure benefits one particular cohort. Eg. Schools benefit all society, but serve the low age portion of the population. These individuals need to be served (schools must have capacity), but are paid for by older segments of society. The engineering planner needs to design the schools to have capacity, but needs to be sure there are enough people to pay for the effort.

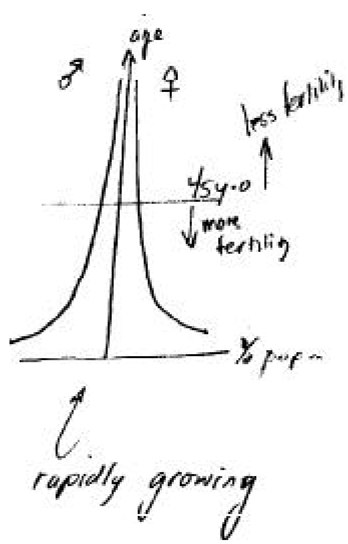

Figure 3. Age Structure Diagram – Rapid Growth Population

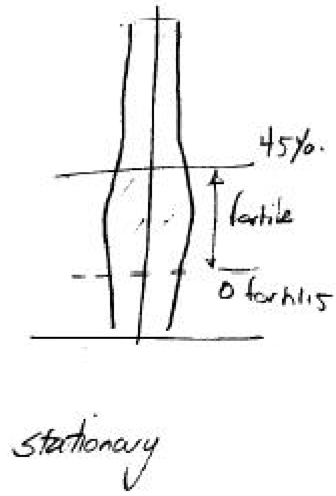

Figure 4. Age Structure Diagram - Stable Population

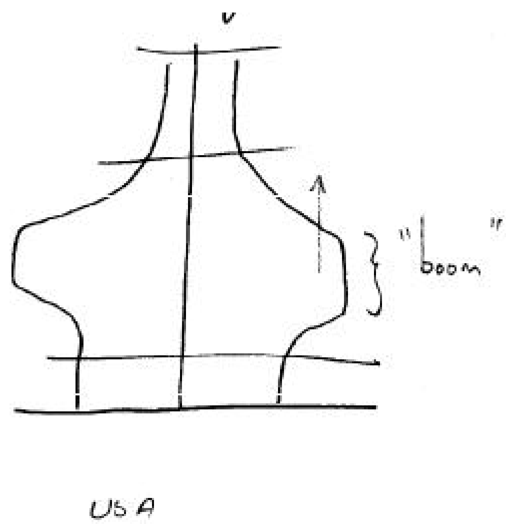

Figure 5. Age Structure Diagram - Boom

Population momentum

Boom-bust scenario.

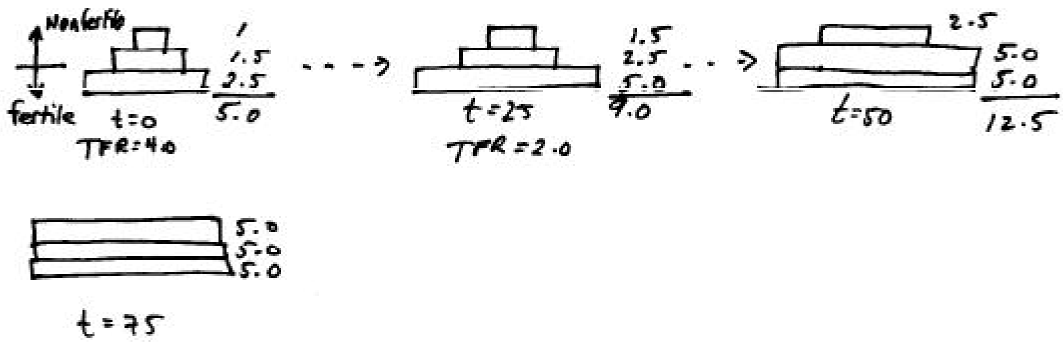

Figure 6. Example of Boom Structure Generation

Even though the TFR dropped from 4.0 to 2.0 in year 25, still have the effect of high TFR 50 years later. This “boom” is called population momentum.

[future revision – typeset example set-up; improve graphic]

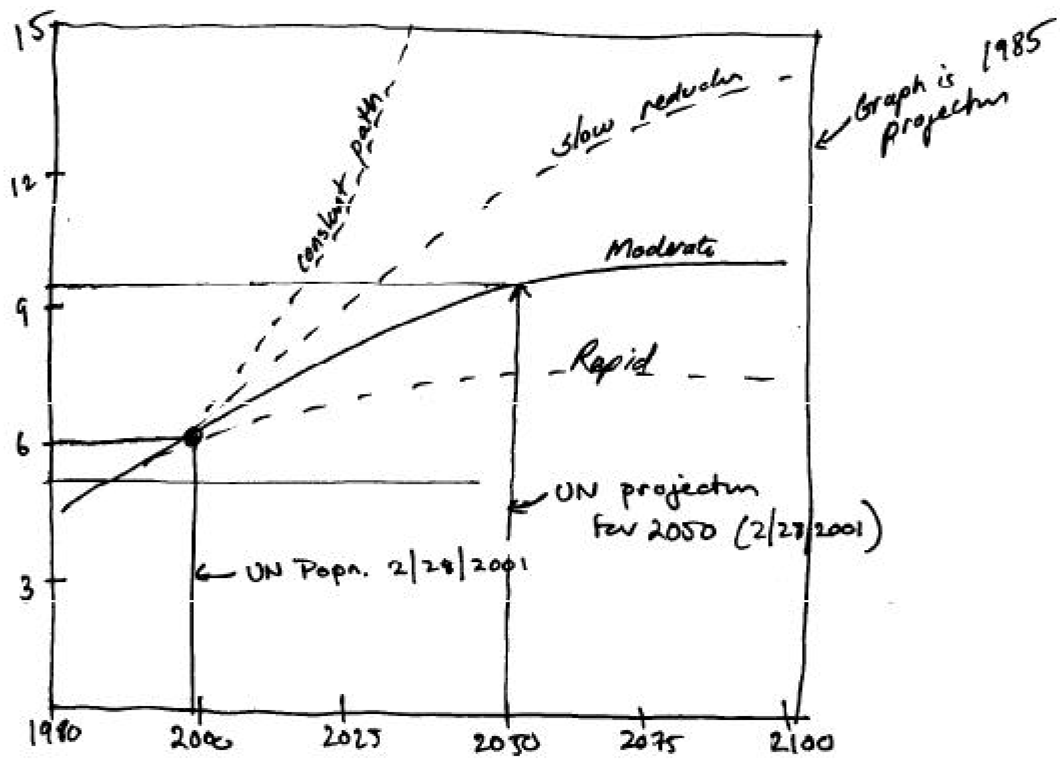

Figure 7. Population Projections from Articles