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From fractal geometry to architecture: Designing a grid-shell-like structure using the Takagi–Landsberg surface[☆]

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ABSTRACT

This paper has applied the concept of fractal geometry in designing a grid-shell-like complex spatial structure. The property of the fractal dimension which characterizes the level of roughness of a shape has been particularly explored in this study for designing a complex-shaped spatial structure by taking a paraboloid as a basic shape of reference. A factor of fractal dimension which is known as the *relative size value* (w) plays the key role in changing the surface texture in accordance with the changing of fractal dimension. In this paper, the *relative size value* (w) has been specifically applied to study the texture-based shape morphogenesis of a paraboloid by using the reference of the Takagi–Landsberg's fractal surface. This research is curious to see how this surface morphogenesis impacts on the structural behavior and unveils an opportunity to develop a new kind of form. For this purpose, we have computationally generated a parametric model of a grid-shell-like structure by making a paraboloid as a basic geometric framework and by adding an extra supporting frame in order to avoid any structural failure during the surface morphogenesis of the outer profile. A structural comparison has been done in between the grid-shell-like structure having the paraboloid-based smooth outer profile and the structure having a fractal-based unsmooth outer profile. A real-scale physical prototype of a fractal-based grid-shell-like structure has been constructed to see its architectural appearance, real-world structural behavior, practical applicability and constructability.

1. Introduction

In architecture, shell, grid-shell and other such regular spatial structures are special stand-alone structures that derive strength from their geometric shapes. If, for a moment, we do not concentrate on the fully self-supporting and stand-alone characteristics of these structures, and consider a design of the partially self-supporting structure, then we can create a possibility to obtain different variations of the feasible shapes of the continuous shell and lattice shell structures. The reason for such a venture is architectural, which means, architecturally new volume of a distinctive lattice shell structure can offer a unique sense of architectural space. In short, it is a quest for searching a new form of a space structure derived from a regular space structure. Such unique and partially self-supporting lattice structures can be developed if we, for an example, exploit the geometric form of a regular grid-shell structure and add a few supporting members to them to ensure structural stability and strength during the shape transformation. This type of structures will neither exactly follow the stress principle of a regular grid-shell nor perfectly follow the structural mechanism of a truss-like space-frame structure. In this paper, we will call such structures as the 'grid-shell-like' structures. This paper focuses on designing a grid-shell-like structure using the rule-based complex geometric system, more precisely, the fractal geometric system in order to develop a unique complex-shaped lattice shell structure.

1.1. Background, motivation and objective

Generally, most shapes of shell, grid-shell and grid-shell-like lattice structures follow the shapes of the quadratic surface, ruled surface, and other regular surfaces, and in few recent occasions, they follow different complex shapes mostly in the Euclidean space [1]. Fractal geometry is one of the unique geometries which mainly deals with a certain category of complex shapes that are

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known as *fractals*. Fractal geometry was systematically first studied by Benoit Mandelbrot in the 1970s. A fractal is commonly characterized by exactly or approximately self-similar repetition of a whole or original shape into its parts [2]. These kind of geometric features are abundant in nature which is mostly and apparently evolved and optimized outcomes that seemingly possess some intrinsic rationality. The natural tree is a nice example of nature's fractal-like shape which tactically uses its iterated branching scheme to meet its biological and mechanical needs. Tree's branching pattern seems to be a structurally optimized configuration supported by a single trunk that enables the tree to achieve wide outer surface for receiving maximum sunlight [3]. Its underlying structural principle is thought to be the hierarchical transformation of the distributed loads from the upper tier to the lower tier through the decreasing number of branching points which lie at each lower tier, and eventually passes through a single trunk. This iterative branching strategy has inspired many architects and engineers to design branching structures for supporting large canopies, such as the canopy structure of the Stuttgart Airport Terminal [3]. Consequently, the iterative design scheme allows working with complex and irregular shapes in a rational way. Few recent researches show its experimental applications in developing complex structures [4–8].

Motivated by the intrinsic structural rationality of the fractal-based iterative design scheme, this paper shows a particular interest in applying fractal geometry in designing a complex yet structurally feasible structure, more specifically a grid-shell-like structure derived from a regular grid-shell structure, and to analyze its structural validity, practical applicability, and constructability. Commonly, a grid-shell is a compressive lattice structure which is derived from the grid lines of a continuous shell structure. For a large-scale structure, the grids are made of one-dimensional discrete members, and therefore the principal stress comes mainly from axial forces. Globally, due to a unique combination of axial forces, a grid-shell acts like a continuous shell structure. In the case of tree-like columns, the complex form of branches is obtained by assembling one-dimensional bars which carry the axial forces that lead to the principal stress of the overall structure. It hints that a strategic arrangement of one-dimensional bars, i.e., the corresponding axial forces in a structure can manipulate the overall structural performance and helps to find a complex structural form as well as optimal or sub-optimal solutions. This idea has inspired us to design a complex form derived from a regular grid-shell structure using iterative scheme. The search for a grid-shell-like complex and unique form is not only an architectural interest from a design and a unique appearance perspectives, but also from a structural curiosity to see that how iteratively designed rule-based complex form beholds mechanical strength. Generally, a grid-shell is constructed by pushing or pulling a flat grid till the desired shape is achieved and then fixing the edges to the ground. However, this method is not the best choice for a discontinuous complex grid-shell-like structure. For the complex-shaped grid-shell-like structures, scaffolding-supported construction of different modules, in which discrete bars are strategically connected to the joints at defined directions, is an efficient choice. This paper is interested to see that how fractal's principle of self-similarity and iterative scheme can be useful for the modular construction of the complex profile of a fractally designed grid-shell-like structure.

Complex shapes are mostly irregular, and in mathematics, the degree of complexity in terms of unsmoothness can be measured by the *fractal dimension*, which is one important property of the fractal geometry [9]. Fractal dimension basically measures the geometric dimensionality of the shapes that lie in between two successive integer dimensional shapes. In a metric space, a factor of fractal dimension can transform a shape from smooth to unsmooth by changing its fractal dimension from integer value to a non-integer value. This concept of fractal dimension can be uniquely applied in designing a parametric spatial structure by which a different mode of smooth to unsmooth volume of a space can be experienced by manipulating the form–dimension relationship. The geometric property of the form–dimension relation and the structural property of the form–force interaction have, therefore, motivated us to observe the gradual transition of the basic form of a grid-shell structure from smooth to unsmooth, and consequently, its impact on the structural behavior.

So far, very few researches have been done where the fractal geometry was used for designing spatial lattice structures, such as a dome [8] and a reticulated shell structure [10] which are limited to the straightforward application of fractal geometry for generating new shapes only without interpreting the factor of fractal dimension. Besides, the transition of a structural shape of a lattice spatial structure from smooth to fractal, and as a result, its impact on structural behavior has not been discussed in those studies. This paper has attempted to design an unsmooth grid-shell-like structure based on fractal geometry. For this, we have taken the Takagi–Landsberg surface as a geometric framework because this surface can be parametrically changed from its smooth paraboloid surface, which is a common shape for a regular grid-shell design, to an unsmooth fractal surface using a factor of fractal dimension, which is known as the *relative size value* (w). To avoid any structural failure, an additional frame has been added to the parametric grid throughout the experiment.

1.2. Structure of the paper

This paper is an outcome of an interdisciplinary research project which combines mathematics, architecture and structural design in order to design a mathematics-derived complex-shaped spatial structure which is architecturally unique, structurally rational, and practically constructible. Therefore, this paper has mainly three parts. The first part is geometric part in which we will concentrate on the outer profile of the proposed structure taking a Takagi–Landsberg's parametric surface as reference geometry. In the second part, the parametric surface will be transformed into a spatial structure with an architectural intention to obtain a complex outer appearance and a unique indoor space volume in terms of visual experience. The third part is mainly structural. In this section, we will consider the overall shape including the added extra supporting members as a complex structure. A structural comparison will be performed between the paraboloid-based regular grid-shell-like structure and the fractal-based complex grid-shell-like structure (both added with extra supporting members). An optimal form of a fractal based grid-shell-like structure will be further searched in this second section. The changing of structural behavior will be further studied with the changing of the outer profile from its smooth version to a complex (i.e., fractal-based) version controlled by the *relative size value*. A real-scale prototype of the fractal-based grid-shell-like structure was constructed to understand its real-world structural feasibility, practical applicability, and constructability.

