CHAPTER 2

An Overview of Optimization Methods

After the completion of this chapter, the student should be able to:

1. Solve optimization problems limited to one or two control variables by graphical means.
2. Recognize that linear mathematical models have the special property that the optimum solution will always occur at an extreme point.
3. Utilize the methods of calculus to find the critical-point solution of nonlinear optimization problems consisting of one control variable and classify the critical point as a minimum, maximum, or point of inflection.

The essential concepts for finding the optimum solution of resource allocation models can be illustrated with graphical means and models limited to two control variables, called *bivariate* models. That is, the control vector $\mathbf{x}$ consists of x_1 and x_2 only. When models consist of more than two control variables, graphical methods cannot be used and mathematical methods will be utilized. First, our discussion will be limited to models with linear objective and linear constraint equations. This is an extremely important class of problems, both from a mathematical and an engineering point of view.

Next, the graphical method will be used to solve bivariate nonlinear problems. Finally, calculus will be used to solve *univariate* mathematical models, models with one control variable. The section on calculus is intended to be a review.

The graphical method is an extremely important method because we can visualize the overall problem more clearly. We not only find the solution to a problem in a straightforward manner, we usually obtain insights that are not apparent by studying the mathematical model alone or by solving with mathematical methods. It is truly unfortunate that the graphical method of solution is not applicable to models with more than two control variables. The conclusions that we draw from solving univariate and bivariate models, however, are incorporated in solving multivariate linear models and the nonlinear programming found in later chapters of this book.

2.1 GRAPHICAL SOLUTION TO LINEAR MODELS

A bivariate linear mathematical model may be written as

$$z = c_1 x_1 + c_2 x_2$$
$$a_{11}x_1 + a_{12}x_2 \{=, \leq, \geq\} b_1$$
$$a_{21}x_1 + a_{22}x_2 \{=, \leq, \geq\} b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 \{=, \leq, \geq\} b_m$$

The number of constraint equations m will depend upon the problem statement. The graphical procedure consists of the following steps.

1. Establish the feasible region from the set of constraint equations.
2. Assume a solution of z^0 and establish the slope of the line $c_1 x_1 + c_2 x_2 = z^0$.
3. Determine the optimum solution z^* by establishing a line that is parallel to z^0 and lies on the boundary of the feasible region.

Consider the mathematical model of Section 1.1. for maximizing total revenue for the building shown in Figure 1.2. The first step is to establish the feasible region. Let us establish the feasible region by investigating each constraint equation separately, and then we shall combine these results to construct the feasible region for the entire constraint set of equations.

The constraints $x_1 \geq 0$ and $x_2 \geq 0$ restrict the feasible region to be all points in the positive quadrant or quadrant I. In Figure 2.1*a*, we see that $x_1 > 0$ restricts the solution to be in quadrants I and IV. Similarly, in Figure 2.1*b*, we see that $x_2 \geq 0$ restricts the solution to be in quadrants I and II. The intersection of both constraints, $x_1 \geq 0$ and $x_2 \geq 0$, will always limit the solution to lie within the positive quadrant, quadrant I, as shown in Figure 2.1*c*.

The constraints $x_1 + x_2 \geq 5000$, $x_1 \leq 5000$, and $x_2 \leq 3000$ are shown in Figures 2.2*a*, 2.2*b*, and 2.2*c*, respectively. The arrows show the half-plane where the feasible region must lie. The intersection of these planes results in the feasible region as shown

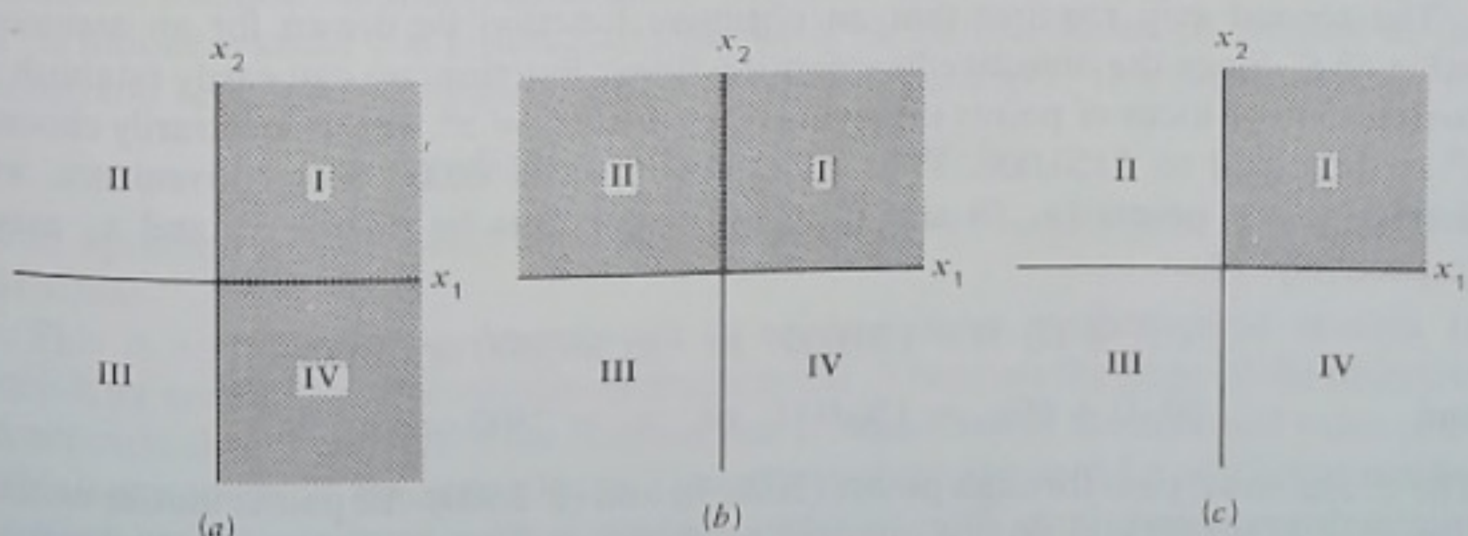

Figure 2.1 Nonnegative constraints $x_1 \geq 0$ and $x_2 \geq 0$.

