# Flow Nets

- Graphical representation of the steady-state velocity potential and stream function.
  - Used to determine flow velocities, flow paths, and travel times.
- Approach is general and can be applied to a variety of fluid problems including compressible, and incompressible ideal flows.
- In porous media, the velocity potential is related to the head and the stream function is related to the path.

# Velocity Potential

- The velocity potential is given by the head or fluid pressure.

$$\phi = Kh \quad \text{or} \quad \phi = Th$$

- The gradient of the velocity potential function is used to recover the velocity value at a point in the flow field.

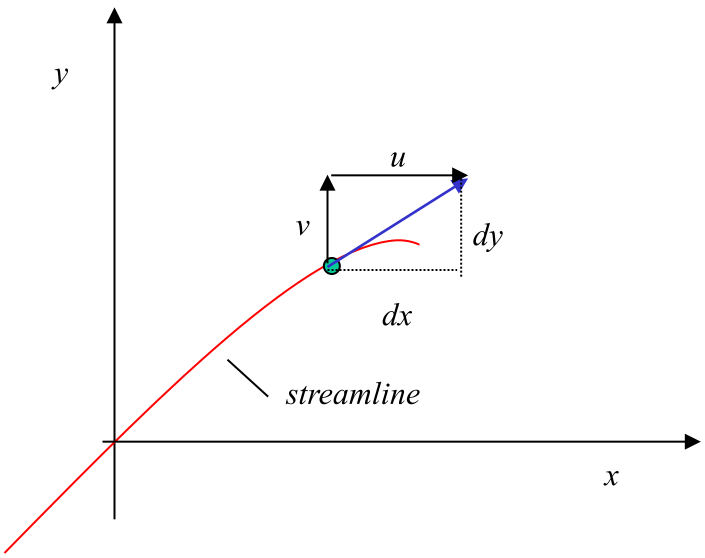
$$\vec{U} = -grad(\phi)$$

- The velocity potential satisfies the governing mass balance equation for steady-incompressible flow.

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 0 \quad \text{or} \quad \nabla^2 \phi = 0$$

# Streamline

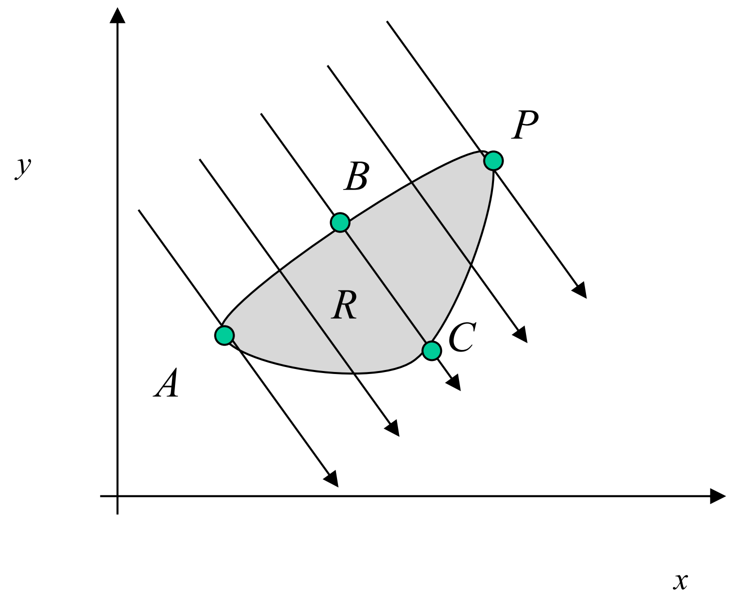
- A streamline is defined as a line that is tangent to the velocity vector in a flow field.