2. Fractal geometry: Takagi–Landsberg surface

2.1. A brief about fractal geometry

A non-strict definition of a *fractal* is a shape or a figure that encapsulates the copies of itself at the infinitely different level of scales, which means it is recursively self-similar as well as rough at every magnifying level. In another definition, fractal shapes are non-integer dimensional and fall in between two successive integer dimensional shapes, i.e., $0 < D\text{H} < 1$, or $1 < D\text{H} < 2$ where $D\text{H}$

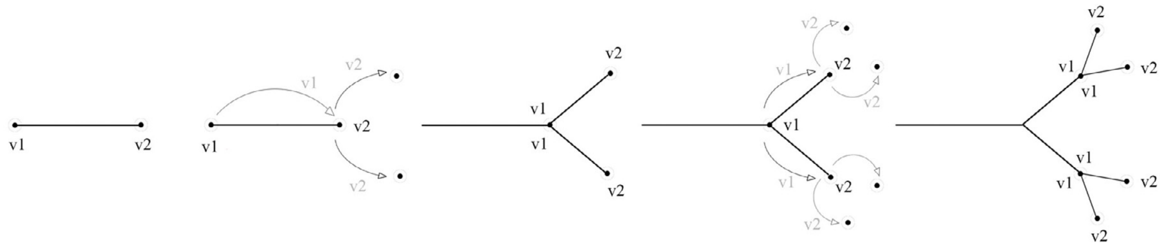


Fig. 1. Construction of a tree-like fractal figure using vector-based recursion method.

is the fractal dimension. According to M. F. Barnsley's description, a *fractal* is a set which is a union of non-overlapped self-similar or self-affine sets [11].

$$F = \bigcap_{n=0}^{\infty} F_n \tag{1}$$

where,

$$F_n = \bigcup_{i=1}^{m} f_i \left(F_{n-1}\right) \tag{2}$$

n is the number iterations and m is the number of self-similar copies produced at each iteration, m is also the number of affine transformations at each new iteration. F_1, F_2, ….., F_n, …are self-similar or self-affine sets of F_0, that are scaled by using the *contractivity factor* λ_i and transformed by using a *transformation* function f_i, such that,

$$F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_{n-1} \supset F_n \supset \cdots \tag{3}$$

Iteration or recursion process is the easy way to generate fractals.[2] In this process, commonly one pattern of geometric transformation of a shape is repeated iteratively by taking the output of the immediate previous generation as a new input. In each stage of repetition, the output shape is mostly the scaled copy of the input shape. The changing of the scale factor and the pattern of transformation changes the overall outcome of the fractal figure. Vector method is generally used in this process to generate the digital model of a fractal. In this method, the original set of vectors creates a new set of vectors after a pattern of transformation, and the same pattern of transformation is repeated by taking the resulted vectors as the input vectors, and thus the process is recursively continued. 'Fig. 1' shows an example of vector-based recursion method to produce a fractal figure.

2.2. Midpoint Displacement Method and Takagi–Landsberg curve

Midpoint Displacement Method is one of the iteration processes that can create different fractal figures by iteratively displacing the midpoints of a straight line. The Midpoint Displacement Method was first developed in the third century BC by Archimedes (287–212 BC) in calculating the area enclosed by a parabola and its baseline, known as 'parabola segment' [9]. Teiji Takagi used the Cartesian expression to represent the Archimedes' Midpoint Displacement Method as a form of Blancmange function for generating a parabolic curve, and he formulated its Cartesian equation on the unit interval as

$$A\left(x\right) = \sum_{n=0}^{\infty} (0.25)^n s(2^n x) \tag{4}$$

[2] Iterated Function System (IFS), Lindenmayer System (L-System), Iterative Subdivisions, Shape Grammars, Midpoint Displacement Method (MDM), etc. are well-known iterative and recursive processes that generates fractals.

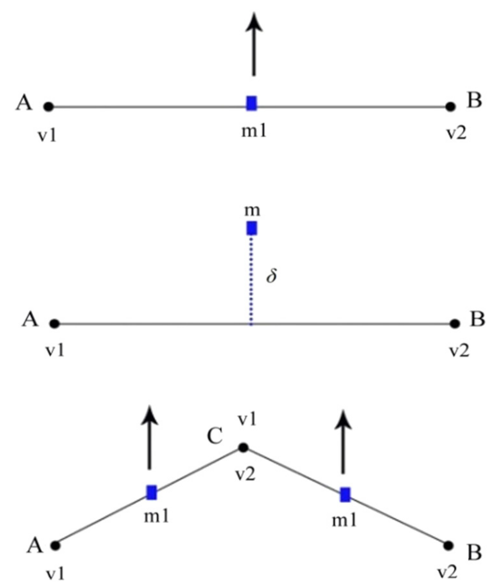


Fig. 2. Displacement of midpoints along global Z-axis.

where, $s\ (2^n x)$ is a 'sawteeth function', in which, $s\left(x\right) = \min_{n \in z} \left|x - n\right|$ is the distance from x to the nearest integer. Here 0.25 in the 'Eq. (4)' is the relative size value, a displacement factor that controls the texture of the curve. Teiji Takagi replaced the value of 0.25 by 0.5, and noticed a remarkable change. The smooth curve of the parabola was transformed into a highly unsmooth curve, later known as Takagi curve. Later, German mathematician George Landsberg used 'w' as the *relative size value* putting the parameter $0.25 < w < 0.5$, thus obtained a highly irregular curve which is nowhere-differentiable but uniformly continuous [12]. This curve is presently known as the Takagi–Landsberg curve which is parametric, and its generalized Cartesian equation on the unit interval is formulated as,

$$A\left(x\right) = \sum_{n=0}^{\infty} (w)^n s(2^n x) \tag{5}$$

However, the vector method for the iterative midpoint displacements is one easier way to digitally model the Takagi–Landsberg fractal curve without depending on the unit interval used in Cartesian equation (Eq. (5)). In this vector method, the midpoint of a straight line (AB) is displaced along the global Z-axis by δ height which produces two new lines (AC and BC). The midpoints of these new two lines are again displaced by height δ_1 along global Z-axis in which $\delta_1 = w.\delta$ and w is a *relative size value* (Fig. 2). This process of midpoint displacements is continued iteratively on each newly born lines such a way that the displacement on the nth step δ_n becomes $w.\delta_{(n-1)}$ which eventually produces the parabolic curve when $w = 0.25$ and the Takagi curve when $w = 0.5$ (Fig. 3).

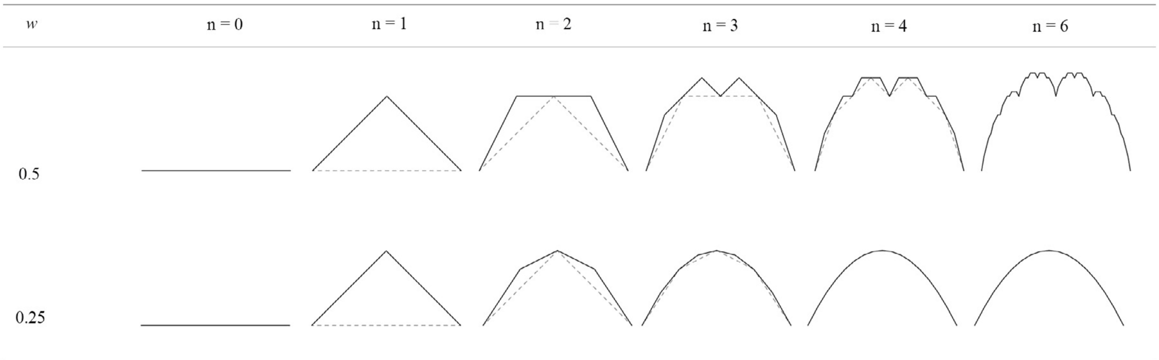


Fig. 3. Iterated midpoint displacement that produces the Takagi curve (top) when $w = 0.5$, the parabolic curve when $w = 0.25$ (bottom).

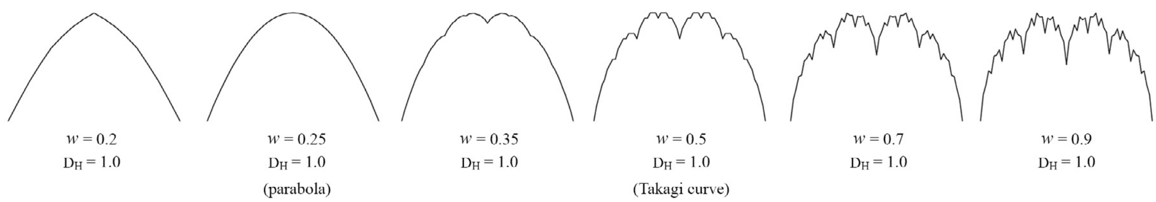


Fig. 4. Takagi–Landsberg curves after six iterations and their fractal dimensions (D_H) with the changing of w value.

In this process, the iteration number '*n*' and the *relative size value* '*w*' are the key variables as inputs that make the model parametric and control its texture. The range of the parameter of '*w*' value has been set from 0.20 to 1.0 so that we can visualize both categories of the curves that are before the parabola (when $w < 0.25$) and after the parabola (when $w > 0.25$). Changing the value of w within this range for more than 4 iterations shows the transformation of the smooth parabolic curve ($w = 0.25$) to unsmooth Takagi–Landsberg ($w \neq 0.25$) curve (Fig. 4). As the w value increases, the roughness of the curve also increases. It is noticeable that the overall height of the Takagi–Landsberg curve is affected by the w value. The constant overall height can be achieved by manipulating the initial triangle height without affecting the degree of roughness with respect to w value.

2.3. Takagi–Landsberg's fractal surface

The two-dimensional counterpart of the Takagi–Landsberg curve is the Takagi–Landsberg fractal surface. Similar to the Takagi–Landsberg curve, the Takagi–Landsberg surface is a parametric surface which can be gradually changed from its smooth paraboloid surface to an unsmooth fractal surface by changing the w parameter. This surface is derived from an equilateral triangular surface as a base using the same vector-based Midpoint Displacement Method. First, the midpoints of all the three sides of the triangle are vertically (along the global Z-axis) elevated by the equal height h. The new midpoints are then connected that gives four new self-similar triangles. The same process is again performed on all the new triangles by elevating their midpoints by h_1 which is equal to wh. This process is repeated for multiple times such a way that at nth iteration h_n becomes $w.h_{n-1}$ which eventually produces a Takagi–Landsberg fractal surface which is parametric and sensitive to the w value (Fig. 5). When $w = 0.25$, the surface becomes a paraboloid, and when $w > 0.5$, it becomes a fractal surface. Benoit Mandelbrot named this paraboloid as 'Mount Archimedes', and the fractal surface having $w = 0.5$ as the 'Mount Takagi' [13]. The *relative size value* (w) controls the roughness of the surface. Higher the value of w, the rougher is the surface. 'Fig. 6' shows the different outcomes of the surface with the changing of w value, and by adjusting the initial height value to obtain a constant overall height of the final outcome without compromising the degree of roughness. Apart from w value, the shape of the original base also determines the different outcomes of such surface if applied the same process. 'Fig. 7' shows the different resulting shapes when the bases are square and pentagon.