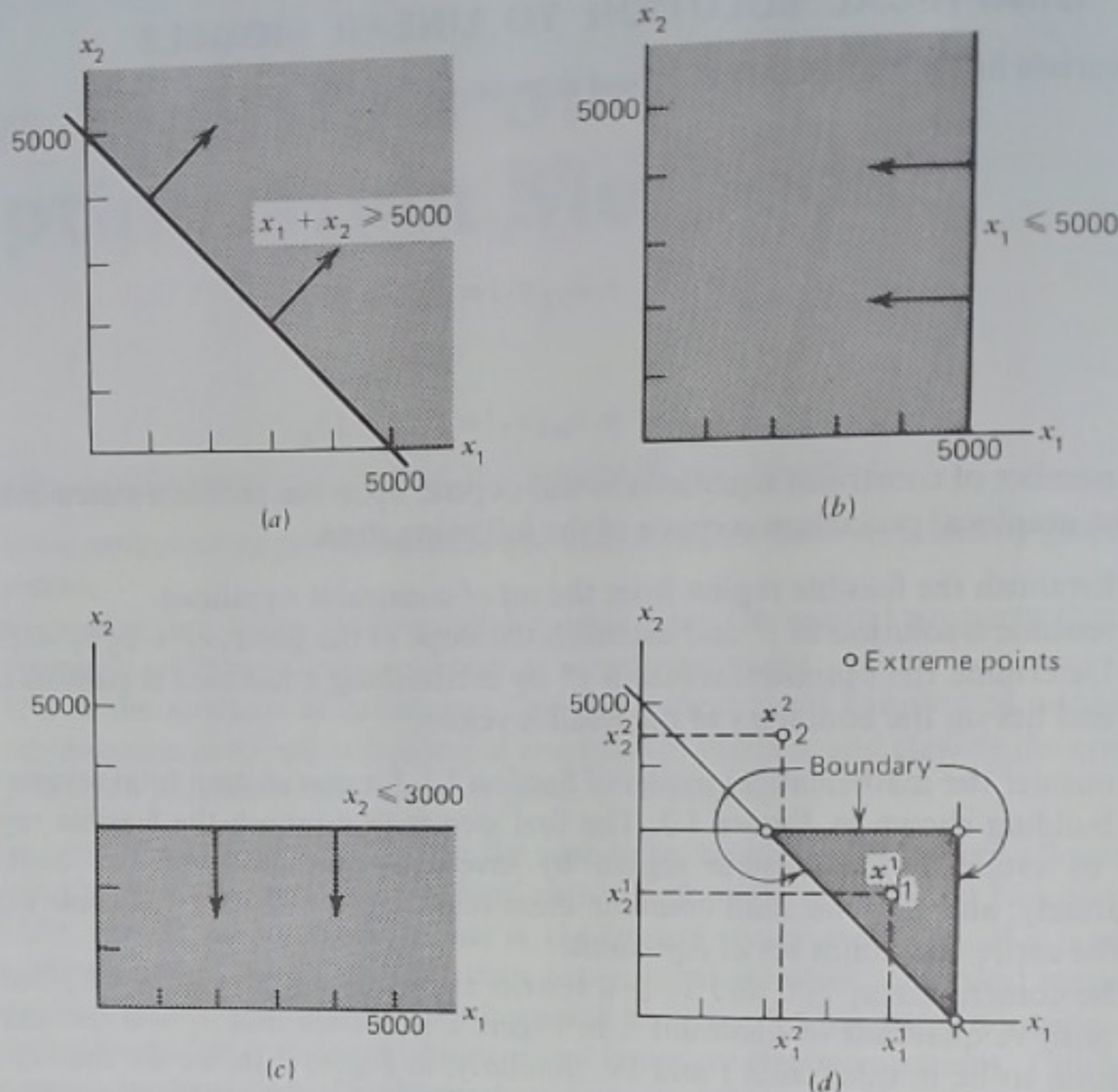

Figure 2.2 Establishing a feasible region.

by the shaded portion of the graph in Figure 2.2*d*. All points that lie within or on the boundary of the feasible region are called *feasible solutions*. All points that lie outside the feasible region are called *infeasible solutions*. Points $x^1 = (x_1^1, x_2^1)$ and $x^2 = (x_1^2, x_2^2)$ are examples of feasible and infeasible solutions, respectively.

The second step requires that an objective function be drawn for an assumed value of z^0. Since the objective function is a linear function, we can easily establish a *contour line* or locus of points satisfying $50x_1 + 60x_2 = z^0$. Let us arbitrarily choose z^0 to be equal to \$150,000. Thus, $50x_1 + 60x_2 = \$150,000$. For convenience, we determine the points $(x_1, 0)$ and $(0, x_2)$. These points lie on the x_1 and x_2 axes, respectively. Thus,

$$50x_1 + 60 \cdot 0 = 150{,}000 \quad \text{or} \quad x_1 = 3000 \text{ or } (3000, 0)$$

and

$$50 \cdot 0 + 60x_2 = 150{,}000 \quad \text{or} \quad x_2 = 2500 \text{ or } (0, 2500)$$

The z^0 line must pass through points (3000, 0) and (0, 2500). All points that lie on the line $z^0 = \$150,000$ and do not intersect as shown in Figure 2.3 are called infeasible solutions. None of these points can be the optimum solution.

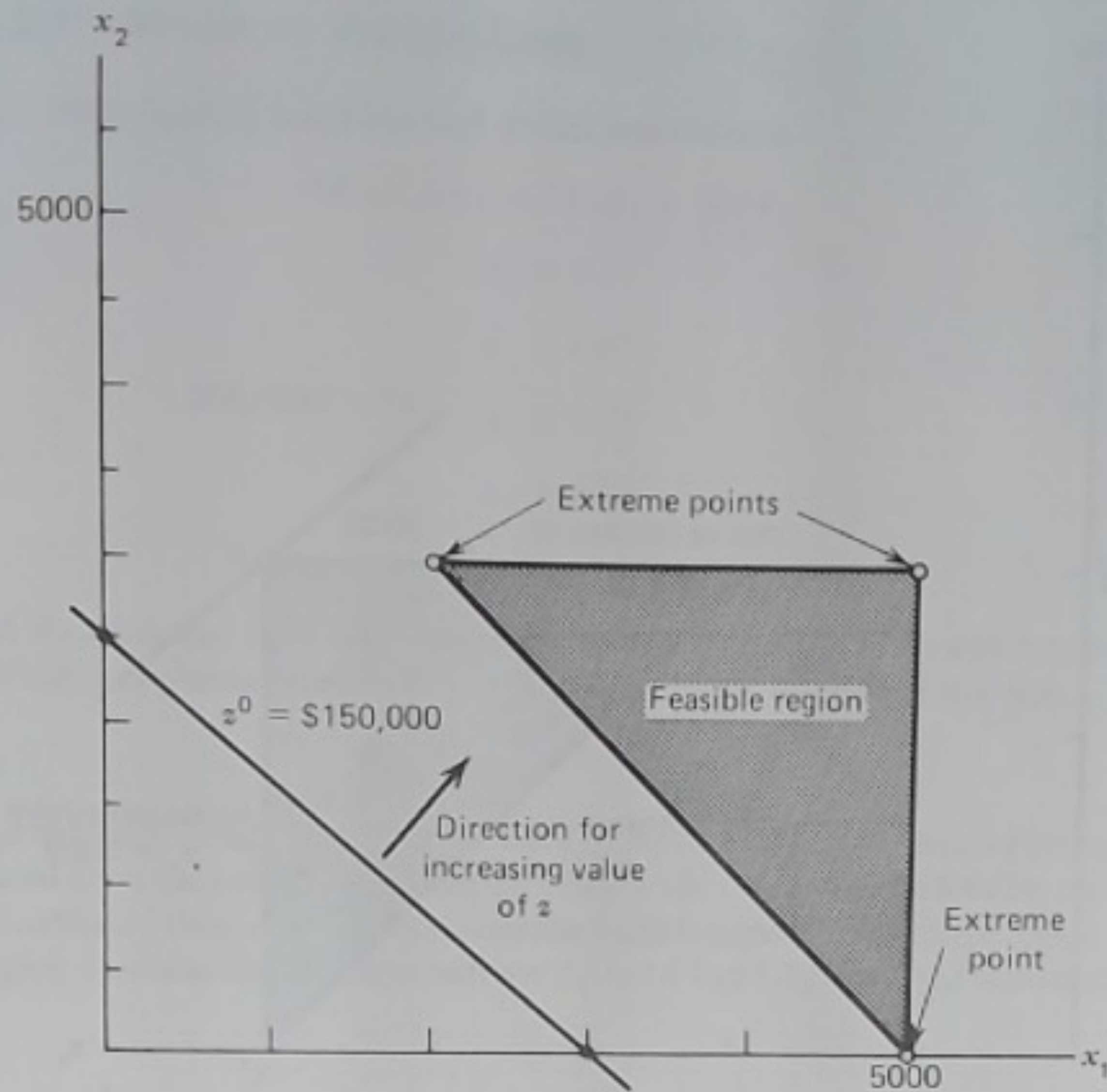

Figure 2.3 Searching for an optimum solution.