- Tangent means: $$\frac{dy}{dx} = \frac{v}{u} \quad \text{or} \quad u\,dy - v\,dx = 0$$

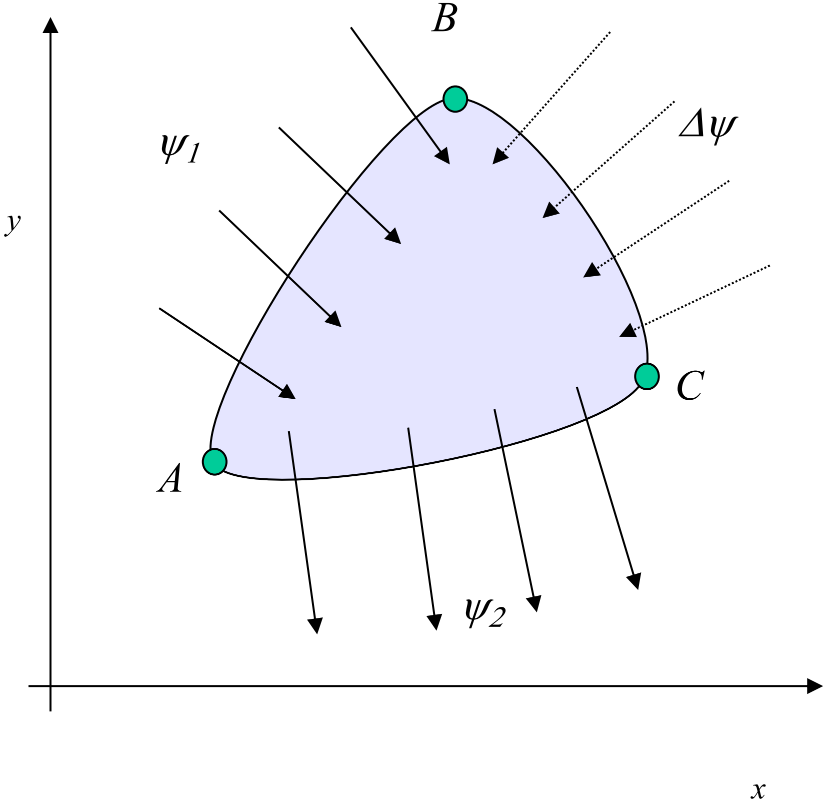
# Stream Function

- Conservation of mass requires that $Q_{ABP}$=$Q_{ACP}$.




- Once *A* is fixed, $Q_R$ depends solely on the location, *P*.
- The volumetric flow through *R* is called the stream function, $\psi = \psi(x, y)$

# Stream Functions and Streamlines




$$Q_{AB} = \psi_1 ; Q_{AC} = \psi_2 ; Q_{BC} = \Delta\psi$$

Let $\psi_A$ be the value of the stream function at A.

Then : $\psi_B = \psi_A + \psi_1 ; \psi_C = \psi_A + \psi_2$

But $\psi_C = \psi_B + \Delta\psi$

$\therefore \Delta\psi = \psi_C - \psi_B = \psi_2 - \psi_1$

If $\Delta\psi = 0$ then $\psi_2 = \psi_1$ and the segment $\overline{\text{BC}}$ is a streamline.

Furthermore, the value of $\psi$ along $\overline{\text{BC}}$ is a constant.

# Potential and Stream Function Relationships

- (1) The velocity is given by the gradient of the velocity potential.

$$u = -\frac{\partial \phi}{\partial x}; v = -\frac{\partial v}{\partial y} \qquad \text{(Darcy's Law)}$$

- (2) Streamlines are tangent to velocity.

$$udy - vdx = -\frac{\partial \phi}{\partial x} dy + \frac{\partial \phi}{\partial y} dx = 0$$

- (3) Lines of constant $\psi$ are streamlines.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

# Flow Net Mathematics

- The last two relations supply the rules to construct a flow net.

$$\frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy = 0 \qquad \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

- Since both equations equal the same constant, then the partial derivatives in each term must be equal.