2.4. Fractal dimension and relative size value (w)

Fractal dimension is a measure of how unsmooth, irregular or detailed a shape is. It is a non-integer dimension that lies in between two successive integer dimensions, and thus measures that how a smaller dimensional figure having less roughness has the tendency to reach the higher dimensional figure of high roughness. In the case of Takagi–Landsberg surface, a smooth paraboloid is transformed into an unsmooth fractal surface. Therefore, based on the concept of fractal dimension, this transformation from smooth to rough can be measured by fractal dimension. Since the transformation from smooth to rough is controlled by the *relative size value*, hence there must be a relation between the *relative size value* and the fractal dimension. Dubuc [14] used the *box-counting method* and presented the relation between the fractal dimension (D_H) of the Takagi–Landsberg's fractal surface and the *relative size value* (w) as follows,

$$D_H = \frac{\log(8w)}{\log(2)}; \quad \text{for } 0.5 < w < 1. \tag{6}$$

when $0.5 < w < 1.0$, then the D_H of the Takagi–Landsberg surface becomes non-integer and floats in between 2.0 to 3.0, which defines that mathematically the Takagi–Landsberg surface is a fractal. The higher the value of D_H, the higher is the unsmoothness of the surface. However, when $0.0 < w < 0.5$, the D_H value becomes 2.0 which is an integer and constant. In this situation, the Takagi–Landsberg surface is mathematically not a fractal. However, in this domain ($0.0 < w < 0.5$), the smooth curve starts becoming unsmooth with the increasing of w value, which clarifies that the Takagi–Landsberg surface is more responsive to w parameter than the parameter of the fractal dimension. Because of its connection to the fractal dimension, we can call the *relative size value* (w)

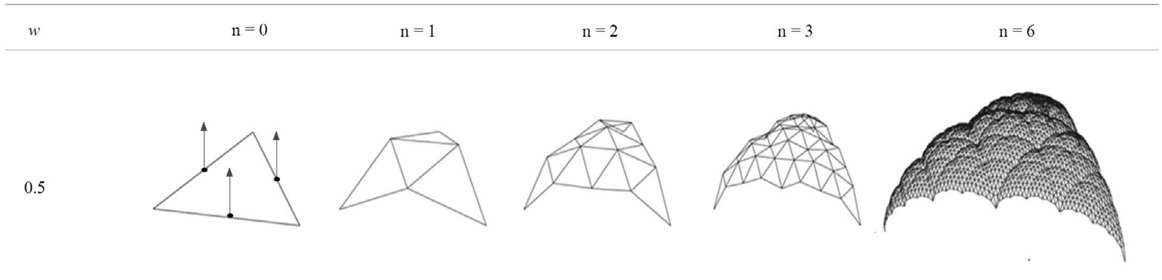


Fig. 5. The construction of a Takagi surface after several iterations when $w = 0.5$.

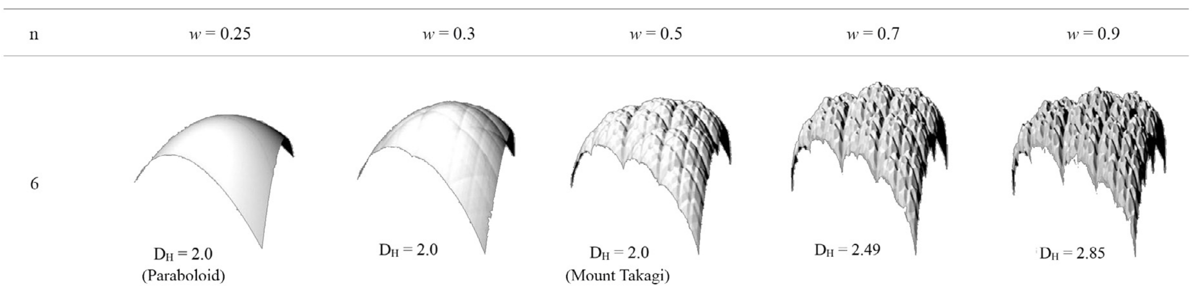


Fig. 6. Transformation of paraboloid's smooth surface to fractal surface at 4th iteration and their fractal dimensions. The initial height value has been adjusted to obtain a constant overall height without affecting the degree of roughness.

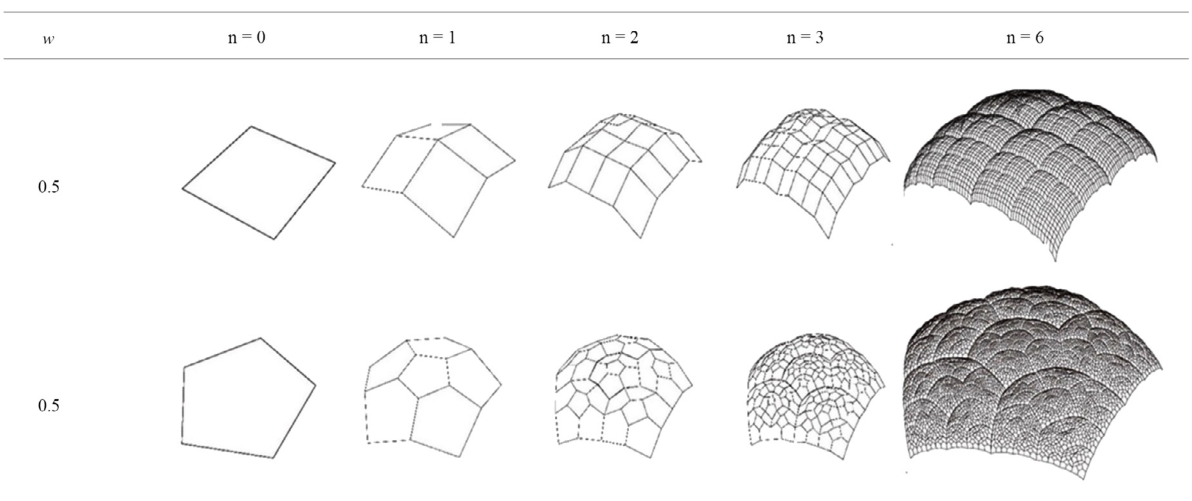


Fig. 7. Construction of Takagi surfaces using different polygonal bases when $w = 0.5$.

value as a *factor of fractal dimension*. 'Fig. 6' shows a paraboloid and different Takagi–Landsberg surfaces based on *w* value and their corresponding fractal dimensions, and 'Fig. 8' represents the relation between the *w* and D_H of the Takagi–Landsberg surface.

The strength of a structure is related to its geometric shape. Structural strength of a two-dimensional surface ($D_H = 2.0$) is different than the strength of a three-dimensional solid ($D_H = 3.0$). Therefore, it is predictable that the structural strength of a non-integer dimensional object having $2.0 < D_H < 3.0$ will have uniquely different strength, possibly higher than the strength of a surface and lower than the strength of a solid. This gives a clear hint that the fractal dimension is a unique geometric parameter which is not only related to the morphing of a continuous or smooth shape into an unsmooth one but also capable of changing its structural strength [7]. Accordingly, the *relative size value* (*w*) which is a factor of fractal dimension, is an effective geometric variable that can manipulate the structural strength of a structure which is based on the Takagi–Landsberg surface.

3. Fractal-based grid-shell-like structure design

The appearance of Takagi–Landsberg surface gives an impression of a unique space structure having multiple self-similar humps on it. This impression has motivated us to explore the applicability of the Takagi–Landsberg surface as a shape of reference in designing a new type of grid-shell or grid-shell-like structure. In this paper, we call this structure as the 'fractal-based grid-shell-like structure'. However, the unique characteristics of the parameter *relative size value* (*w*) is another key reason for this design experiment because this parameter controls the texture of the Takagi–Landsberg surface, thus influences the structural strength of the fractal-based grid-shell-like structure. When $w = 0.5$, the

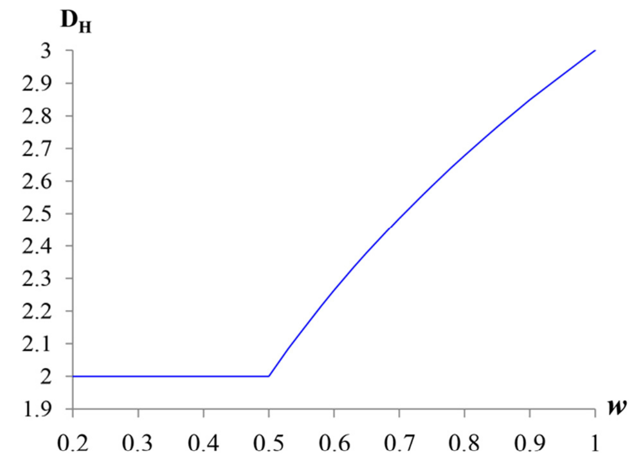


Fig. 8. Relation between the *relative size value* (w) and the fractal dimension D_H of Takagi–Landsberg surface. This relation is useful for representing the degree of roughness measured by the value of the fractal dimension.

Takagi–Landsberg surface is a paraboloid and its corresponding grid-shell is a regular grid-shell, but when the w value is changed from 0.5, then it gradually transforms the regular paraboloid grid-shell into a fractal-based grid-shell-like structure. In this transformation, because it is derived from a paraboloid grid-shell structure, the fractal-based grid-shell-like structure may partially possess the structural mechanism similar to a regular grid-shell. In the following section, the structural mechanism of a regular grid-shell structure has been briefly discussed to give an idea that how a grid-shell-like lattice shell structure based on Takagi–Landsberg surface may structurally behave.