Utilizing the third step, the optimum solution may be found by drawing a contour line parallel to z^0 that lies at an extreme point of the feasible region. The direction for increasing the values of z is shown in Fig. 2.3. Any new contour line of z must intersect the feasible solution to be a candidate for an optimum solution. For linear models, an *extreme point* is defined to be the intersection of two or more constraint equations. An extreme point will lie on the boundary of the feasible region; therefore, it is a feasible solution and a candidate for the optimum solution. The contour line marked z^* is parallel to z^0 and it passes through the extreme point x^*, the location of the optimum solution for this maximization problem as shown in Figure 2.4. The point x^* is unique because z is a maximum and satisfies the conditions established in the constraint set of the problem. Thus, the optimum solution to this problem is

$$x_1^* = 5000 \text{ ft}^2 \text{ and } x_2^* = 3000 \text{ ft}^2$$

with optimum or maximum total revenue equal to $z^* = \$50 \cdot 5000 + \$60 \cdot 3000 = \$430{,}000$.

This is a straightforward approach to solving linear mathematical models. In step 3, we are able to establish the optimum point z^* because the slope of the objective function is always parallel to the contour line z^0 regardless of the assumed value of z^0. This is a property of linear functions. This approach can be used for any linear mathematical model restricted to two control variables with a minimum or maximum objective function.

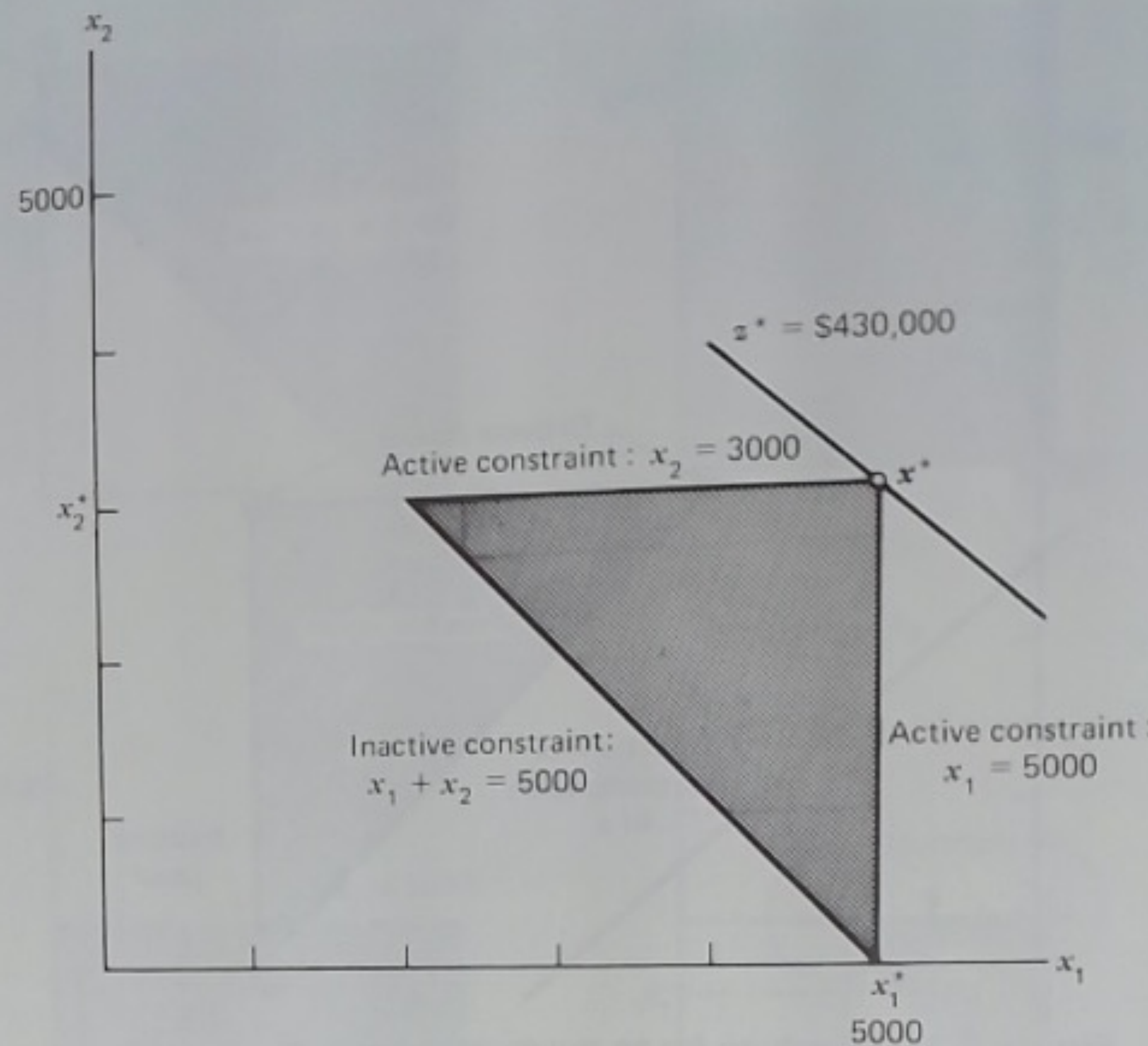

Figure 2.4 The optimum solution.

Active and Inactive Constraints

When identifying whether or not a solution is located on the boundary or at an extreme point of the feasible region, the terms active and inactive constraints are used. By definition, an *active constraint* will occur on the *boundary line* or at *an extreme point* of the feasible region. The constraint equation where $g(x)$ is either a linear or non-linear function of x,

$$g(x) \leq b$$

is said to be active constraint when the values of x satisfy the strict equality as:

$$g(x) = b$$

On the other hand, when the solution x is such that the inequality is satisfied,

$$g(x) < b$$

the constraint equation is said to be an *inactive* constraint. In Figure 2.4 the active and inactive constraints are indicated.

EXAMPLE 2.1 Minimum-Weight Truss

In Example 1.1, the following mathematical model was derived.

$$\text{Minimize } z = 612A_1 + 442A_2$$

$$A_2 \geq 4.17$$

$$A_1 \geq 6.67$$

$$A_1 \geq 8.33$$

$$A_2 \geq 6.25$$

$$A_1 \geq 16.7$$

where A_1 and A_2 represent the cross-sectional areas of the compression and tension members, respectively. Find the optimum member sizes using a graphical method of solution.