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \, and \, \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x}$$

# Cauchy-Riemann Conditions

- These equalities are called the Cauchy-Riemann Conditions for Ideal Flow. They are further expanded using Darcy's Law as:

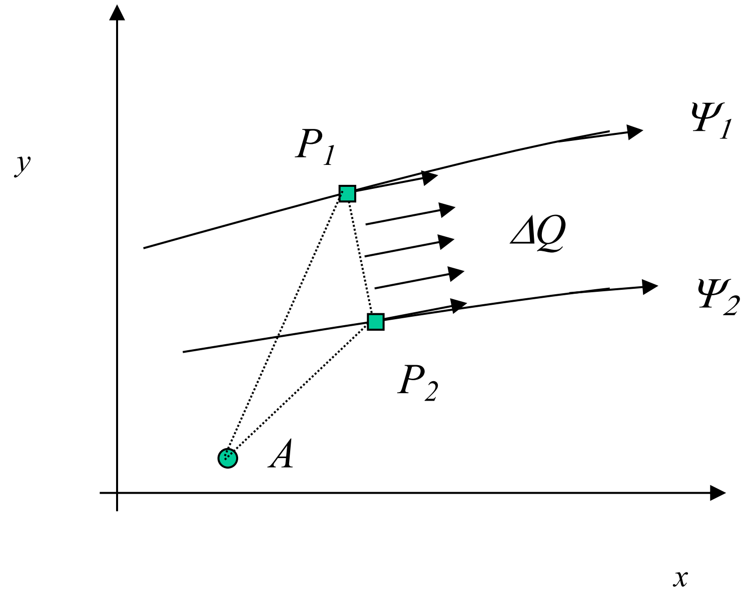
$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = K_y \frac{\partial h}{\partial y} \text{ and } \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x} = -K_x \frac{\partial h}{\partial x}$$

- Or:

$$\frac{\partial h}{\partial y} = \frac{1}{K_y} \frac{\partial \psi}{\partial x} \text{ and } \frac{\partial h}{\partial x} = -\frac{1}{K_x} \frac{\partial \psi}{\partial y}$$

# Streamtubes

- Flow bounded by two streamlines is called a streamtube.




$$\Delta Q = \psi_{P_2} - \psi_{P_1} = \psi_2 + \psi_A - \psi_1 - \psi_A = \psi_2 - \psi_1 = \Delta \psi$$

- Discharge in a streamtube is the <u>***difference***</u> in the in the values of the bounding stream functions.

# Irrotational Flow

- Irrotational flow means that: $\frac{\partial}{\partial x}(\frac{\partial h}{\partial y}) - \frac{\partial}{\partial y}(\frac{\partial h}{\partial x}) = 0$

- Substitute Cauchy-Reimann conditions to obtain

$$\frac{\partial}{\partial x}(\frac{1}{K_y}\frac{\partial \psi}{\partial x}) - \frac{\partial}{\partial y}(-\frac{1}{K_x}\frac{\partial \psi}{\partial y}) = 0$$

- Or, in compact notation:

$$div(\frac{1}{\mathbf{K}} grad(\psi)) = \nabla \bullet (\frac{1}{\mathbf{K}} \nabla \psi) = 0$$

# Results

- Compare to the steady groundwater flow equation.

$$div(\mathbf{K}_{ij}\,grad(\phi)) = \nabla \bullet (\mathbf{K}_{ij} \nabla \phi) = 0$$

$$div(\frac{1}{\mathbf{K}_{ji}}\,grad(\psi)) = \nabla \bullet (\frac{1}{\mathbf{K}_{ji}} \nabla \psi) = 0$$

- These two PDEs are the basis of numerical generation of flow nets.

# Application

- Numerical generation of flow nets is accomplished by
  - Generating discrete distributions of potential and stream functions over the entire problem domain
  - Contouring the results to create a picture of the flow net.
- Practical aspects:
  - Both governing PDEs are LaPlace equations. Thus a tool that solves LaPlace problems will suffice for both equations (although boundary conditions will be different)