3.1. Grid-shell structures

Grid-shells are doubly-curved, large-span and shape-resistant spatial structures resulting from the discretization of the continuous shell into a grid of one-dimensional linear elements that are joined to each other by means of structural nodes [15] (Fig. 9a). A continuous shell (Fig. 9c) can resist normal and in-plane shear stresses, while a grid-shell (Fig. 9d) can only resist forces in the direction of the bar members, i.e., axial forces (Fig. 9b). Besides, continuous shell carries loads generally by membrane force, whereas grid-shells carry their loads mainly by compressive forces [16]. In order to create true shell behavior in terms of obtaining shear stiffness in a grid-shell having square grids, diagonal bracing is done which triangulates the grid. In this sense, the shells with triangular grids seem to be preferable, although the advantage of the shape of grid units depends on the span–height ratio, especially with respect to the buckling capacity [17]. However, different shapes of grid units have different advantages and disadvantages [18,19]. From a broader perspective, the continuous members in a doubly curved grid-shell act as arches, and thus, the final structure benefit from the efficiency of both shell and arch schemes [20]. The lattice of the grid-shells can be made of steel, wood or composites. The construction of traditional grid-shells is much easier as compared to the continuous shell structures. In the construction of a grid-shell, a lattice is first prepared and laid flat on-site and then pushed, pulled or deformed into a doubly curved shape. However, the construction of non-traditional grid-shells (such as reticulated shells) with the desired curve or having surface irregularity is rather difficult, for which specially designed scaffolding or ladders are strategically framed and erected.

During the pre-construction design stage, a statically sound, robust and free-form grid-shell can be obtained if one uses a form-finding process that aims to find a funicular surface [23]. Thrust Network Analysis method [24] and its extension [25] based on graphic statics are the recently advanced approaches for grid-shell form finding. Apart from funicular shapes, paraboloid grid-shells, mainly of positive Gaussian curvature, are also popular because of their elegant appearance from an architectural point of view. The advantage of a paraboloid is that the bending stresses are confined to narrow zones along the boundaries and are negligibly small, especially under the uniformly distributed vertical loads. Thus the membrane theory provides already a good approximation for the stress condition of the surface [26]. Curvature plays an important role in the design of the grid-shell structure. Similar to continuous shells, grid-shells are particularly sensitive to buckling [16]. Increasing the curvature of a shell increases its buckling capacity [27]. The choice of grid units and their sizes has also impact on the buckling of grid-shells [17].

From a geometric point of view, in grid-shells, one-dimensional elements that carry axial forces are assembled in such a way that they produce a two-dimensional doubly curved profile which offers compressive-only load-bearing efficiency. In the fractal concept, an iterative assembling of one-dimensional elements can produce not only a two-dimensional or a three-dimensional profile, but also in between them. Based on this concept, a grid-shell can be iteratively designed whose overall two-dimensional smooth shape can be gradually transformed into a non-integer dimensional unsmooth shape. From the perspective of form–forces relation, this transformation will reconfigure the network (i.e., the grid) of axial forces resulting from a doubly-curved grid-shell structure, and eventually lead to change its overall structural behavior. This transformation may offer a unique optimal solution with an uncommon structural appearance, and this curiosity motivated us to search for such unique and structurally feasible (even perhaps optimal or suboptimal) form of a grid-shell structure.

3.2. Grid-Shell-like structure from Takagi–Landsberg surface

In the field of architecture, the Takagi–Landsberg curve and its surface were recently used by Buser et al. [28] to find their scope in designing some shapes for architectural structures (Fig. 10a, b). However, their study was limited to the development of some geometric models only, but they did not apply the models in designing some structural models to understand their structural behavior. Gentil and Neveu [29] applied Takagi–Landsberg surface for developing a mixed-aspect fractal surface (Fig. 10c). But, their study was confined within the geometric experiment only, but not extended to explore its possibility in architectural applications. Both of these studies lack the discussion about the effect of fractal dimension or the factor of fractal dimension on their models.

Our intention is to extend the geometric model of the Takagi–Landsberg surface to a finite element grid-shell model for studying its structural behavior. Following assumptions are taken for preparing the model:

a. An equilateral triangle with each side of 4 m has been taken as a base for generating the Takagi–Landsberg surface.
b. Only the finitely iterated models are valid for the real world applications. So, we have taken the third iterated model of the Takagi–Landsberg surface.
c. Only the triangular mesh edges of the Takagi–Landsberg surface are taken as a grid and call it as a 'Takagi–Landsberg grid-shell' (Fig. 11b).
d. A supporting frame has been added to the grid-shell for ensuring enough strength to avoid collapsing (Fig. 11a).
e. All lines of the Takagi grid and the additional frame are wooden bars (teak wood) having a square cross-section with each side 2.4 cm.

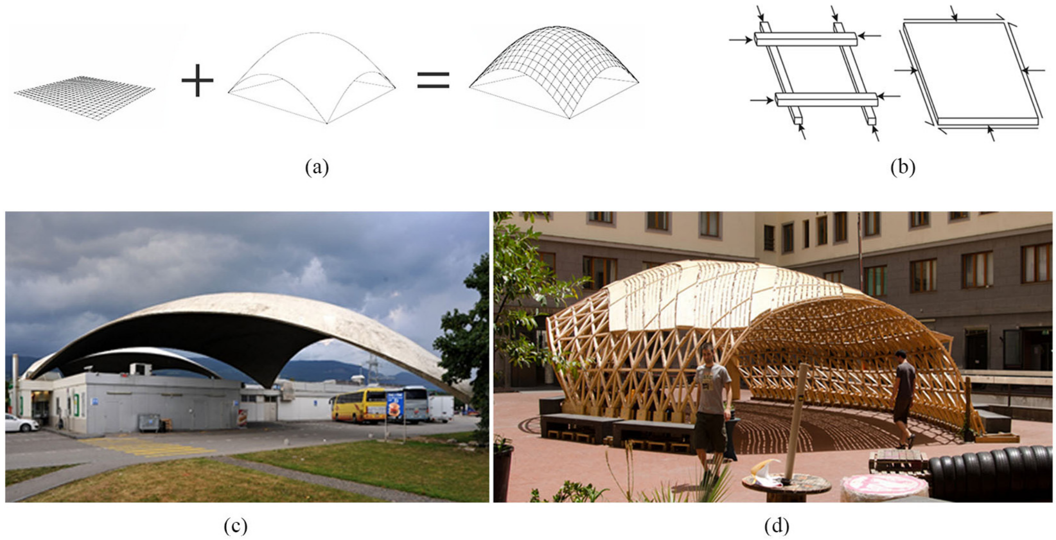


Fig. 9. (a) A grid-shell is made of a grid that takes a shape of continuous shell, (b) A continuous shell can resist normal and shear forces (left) while the grid-shell can only resist forces in the direction of the lath (right), (c) A continuous thin shell - Service Station in Deitingen by Heinz Isler (1968) [21]. (d) A freeform grid-shell–Grid-shell pavilion in the University of Naples "Federico II" (2012) [22].

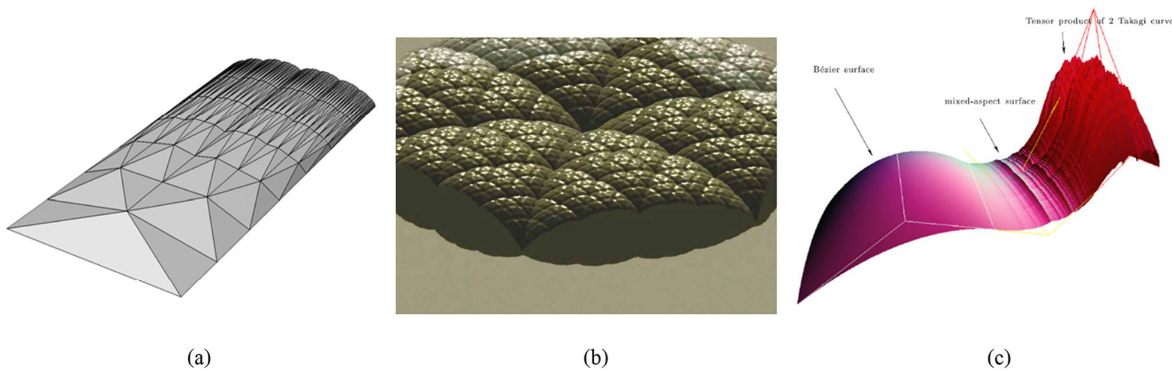


Fig. 10. (a) A geometric model of a fractal-based space structure developed by a set of differently iterated Takagi–Landsberg curves proposed by Buser et al. [28]. (b) A model of a fractal dome developed by assembling six Takagi–Landsberg surfaces proposed by Buser et al. [28]. (c) Mixed aspect fractal surface as a result of combining Takagi surface, Takagi curve surface and Bézier surface developed by Gentil and Neveu [29].

f. All these bars are connected at the joints by a wooden sphere having 9 cm diameter. We have ignored the weight of each ball during the structural analysis.
g. Three feet of the final structure is vertically and horizontally restrained on the ground.
h. The outer profile has been covered by flat panels for expressing its surface discontinuity (Fig. 11d).

The Takagi–Landsberg surface has inward pointed depression at several key nodes which make the surface structurally insecure. To cope up with this problem, first, we have quickly performed a finite element nonlinear analysis on the grid-shell when $w = 0.6$ for finding the most affected nodes under the gravity and uniformly distributed vertical loads, and marked them. Then, an additional frame has been prepared to support most of those affected nodes in such a way that it cannot occupy the inner space and consume as fewer bars as possible. (Fig. 11a), and can approximately blend with the curved surface of the main layer. (Fig. 11c) at this stage, we have named the overall structure with the additional frame as a 'grid-shell-like' structure. In architecture, using a supporting frame for improving the stability of a grid-shell is not very uncommon. We see the use of the additional supporting frame in the world's first steel-made grid-shell structure for the Plate Rolling Workshop in Vyksa designed by Vladimir Shukhov in 1897 (Fig. 12). In Shukhov's grid-shell, these supporting structures are trussed arches whose polygonal top chords are formed after a parabola [30]. In this structure, the evenly spaced three-hinged trussed frames were introduced to support the edges of five barrel vaulted grid-shells, in which each trussed frame supports two grid-shells.