Solution

The first step is to establish the feasible region. Each equation has been plotted in Figure 2.5. Arrows have been placed upon each constraint equation to show the location of the feasible region for each one. The intersection of these constraints specify the feasible region. The feasible region is shown as the shaded region. It is bounded by the equations $A_1 = 16.7$ and $A_2 = 6.25$ as shown in Figure 2.6.

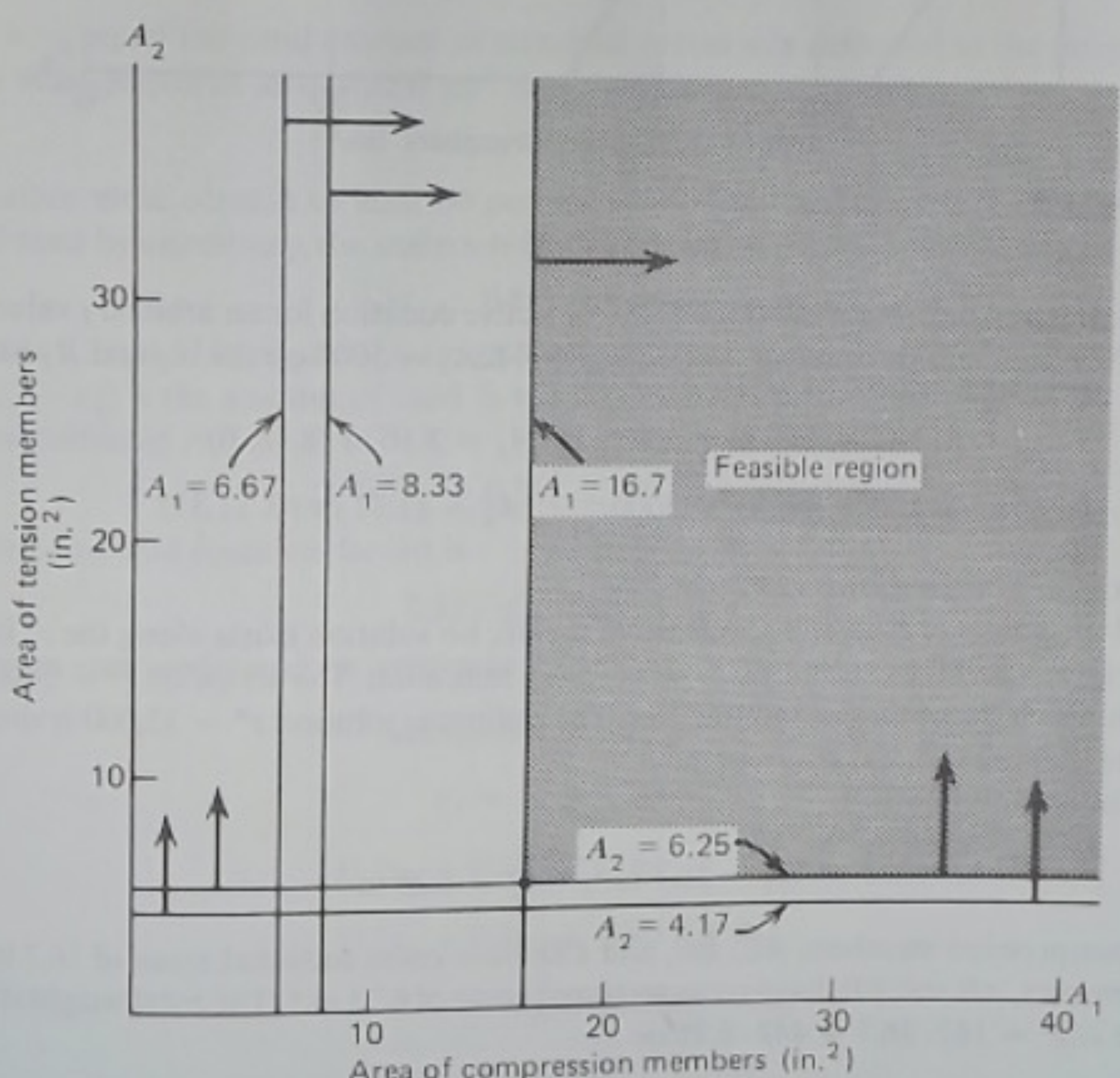

Figure 2.5

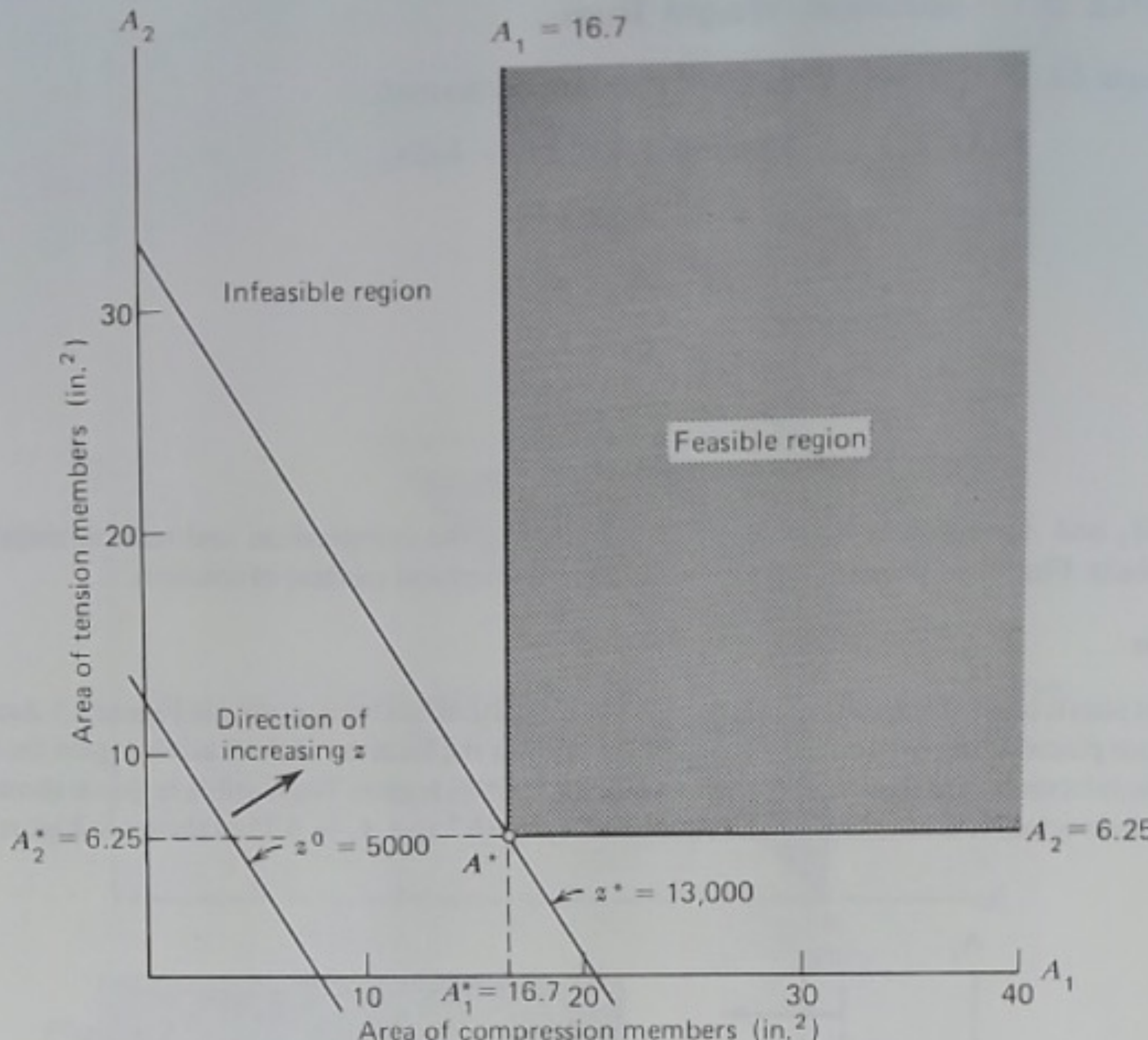

Figure 2.6

The next step is to determine the slope of the objective equation for an arbitrary value of z^0. Let $z^0 = 5000$. We shall find the intercepts of $612A_1 + 442A_2 = 5000$ on the A_1 and A_2 axes.

$$612A_1 + 442 \cdot 0 = 5000; \qquad A_1 = 8.16 \text{ or } (8.16, 0)$$

$$612 \cdot 0 + 442A_2 = 5000; \qquad A_2 = 11.31 \text{ or } (0, 11.31)$$

The line for $z^0 = 5000$ is shown in Figure 2.6.

Since all the points of z^0 lie in the infeasible region, no solution exists along the z^0 line. By increasing z, we can find the extreme point where z is a minimum. The arrow on $z^0 = 5000$ show the direction in which the optimum solution lies. The optimum solution $z^* = 13{,}000$ is shown in the figure. Note that lines z^* and z^0 are parallel.

The optimum solution is

$$A_1^* = 16.7 \text{ in.}^2, \quad A_2^* = 6.25 \text{ in.}^2$$

Thus, all compression members, AC, BC, and CD, have cross sectional areas of 16.7 in.^{2} and all tension members, AB and BD, have cross sectional areas of 6.25 in.^{2}. The total weight of the minimum truss is $z^* = 162 \cdot 16.7 + 442 \cdot 6.25$ or

$$z^* = 13{,}000 \text{ lb}$$

EXAMPLE 2.2 Minimum-Cost Aggregate Mix Model

A contractor is considering two gravel pits from which he may purchase material to supply a project. The unit cost to load and deliver the material to the project site is \$5.00/yd^3 from pit 1 and \$7.00/yd^3 from pit 2. He must deliver a minimum of 10,000 yd^3 to the site.

The mix that he delivers must consist of at least 50 percent sand, no more than 60 percent gravel, nor more than 8 percent silt. The material at pit 1 consists of 30 percent sand and 70 percent gravel. The material at pit 2 consists of 60 percent sand, 30 percent gravel, and 10 percent silt.

(a) Formulate a minimum-cost model.
(b) Determine the optimum solution by the graphical method.
(c) Determine the active and inactive constraint equations for the optimum solution.
(d) Determine the proportions of sand, gravel, and silt in the optimum solution.

Solution

(*a*) ***Formulation*** Since the gravel from pit 1 does not contain the minimum amount of sand to meet project requirements, the contractor may not utilize the cheaper material exclusively. He must mix the material from pits 1 and 2 to produce the required proportions.

We define the control variables to be

x_1 = amount of material taken from pit 1 (in cubic yards)

x_2 = amount of material taken from pit 2 (in cubic yards)

The cost function is

$$\text{minimize } c = \$5.00x_1 + \$7.00x_2$$

Let $x_1 + x_2$ equal the total amount of standard gravel mix delivered to the project site. The contractor must deliver at least 10,000 yd^3, thus the delivery constraint is

$$x_1 + x_2 \geq 10{,}000$$

The mixture must contain at least 50 percent sand. The contractor may obtain the desired amount of sand by combining the materials from each pit.

$$0.3x_1 + 0.6x_2 \geq 0.5(x_1 + x_2)$$

The products $0.3x_1$ and $0.6x_2$ are the amounts of sand taken from pits 1 and 2, respectively. The term $0.5(x_1 + x_2)$ is the amount of sand in the mix. Similarily, the constraint on the amount of gravel to be delivered is

$$0.7x_1 + 0.3x_2 \leq 0.6(x_1 + x_2)$$

Finally, the constraint equation for silt is

$$0.1x_2 \leq 0.08(x_1 + x_2)$$

The minimum cost model may be written as

$$\text{Minimize } c = 5x_1 + 7x_2$$

$$x_1 + x_2 \geq 10{,}000 \qquad \text{(delivery)}$$

$$0.3x_1 + 0.6x_2 \geq 0.5(x_1 + x_2) \qquad \text{(sand)}$$

$$0.7x_1 + 0.3x_2 \leq 0.6(x_1 + x_2) \qquad \text{(gravel)}$$

$$0.1x_2 \leq 0.08(x_1 + x_2) \qquad \text{(silt)}$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

or in standard form:

$$\text{Minimize } c = 5x_1 + 7x_2$$

$$x_1 + x_2 \geq 10{,}000 \qquad \text{(delivery)}$$

$$-2x_1 + x_2 \geq 0 \qquad \text{(sand)}$$

$$-x_1 + 3x_2 \geq 0 \qquad \text{(gravel)}$$

$$4x_1 - x_2 \geq 0 \qquad \text{(silt)}$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

The second cost model will be utilized for plotting. The first cost model will be used to answer part d and for checking the results.

(*b*) *Graphical Solution*

Step 1. *Establish the feasible region.* Since $x_1 \geq 0$ and $x_2 \geq 0$, the feasible region is restricted to be in a positive quadrant. Since only two points are needed to determine a line, each constraint equation is assumed to be a strict equality, and then the boundary of the constraint equation is found. For instance, for the equation:

$$x_1 + x_2 = 10{,}000$$

when $x_1 = 0$, x_2 must be equal to 10,000 or $x_2 = 10{,}000$ and when $x_2 = 0$, x_1 must be equal to 10,000 or $x_1 = 10{,}000$. Thus, the points (0, 10,000) and (10,000, 0) are sufficient to determine the boundary of the constraint, $x_1 + x_2 \geq 10{,}000$. Next, the boundary of the feasible region for the inequality constraints of sand, gravel, and silt is determined. In Figure 2.7*a* the arrows show the direction of the feasible region.

Since the gravel constraint $-x_1 + 3x_2 \geq 0$ lies outside the feasible region, it will not be considered in the search for the optimum solution.

Steps 2 *and* 3. Estimate an optimum solution, and test the condition of optimality. The minimum cost was assumed to be equal to \$80,000 or $c^0 = \$80{,}000$. Figure 2.7*b* shows this estimate to be too high. This estimate does not satisfy the condition of optimality.

The optimum-cost line will be parallel and less than the initial estimate of $c^0 = \$80{,}000$. Furthermore, for linear mathematical models, we should search for the optimum solution at an extreme point. For this problem, the optimum point is equal to

$$x_1^* = 3300 \text{ ft}^3, \quad x_2^* = 6700 \text{ ft}^3$$

with minimum cost $c^* = \$63{,}400$.

Note that the optimum-cost line is parallel to the initially estimated cost line, and it satisfies the conditions of optimality. In addition, the optimum solution occurs at an extreme point.