The final configuration of the grid-shell-like structure shown in 'Fig. 11d' is made parametric such a way that the changing of w value automatically transforms the whole setup (Fig. 11c) which is a combination of the Takagi–Landsberg grid (Fig. 11b) and the additional frame structure (Fig. 11a). This parametric system has allowed us to perform quick and multiple structural analyses on the same model whenever its shape is changed with the changing of w value under different loads. When the w value gets higher, the density of the unsmoothness also gets higher and as a result the weight of the grid-shell-like structure gets increasing, (Fig. 13) which eventually impacts on the structural behavior when the gravity load is considered.

4. Structural characteristics

For structural analysis, if we consider only the single layer of the grid-shell-like structure without considering any structural

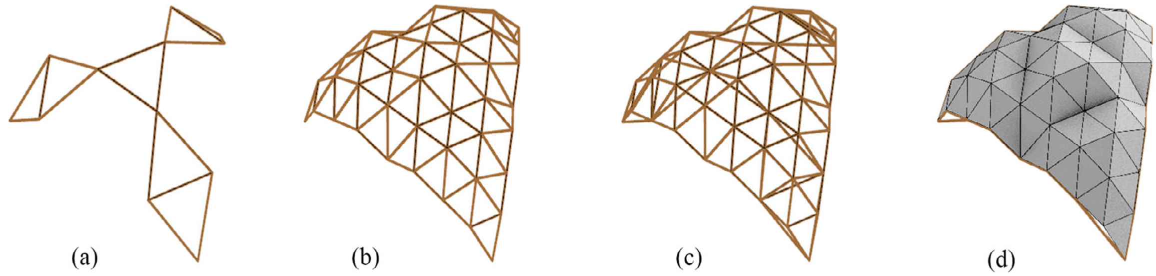


Fig. 11. Fractal-based grid-shell-like structure design when $w = 0.6$; (a) Additional frame (b) Takagi–Landsberg grid obtained after 3 iterations; (c) Fractal-based grid-shell-like structure design by combining the Takagi–Landsberg grid and the additional frame structures; (d) Flat triangular panels covered the grid to create the appearance of unsmoothness.

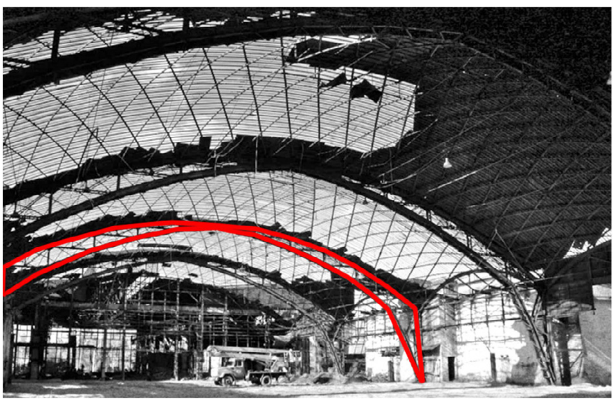


Fig. 12. Plate Rolling Workshop in Vyksa by Shukhov (1897): View into the three western bays [30]. An example of a grid-shell having an external supporting frame (red) to ensure the structural stability and strength. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

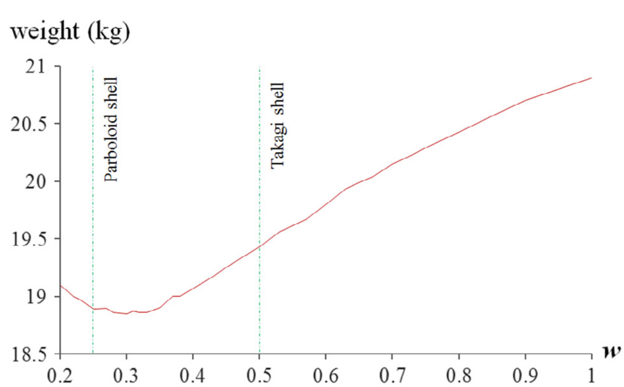


Fig. 13. The relation between the *w* value and the weight of the parametric model of the proposed grid-shell-like structure having an extra frame added to ensure structural stability.

support, then the stand-alone paraboloid grid-shell will be incomparable with the non-paraboloid (when $w \neq 0.25$) grid-shells because an immediate linear buckling analysis shows that the non-paraboloid grid-shells are extremely weak or fragile while the single-layer parabolic grid-shell is strong enough under gravity and uniformly distributed vertical loads. Accordingly, adding external support will allow us to perform a comparative structural analysis with the changing of *w* value, and also enable us to understand the changing of structural behavior under asymmetric and horizontal loads.

4.1. Finite element analysis

For performing the finite element analysis using FEM solver, we have taken the finite element model of the Takagi–Landsberg grid-shell-like structure prepared in the 'Section 3.2'. All the joints in the model are hinged; hence, all the lines act as structural bar elements. Its feet are vertically and horizontally restrained at the ground. Three different loads are applied (a) a combination of self-weight (1 kN/m^2) and uniformly projected vertical load (1 kN/m^2) (b) uniformly projected asymmetric load (1 kN/m^2) and (c) uniformly distributed the horizontal load (0.1 kN/m^2) (Fig. 14). Here, the horizontal load is considered as a reference of lateral wind load. The real impact of the wind such as suction has not been considered because of the simplification of the analysis. After this preprocessing stage, a nonlinear static analysis has been performed on the finite element model. The parametric modeling tool and the FEM solver have been connected in such a way that they can act interactively. In this way, FEM solver gives prompt feedback of structural behavior with the changing of *w* value. Our main objective of using the FEM solver is to see the axial forces behavior and the maximum displacements of the variations of the grid-shell-like structure under different applied loads. With this arrangement, first, we performed the nonlinear static analysis on the paraboloid-base smooth grid-shell-like structure when $w = 0.25$ and then on the fractal-based unsmooth grid-shell-like structure when $w = 0.6$, and see their results to compare the structural behavior between the smooth form and the fractal form of both the grid-shell-like structure (Figs. 15 and 16). Later, the maximum displacements δ of the different variations of the parametric grid-shell-like structure with respect to *w* value have been recorded under three different load conditions in order to find a relation between the *w* vs. δ.

4.2. Results and discussions

After the analysis, we find a significant effect of *w* value on the overall structural behavior of the Takagi–Landsberg grid-shell-like structure. The domain of *w* parameter has been fixed starting from 0.2, not from 0.25, just to see its effect even before its outer profile reaches to a paraboloid shape ($w = 0.25$). Under the **load case 1**, the maximum displacement with respect to *w* is non-linear. (Fig. 17) In the beginning, we expected the lowest displacement in the paraboloid-based grid-shell-like structure. But unpredictably we observed the lowest displacement at $w < 0.2$. After $w = 0.2$ to 0.24, the maximum displacements are almost the same except at $0.37 < w < 0.45$ where the displacements are the lowest. Based on the *w*-weight relation shown in Fig. 15, the weight gets almost linearly increasing after $w = 0.3$. In this situation, the overall load (self-weight + uniformly projected loads) in the load case 1 becomes higher when *w* is higher after $w = 0.3$; yet we see the

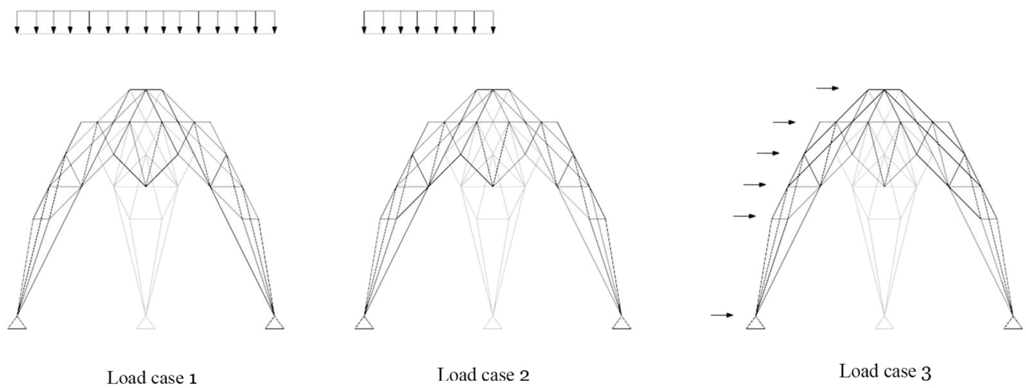


Fig. 14. Different load cases; Load case 1—Uniformly projected vertical loads and self-weight; Load case 2—Uniformly projected asymmetric vertical load; Load case 3—Uniformly distributed the horizontal load.

fewer displacements at $w > 0.3$ as compared to the $w = 0.3$ including the paraboloid ($w = 0.25$). If we compare the paraboloid-based grid-shell-like structure ($w = 0.25$) and a fractal-based grid-shell-like structure ($w = 0.6$), then we find the fractal-based structure has the less displacement than the displacement of the paraboloid-based structure under the load case 1. (Fig. 15) In this load condition, most members are under compression. Only a few members (red in Fig. 16, left column) experience tension due to their active involvement with the additional supporting frame. In fact, because of the supporting frame, a part of each grid-shell-like structure having different w value exhibits the truss-like behavior, and as a result, few members enjoy tension.