(*c*) *Active and Inactive Constraints* Since the optimum solution passes through the intersection of the lines marked delivery and sand, the equations $x_2 - 2x_1 = 0$ and $x_1 + x_2 = 10{,}000$ are both active constraints. The remaining equations are inactive constraints. These are the lines labeled silt and gravel. Since the constraint equation for gravel lies outside the feasible region, it will always be an inactive constraint.

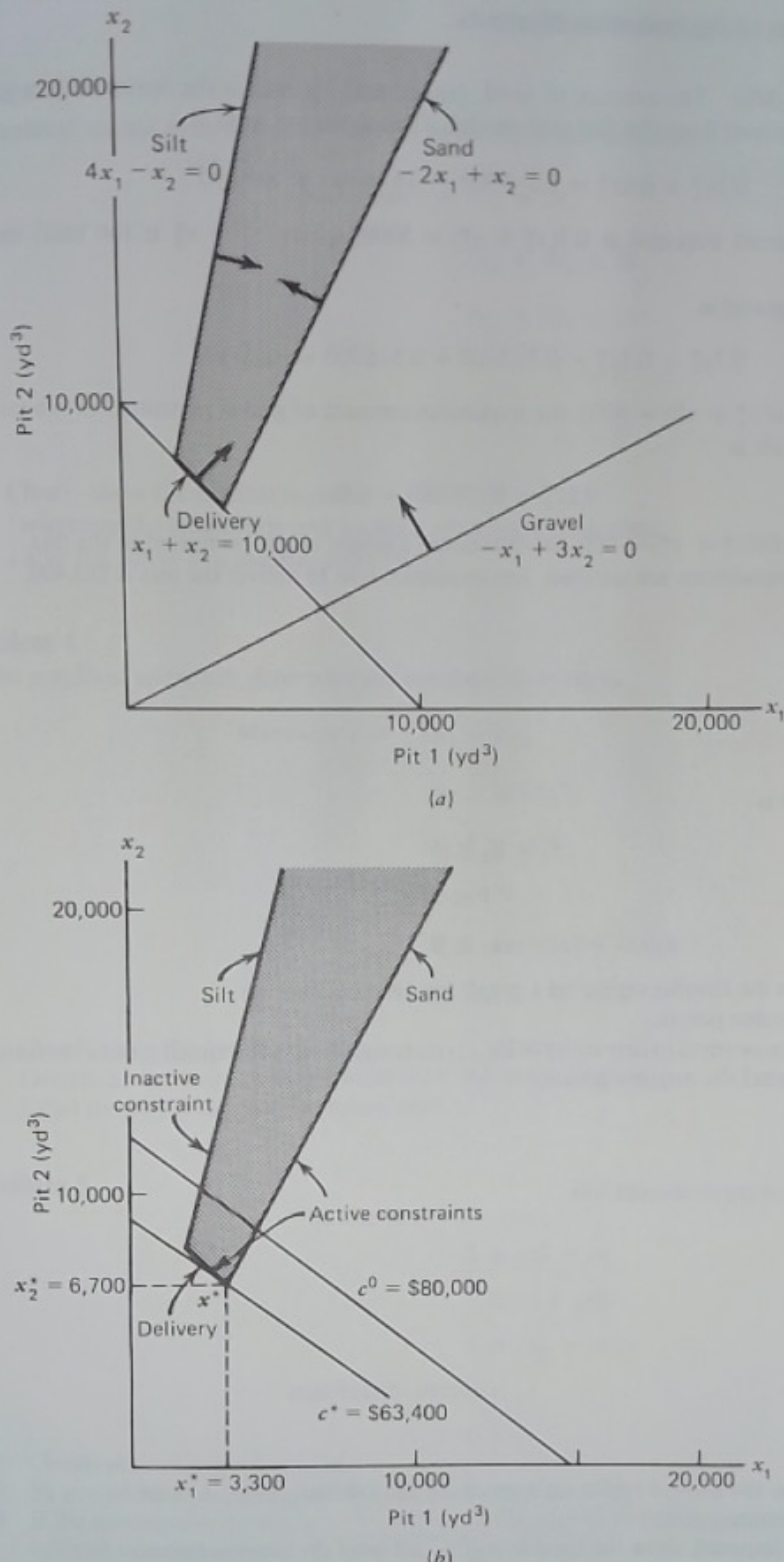

Figure 2.7 (*a*) Feasible region. (*b*) Optimum solution.

(*d*) ***The Optimum Mix*** The amount of sand, gravel, and silt that is the optimum mix is most conveniently determined from the first cost model. The amount of sand is

$$0.3x_1^* + 0.6x_2^* = 0.3 \cdot 3300 + 0.6 \cdot 6700 = 5010 \text{ yd}^3$$

The minimum amount required is $0.5(x_1^* + x_2^*) = 5000$, where $x_1^* + x_2^*$ is the total amount delivered.

The amount of gravel is

$$0.7x_1^* + 0.3x_2^* = 0.7 \cdot 3300 + 0.3 \cdot 6700 = 4320 \text{ yd}^3$$

This is less than $0.6(x_1^* + x_2^*) = 6000$, the maximum amount of gravel permitted in the mix.

The amount of silt is

$$0.1x_2^* = 0.1(6700) = 670$$

This is less than $0.08(x_1^* + x_2^*) = 800$, the maximum amount of silt permitted in the mix.

The constraint conditions are satisfied. The minimum cost to deliver the mix is $63,400.

PROBLEMS

Problem 1

The constraint set is

$$x_1 + x_2 \leq 3 \qquad (1)$$

$$1 \leq x_2 \leq 2 \qquad (2)$$

$$x_1 \geq 0 \qquad (3)$$

(a) Clearly show the feasible region on a graph for the constraint set.
(b) Label all extreme points.
(c) If the less than or equal to constraint of Eq. (1) is changed to a strict equality, show the feasible region and label the extreme points.

Problem 2

Consider the following constraint sets:

$$x_1 - 2x_2 \leq 2$$

$$2x_1 + x_2 \leq 9$$

$$-3x_1 + 2x_2 \leq 3$$

$$x_1 \text{ is unrestricted in sign}$$

$$x_2 \geq 0$$

(a) Clearly show the feasible region on a graph for the following constraint set.
(b) Label all extreme points.
(c) If $x_1 \geq 0$ is imposed, show the feasible region and label the extreme points.

Problem 3

By graphical means, determine the location of the optimum solution.

$$\text{Maximize } z = 3x_1 + 2x_2$$

$$2x_1 + 4x_2 \le 21$$

$$5x_1 + 3x_2 \le 18$$

$$x_1 \ge 0$$

$$x_2 \ge 0$$

(a) Clearly show the feasible region.
(b) Determine the magnitude and location of the optimum point.
(c) Label the active and inactive constraints.

Problem 4

By the graphical approach, determine the optimum solution to

$$\text{Maximize } z = 5x_1 - 2x_2$$

$$x_1 - 2x_2 \le 0$$

$$2x_1 + x_2 \le 9$$

$$-3x_1 + 2x_2 = 3$$

x_1 is unrestricted in sign

$$x_2 \ge 0$$

(a) Clearly show the feasible region.
(b) Determine the magnitude and location of the optimum point.
(c) Label the active and inactive constraints.