The structural behaviors of the grid-shell-like structure with respect to the w value under the **load cases 2** and **load case 3** are almost the same. Under both load conditions, paraboloid-based smooth structures are weaker than others except $w < 0.25$. The lowest displacements are observed again at $0.37 < w < 0.45$. We have marked this set of w values ($0.37 < w < 0.45$) as an 'optimal zone' (Fig. 17). According to Mandelbrot [13], mathematically, the outer profiles (or, the basic surfaces) of the grid-shell-like structure that lie in the 'optimal zone' follow Lipschitz function, which implies that these surfaces are rectifiable, with the fractal dimension saturated at the value 2.0. If we separately compare the maximum displacements of paraboloid-based grid-shell-like structure and the fractal-based grid-shell-like structure ($w = 0.6$) under the load cases 2 and 3, then we find the fractal-based unsmooth structure performs better than the paraboloid-based smooth structure (Fig. 15). It is obvious that both the structures experience enough tension because under the load cases 2 and 3, their outer profiles (the main/top layer of grids) heavily rely on the supporting frame (Fig. 16).

In this context, it can be argued that the grid-shell-like structures analyzed here do not show pure grid-shell action, instead, they partly show a truss-like behavior. This unique behavior is due to the addition of supporting members, especially due to the irregularity of the grid surface in the fractal-based unsmooth structures. If no additional support is added, then under the load case 1, the axial forces in a grid-shell having $w > 0.25$ do not get any continuous path within the grid surface because of the surface irregularity, and therefore, the structure becomes much weaker than the funicular or paraboloid shapes in which axial forces follow a continuous path towards the ground. Nevertheless, under the load cases 2 and 3, both the paraboloid and fractal grid-shells become very weak and unstable if no external supports are added. Once, the irregularity starts increasing with the increasing of w value, the overall structure in addition to the support frame starts becoming more truss-like assemblage, and perhaps, it might be the reason for less maximum displacement of the fractal-based grid-shell-like structure ($w > 0.25$) than that of the paraboloid-based grid-shell-like structure (Fig. 17). In this sense, from a structural point of view, we may call the fractal-based grid-shell-like structure not as pure 'grid-shell' structures [31], but as 'non-single layered reticulated shell' structures, while, architecturally we perhaps can use the general term, a 'grid-shell' structure.

Stability failure is one of the major issues concerning grid-shell design. Member slenderness in small-scale grid-shells having wooden laths is usually high, and structural collapse is generally caused by buckling instead of material failure. Stability check in terms of buckling analysis is therefore of great importance in a proper design process. For this purpose, a non-linear buckling analysis has been performed in the parametric model of Takagi–Landsberg grid-shell-like structures. From a general perspective, the buckling capacity is high in the grid-shell-like structures due to their high span–height ratio and triangular grid-units [17]. However, the degree of unsmoothness of their outer profiles or basic surfaces has an impact on their buckling behavior. 'Fig. 19' shows the relation between the buckling load capacity of Takagi–Landsberg grid-shell-like structure with respect to the w value. 'Fig. 18' shows the first three buckling modes of the paraboloid-based ($w = 0.25$) and the fractal-based ($w = 0.6$) grid-shell-like structures. 'Fig. 18' and the 'Fig. 19' explains that the buckling load capacity in the fractal-based complex structures are higher than the paraboloid-based smooth structure, and the possible reason could be the active involvement of the additional supporting frame in the fractal-based structures. In fact, buckling analysis is a complex process and especially if the lattice shell shape is also complex. Different aspects, such as the torsional buckling, local buckling, local and global constraints, etc. should be considered for understanding the proper buckling behavior of the grid-shell-like structure. However, in this paper, the buckling analysis has considered only the global buckling under the uniformly distributed vertical load as a brief study.

5. Prototype construction

After the finite element analysis of the digital model of the fractal-based grid-shell-like structure, we planned to verify its practical constructability and structural stability by making its real scale physical prototype. The key advantage of making a fractal-based complex structure is its scheme of self-similar repetitions which allowed us to make a complex assembly of a single module first, and then repeat the same assembly for making other self-similar modules. In Takagi–Landsberg curve, self-similar copies exist at the humps of the curve shown in 'Fig. 20a'. These humps are not limited to the central part of the graph; it happens everywhere where each copy (T_i) is the 1/4th-scaled copy of its immediate larger copy (T_{i+1}) [32]. Generally, in exact self-similar fractals, each

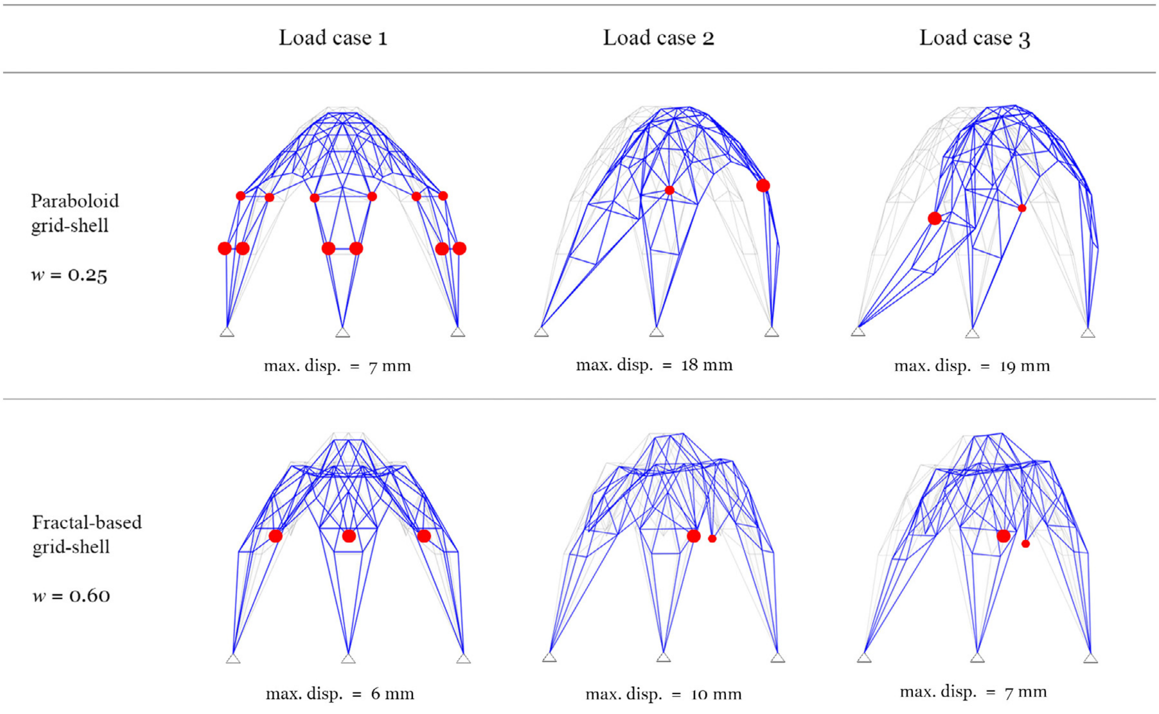


Fig. 15. Maximum nodal displacements under three different load cases.

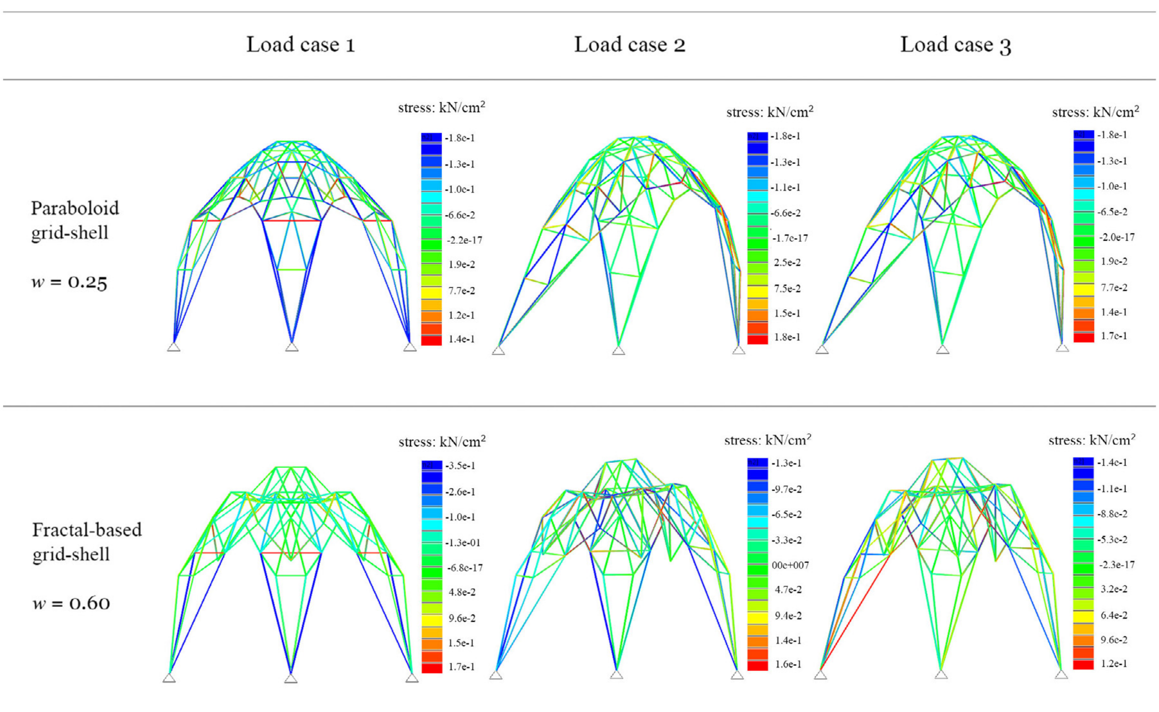


Fig. 16. Maximum displacements and axial stresses of the paraboloid-based grid-shell-like structure ($w = 0.25$) and Takagi grid-shell-like structure ($w = 0.5$) under 'Load case 2'. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

part is a scaled copy of the overall appearance, while in the Takagi–Landsberg curve, only the humps are scaled copy of the whole, but remaining parts are the scaled copy of the non-hump parts of the Takagi curve. This is due to the self-affine nature of Takagi–Landsberg curve [32], which, in fact, displays a modular scheme. 'Fig. 20b' illustrates three different modules that are repeated throughout the curve. The Takagi–Landsberg surface follows the same self-affine nature which enables an easy scheme for modular construction.