Problem 5

$$\text{Minimize } z = 3x_1 + x_2$$

$$2x_1 + 2x_2 \le 9$$

$$2x_1 - 4x_2 = 6$$

$$x_1 \ge 0,$$

$$x_2 \ge 0$$

(a) Clearly show the feasible region.
(b) By graphical means determine the optimum solution.
(c) If the nonnegative restriction on x_2 (i.e., $x_2 \ge 0$) were removed, determine the effect on the optimal value of z.

Problem 6

$$\text{Maximize } z = 2x_2$$

$$x_1 \le x_2 \quad (1)$$

$$-x_1 + 2x_2 \le 2 \quad (2)$$

$$2x_1 + 2x_2 = 3 \quad (3)$$

x_1 is unrestricted in sign

$x_2 \ge 0$

(a) Clearly show the feasibility region.
(b) Determine the magnitude of the optimum point.
(c) Label the active and inactive constraints.
(d) How is the optimum point affected if constraint (2) is removed. Why?

Problems 7, 8, 9, and 10 were formulated in Section 1.1 as problems 1, 2, 3, and 4, respectively. The question here deals with the solution of the problems.

Problem 7

A contractor has two sand and gravel pits where he may purchase material. The unit cost including delivery from pits 1 and 2 is \$5 and \$7 per cubic yard, respectively. The contractor requires 100 yd^3 of mix. The mix must contain a minimum of 30 percent sand. Pit 1 contains 25 percent sand, and pit 2 contains 50 percent sand.

The object is to minimize the cost of material.

(a) Draw the feasible region.
(b) Determine the optimum solution by the graphical approach.
(c) Label the active and inactive constraints.

Problem 8

An aggregate mix of sand and gravel must contain no less than 20 percent nor more than 30 percent gravel. The in situ soil contains 40 percent gravel and 60 percent sand. Pure sand may be purchased and shipped to site at \$5.00/yd^3. A total mix of 1000 yd^3 is needed. There is no charge to use in situ material.

The goal is to minimize cost subject to the mix constraints.

(a) Draw the feasible region on the graph.
(b) Determine the optimum solution.
(c) Label the active and inactive constraints.

Problem 9

There are two suppliers of pipe:

SOURCE	UNIT COST (\$/LINEAR FOOT)	SUPPLY (LINEAR FOOT)
1	\$100	100 ft maximum
2	\$125	Unlimited

Nine hundred feet of pipe is required. The goal is to minimize the total cost of pipe.

(a) Find the optimum solution.

(b) Formulate a mathematical model with the supply of pipe from source No. 2 limited to 700 linear feet.

(c) Does a solution to part b exist? Use the graph to prove your answer.

Problem 10

A company requires at least 4.0 Mgal/day more water than it is currently using. A water-supply facility can supply up to 10 Mgal/day of extra supply. A local stream can supply an additional 2 Mgal/day. The concentration of pollution must be less than 100 mg/l BOD, the biological oxygen demand. The water from the water-supply facility and from the stream has a BOD concentration of 50 mg/l and 200 mg/l, respectively. The cost of water from the water supply is \$100/Mgal and from the local stream is \$50/Mgal. The goal is to minimize cost of supplying extra water that meets water quality standards.

By graphical means, show the feasible region, and determine the location of the optimum solution.

Problem 11

A treatment plant has a capacity of 8 Mgal/day and an operating efficiency of 80 percent. The operating efficiency is defined as the amount of BOD_5, the 5-day biological oxygen demand, removed by the treatment plant facility. For example, if 200 mg/l of BOD_5 enters the plant, the amount of BOD_5 leaving it is

$$200(1 - 0.8) = 40 \text{ mg/l of } BOD_5$$

Owing to regional population growth, the rate of flow of wastewater has increased to 20 Mgal/day. The BOD_5 of the wastewater is 200 mg/l. Since the plant can only treat Mgal/day, 12 Mgal/day is being diverted from the plant and is entering the river without treatment (Figure 2.8*a*).

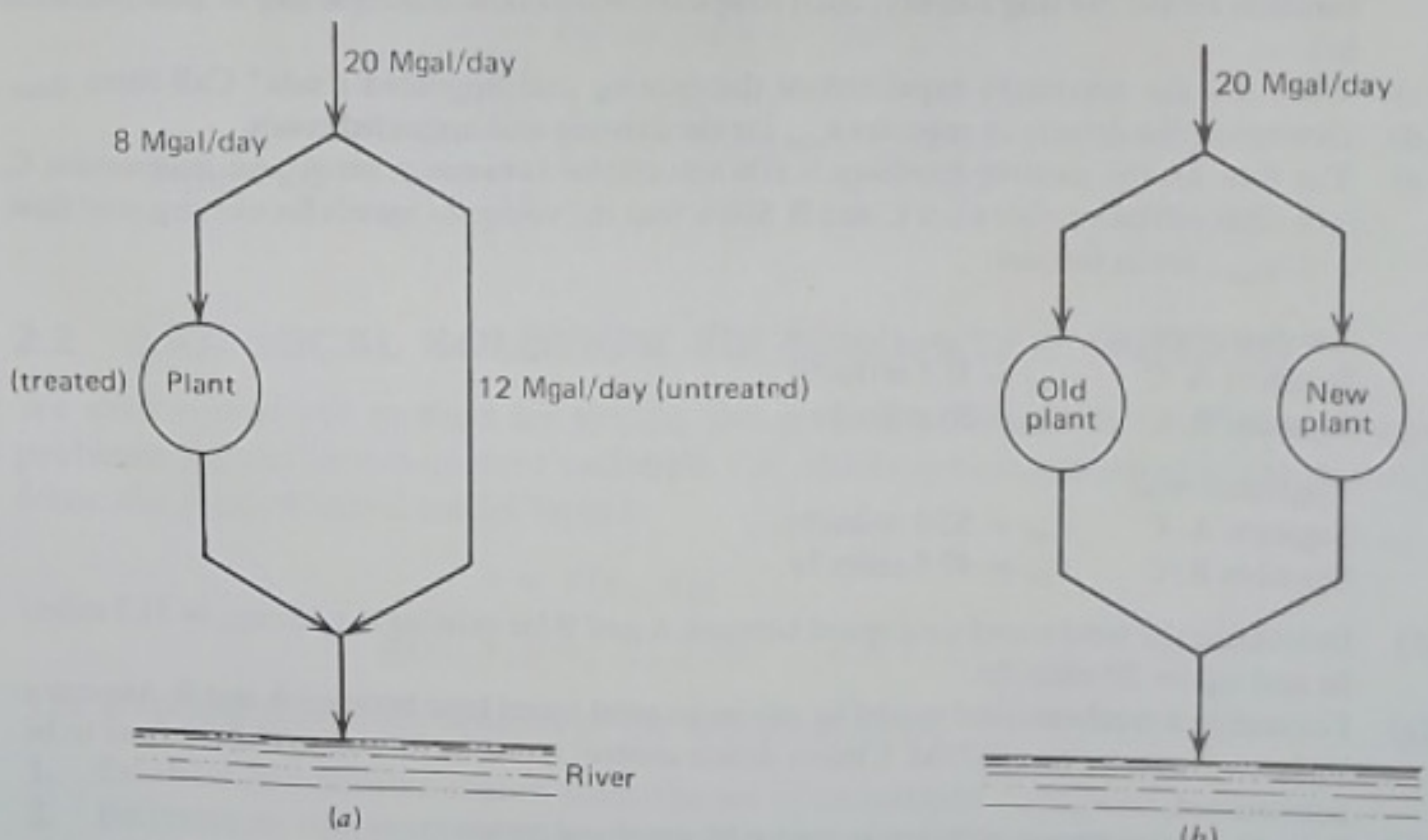

Figure 2.8

(a) Determine the BOD_5 of the combined flow of treated and untreated water entering the river (BOD_5 in mg/l).

(b) The cost of building a new treatment plant (Figure 2.8*b*) is

$$C = \$2.88 \times 10^6 Q \qquad (Q = \text{flow})$$

Define all control variables and formulate a mathematical model to treat 20 Mgal/day of wastewater which has 200 mg/l of BOD_5. The combined flow of treated water entering the river from the old and new plants must not exceed 25 mg/l of BOD_5. The efficiency of the existing plant is 80 percent, and of the new plant is 90 percent.

(c) Determine the minimum-cost solution by graphical means.

Problem 12

A 30-mile stretch of roadway (Figure 2.9*a*) is considered to have a poor level of service. Twenty-five million dollars ($25M) has been allocated for the project. The cost to improve the roadway is $1M/mile; therefore, the entire roadway cannot be improved.

The existing and proposed upgraded roadways are assumed to have a speed–density relationship as shown in Figure 2.9*b*.

(a) Derive speed–density functions for the existing and upgraded roads. The speed u is assumed to be a linear function of density k.

$$k = \text{density (vehicles/mile)}$$

$$u = \text{speed (miles/hr)}$$

(b) Derive a flow-density function for the existing and upgraded roads, $q = ku$, and draw the function for the existing and upgraded roadways, where flow is defined as q = flow (vehicles/hr).

(c) What are the maximum capacities of the existing and upgraded roads? Call them q_{max}.

(d) Determine the density at capacity k_{max} for the existing and upgraded roads.

(e) The flow on the existing roadway is 800 vehicles/hr between point A and intersection C, and 1200 vehicles/hr between C and B. Show that the vehicular speeds for uncongested flow, $q \le q_{max}$, are as follows:

Existing road
Segment A–C $u_{AC} = 31.5$ miles/hr
Segment B–C $u_{BC} = 20$ miles/hr

Upgraded road
Segment A–C $u_{AC} = 52.4$ miles/hr
Segment B–C $u_{BC} = 47.4$ miles/hr

(f) Determine the total travel time speed between A and B for existing road or $u_{AC} = 31.5$ miles/hr and $u_{BC} = 20$ miles/hr.

(g) Formulate a mathematical model to minimize total travel time between A and B. Assume a budgetary constraint of $25M. Clearly define control variables for mileage of roadway to be constructed.

(h) Solve for the optimum solution in part g by graphical means.

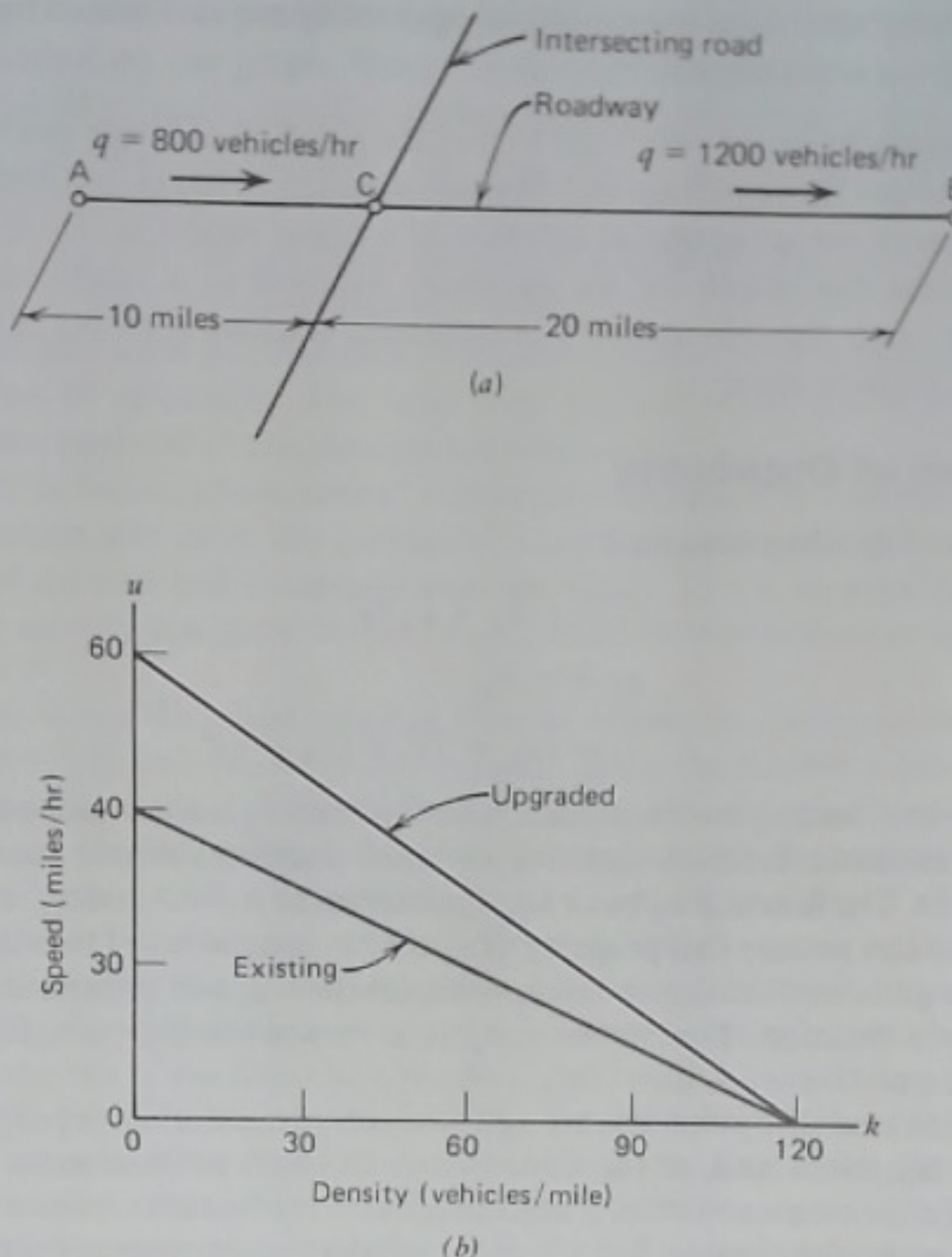

Figure 2.9

2.2 GRAPHICAL SOLUTION TO NONLINEAR MODELS

We shall extend our method for finding the optimum solution to nonlinear design problems limited to two control variables. For optimization problems of the general form, the mathematical model form is

$$z = f(x_1, x_2)$$

$$g_i(x_1, x_2)\{=, \leq, \geq\}bi \qquad i = 1, 2, \ldots, m$$

The graphical procedure consists of the following steps.

1. Establish the feasible region from the set of constraint equations.
2. Estimate an optimum solution, $z = z^0$, and draw the function, $z^0 = f(x_1, x_2)$.