For the prototype construction, we selected the fractal-based grid-shell structure whose *relative size value* (w) is 0.6, i.e., the fractal dimension (DH) is 2.26. Although, this is not the best shape

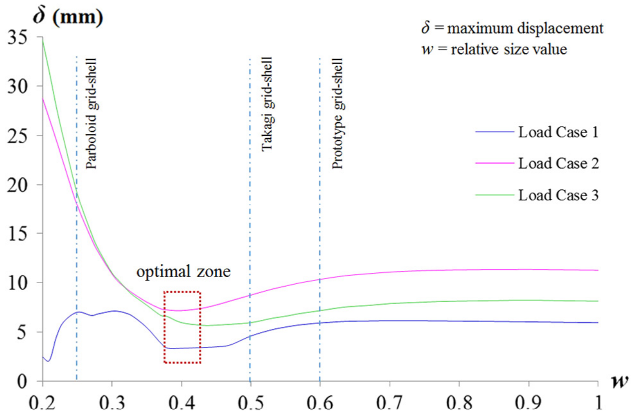


Fig. 17. Maximum displacements and axial stresses of the paraboloid-based grid-shell-like structure ($w = 0.25$) and Takagi's fractal surface based grid-shell-like structure ($w = 0.5$) under 'Load case 3'.

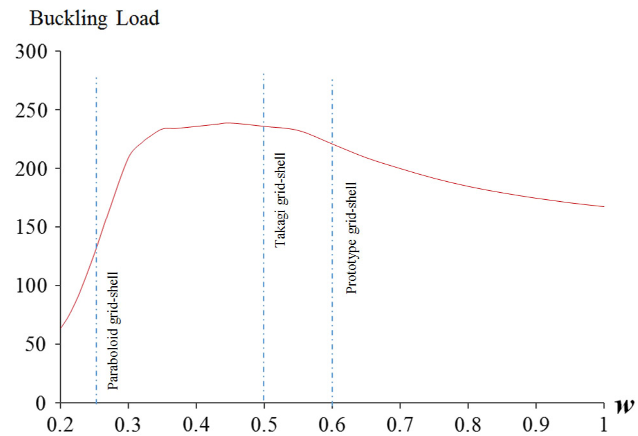


Fig. 19. A relation between the buckling load vs. w value.

in terms of less deformation under self-weight and horizontal loads, yet we selected this one just to obtain a shape which is truly a fractal in terms of Hausdorff fractal dimension ($2.0 < D_H < 3.0$). Besides, it also offers a fancy outcome that expresses a typical complex yet elegant appearance of the Takagi–Landsberg's fractal surface. In this structure, we had three sets of self-similar modules ('1', '2' and '3') as three legs and one unique module ('4') in the center (Fig. 21, Right). The additional frame (Fig. 11a) has some bars which occupy the usable space inside the structure (Fig. 21, Left).

For the construction, we started fabricating the modules first, and then finally assembled them to get the final structure. The whole fabrication process was intended to be done by the manual process. The purpose of such manual construction was to present that such a complicated shape, such as fractal, could be realized without any robotic and computer-aided manufacturing tools which consume considerable energy.

The structure was completely made of wood. The joints were prepared by wooden balls so that each bar member coming from different directions can be easily connected at a specific point (Fig. 22a). The main challenge was to locate the points for holes on each ball so that the bars coming from different directions can be fixed at specific holes by maintaining their vectors. From the top, the angle between two adjacent bars near to the ball is always 30° (Fig. 22a, bottom left), but vertically their angles are different and that was the main reason of difficulty to locate each hole on the ball surface. For this, we have developed our own coordinate measuring system (Fig. 22b) that can locate the points on each ball using both the vertical and horizontal coordinates. These locations were first calculated and market in the digital

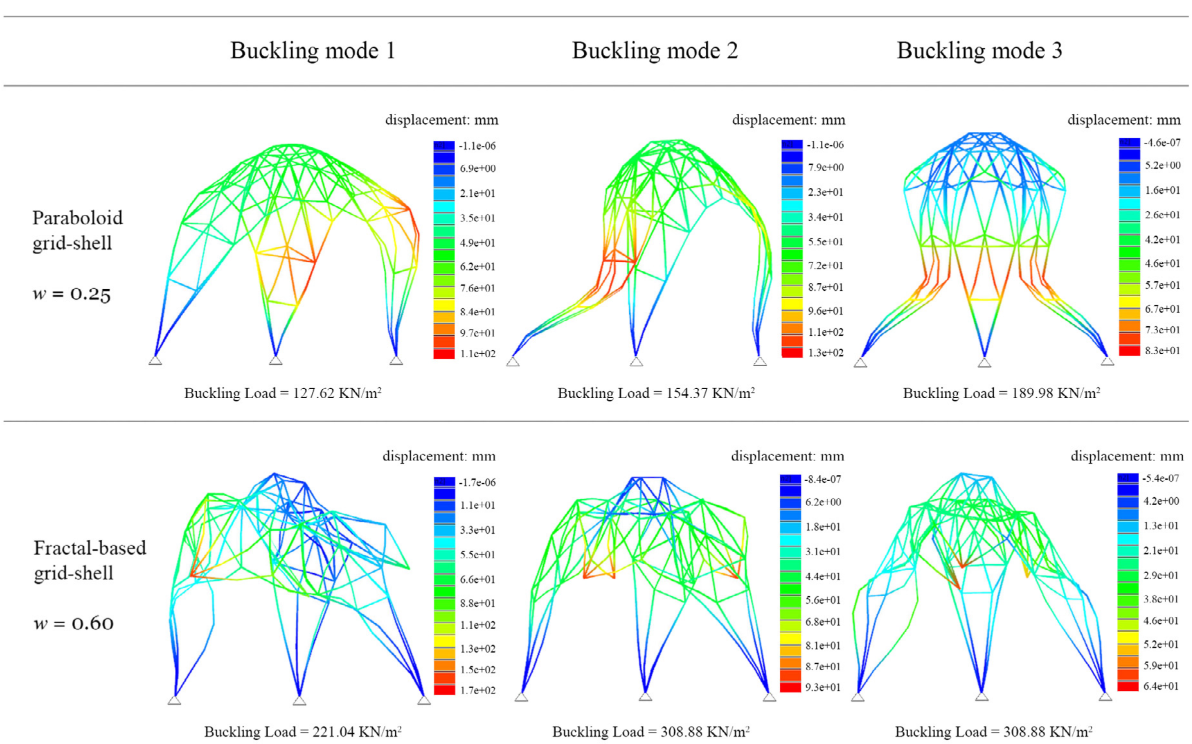


Fig. 18. The first three buckling modes of the Paraboloid-based ($w = 0.25$) grid-shell-like structure (top) and the fractal-based ($w = 0.6$) grid-shell-like structure (bottom) both added with the additional frames.

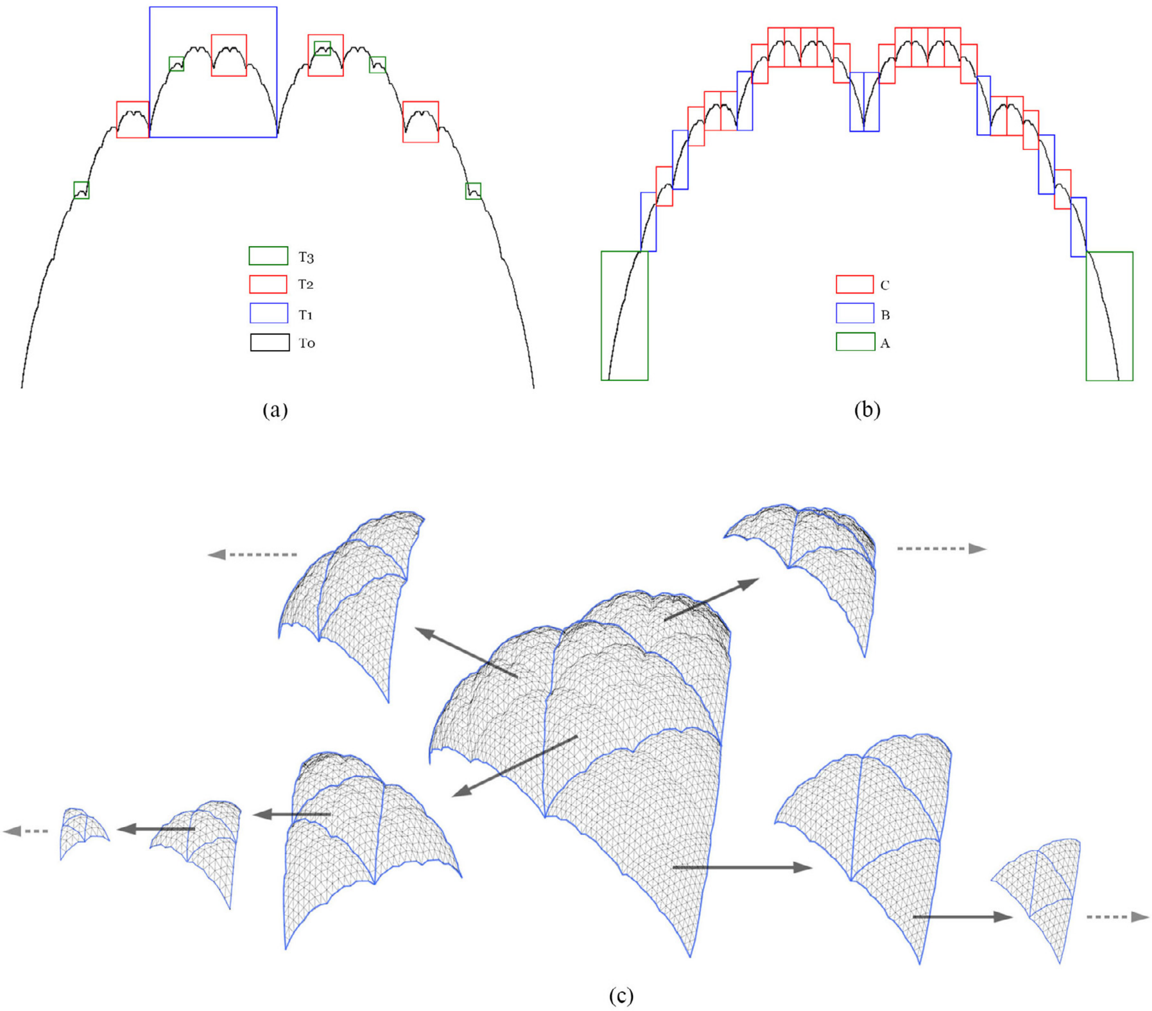


Fig. 20. (a) Self-similar repetition of the whole Takagi–Landsberg curve (T_0) into its parts ($T_1, T_2, T_3, \ldots, T_n$) in different scales. (b) Self-similar modules (A, B, and C) of a Takagi–Landsberg curve. (c) Self-similar modules of the Takagi–Landsberg fractal surface, and reflection of the whole into its parts at different scales.

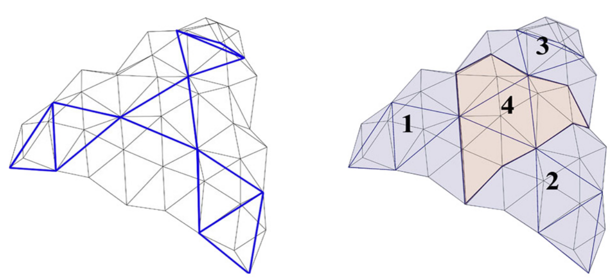
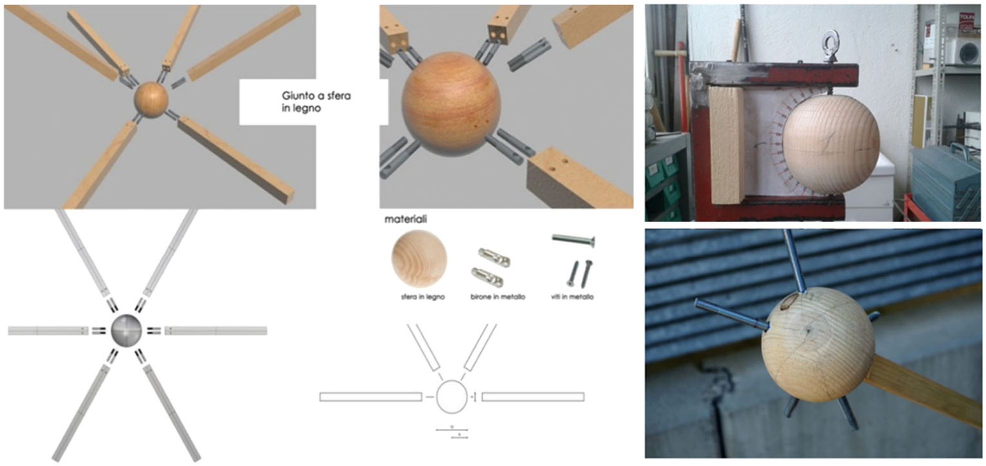


Fig. 21. Left—a Simplified assemblage of the additional frame structure; Right—Four structural modules, three of them (1, 2 and 3) are perfectly self-similar.

model on the computer. Each ball is connected first making a single unit and then each unit is assembled to make a single module. This way, three identical modules (1, 2 and 3 of Fig. 21 Right) were prepared (Fig. 23a), and finally, all modules were connected to get the final structure (Fig. 23b). After the final assemblage of the grid-shell-like structure (Fig. 24a), we fixed lightweight transparent PVC panels on the top to express the fractal character as an unsmooth but an approximately self-similar surface. Finally, the whole structure was erected on the ground (Fig. 24b). The structure was standing there successfully for couples of months without any visible deformation.

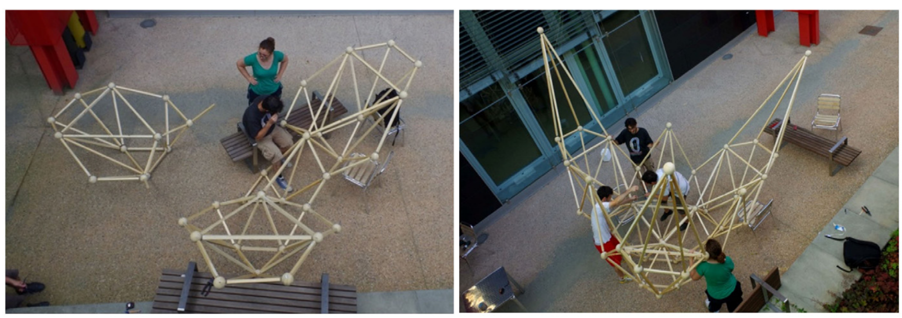
6. Conclusion

Architects and structural designers generally show their interests in finding new l forms that are unique by appearance and structurally optimal or suboptimal. In finding the innovative and optimal forms through the computational modeling and optimization process, geometric variables play a key role in shape morphogenesis. In practice, mostly the shape morphogenesis does not affect the geometric dimension of the shape. There are very rare attempts where the geometric dimension of a shape has been deliberately changed in order to search for new forms in design. Our study has focused on finding an innovative and optimal (and suboptimal) forms extracted from the passage through which a shape is transformed from its one geometric dimension to another geometric dimension. For this study, we have taken a paraboloid-based grid-shell-like structure as a benchmark example whose geometric dimension has been changed to the higher but non-integer dimensions within the domain of 2.0 to 3.0, which results in the changing of its surface texture. We have adopted here the Takagi–Landsberg surface which is a parametric surface and can be gradually transformed its paraboloid version into a fractal version through the fractal process where the *relative size value* (w) is the key geometric variable. When $w = 0.25$, it results in a paraboloid, and when $0.5 < w < 1.0$, it produces a fractal. We have used the finite element analysis method to understand the changing of the structural behavior of the grid-shell-like structure when its



(a) (b)

Fig. 22. (a) Initial strategy for the detailing of joints to connect bars; (b) Actual details of joints after simplifying the initial connecting scheme.



(a) (b)

Fig. 23. (a) Making three self-similar modules, (b) Final assembly of three self-similar modules and one central unique module.

(a) (b)

Fig. 24. Fractal-based grid-shell structure (a) without covering panels and (b) with the covering panels.

outer shape passes through a transition from a paraboloid to a fractal, and find the optimal shape under different load conditions. Unpredictably, we have found the optimal outer shapes in terms of less deformation at $w \approx 0.4$, which means the outer shapes that are neither paraboloids nor pure fractals, but unsmooth shapes which follow the Lipschitz function, i.e., the shapes that are rectifiable with the fractal dimension saturated at the value 2.0 [13]. However, after comparing the paraboloid-based smooth grid-shell-like structure with a fractal-based grid-shell-like structure ($w = 0.6$), we have found that the fractal-based complex structure performs better only if a supporting frame is always an integral part of each structure.

To check the practical constructability of the fractal-based grid-shell-like structure ($w = 0.6$), we constructed a real-scale physical prototype based on the self-similar modular method and erected it on a defined ground in the Politecnico di Torino's Mirafiori campus. It stood there for almost six months. But, a heavy storm partly damaged the structure. In fact, only the leg part was affected, but the remaining structure was unharmed which indicates that the weak connection of the supports to the ground might be the main the reason for this part failure. Because of this damage, we missed the chance to perform a practical load test on the prototype structure. Nevertheless, the failure by natural storm has raised few practical questions and unavoidable challenges which encourage future study on the following issues.

a. A detailed structural analysis under different environmental loads such as heavy wind, rain, snowfalls and seismic loads: Grid-shells and such spatial structures are prone to experience wind suction, and hence, a wind suction analysis is important prior to constructing the fractal-based grid-shell-like structure.
b. Practical load testing to understand the real-world structural behavior of the fractal-based grid-shell-like structure.
c. Stability checks of the fractal-based grid-shells-like structure: Grid-shells are sensitive to buckling and it depends on several factors including geometric ones. Surface irregularity is an uncommon geometric factor in the fractal-based grid-shell-like structures. One specific study could be dedicated only to the detailed study of the buckling behavior of the fractal-based structures with respect to w value. Torsional buckling should be carefully studied in such fractal-based complex lattice shells.
d. Non-linear static analysis for the fractal-based form-finding process: One particular research could be dedicated to generating a wide range of fractal-based grid-shell-like structure using non-linear static analysis method. A complicated static analysis procedure can be applied to enrich the fractal-derived design space as a future research.
e. Developing an interactive process or tool that integrates a force-based or equilibrium-based form-finding with the iteration-based or w-based fractal-generation to obtain a statically sound fractal-derived grid-shell structure.
f. A new study can be done on applying fractal geometry for finding an optimal form for a purely stand-alone spatial structure without adding any extra supporting frame.

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