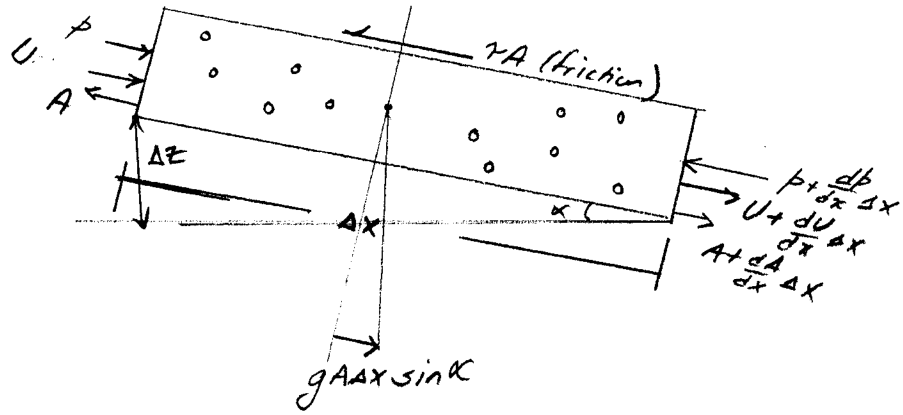
Porous Medium Flow



Conservation of Mass

$$0 = \frac{d}{dt}\int_{CV} \varphi\, d\forall + \int_{CS} \varphi \underline{V}\cdot d\underline{A} = A\Delta x \frac{d\varphi}{dt} - \varphi(UA) + \varphi\left(U + \frac{dU}{dx}\Delta x\right)\left(A + \frac{dA}{dx}\Delta x\right)$$

$$UA + \frac{AdU}{dx}\Delta x + \frac{UdA}{dx}\Delta x + \cancel{\frac{dU}{dx}\frac{dA}{dx}\Delta}$$

neglect

$$0 = A\Delta x \frac{d\varphi}{dt} - \cancel{\varphi UA} + \cancel{\varphi UA} + \varphi \frac{dUA}{dx}\Delta x$$

$$\therefore \quad \frac{d\varphi}{dt} = \frac{1}{A}\frac{dUA}{dx}$$

if $\frac{dA}{dx} = 0$ (constant cross section)

then $\frac{d\varphi}{dt} = -\frac{\partial U}{dx}$

Conservation of Momentum

$$\Sigma F = \frac{d}{dt}\int_{CV} \varphi \underline{V}\, d\forall + \int_{CS} \varphi \underline{V}(\underline{V}\cdot d\underline{A})$$

$$= \varphi\Delta x \frac{\partial UA}{\partial t} - \varphi U^2 A + \underbrace{\varphi\left(U + \frac{dU}{dx}\Delta x\right)\left(UA + \frac{dUA}{dx}\Delta x\right)}_{U^2A + UA\frac{dU}{dx}\Delta x + U\frac{dUA}{dx}\Delta x + \cancel{\frac{dU}{dx}\frac{dUA}{dx}}\Delta x}$$

Neglect

$$= \varphi\Delta x \frac{\partial UA}{dt} - \cancel{\varphi U^2 A} + \cancel{\varphi U^2 A} + \underbrace{\varphi UA\frac{dU}{dx}\Delta x + \varphi U\frac{dUA}{dx}\Delta x}_{}$$

$$\frac{dU^2A}{dx}\Delta x = \frac{dUUA}{dx} = U\frac{dUA}{dx} + UA\frac{dU}{dx}$$

$$\Sigma F = \varphi\Delta x \frac{\partial UA}{\partial t} + \rho\frac{\partial U^2 A}{\partial x}\Delta x$$

Forces

pressure : $pA - \underbrace{(p + \frac{dp}{dx}\Delta x)(A + \frac{dA}{dx}\Delta x)}$

$$pA + A\frac{dp}{dx}\Delta x + p\frac{dA}{dx}\Delta x + \cancel{\frac{dp}{dx}\frac{dA}{dx}\Delta x^2}$$ Neglect

: $- \frac{dpA}{dx}\Delta x$

gravity : $\rho g A \Delta x \sin\alpha = \rho g A \Delta z$ ($\sin\alpha \approx \tan\alpha \approx \alpha$ for small

friction : $- \tau A_f$

In porous flow, the flow is almost always laminar — regardless of pore geometry the friction term will be of the form

$cA\tau\Delta x$ where c incorporates geometry and characteristic length

$\therefore$ The momentum balance is

$$- \frac{dpA}{dx}\Delta x - cA\tau\Delta x - \rho g A\Delta z = \rho\Delta x \frac{\partial uA}{\partial t} + \rho\frac{\partial u^2 A}{\partial x}\Delta x$$

For constant cross sectional area & divide by $\rho g A$

$$- \frac{\partial p}{\partial x}\frac{\Delta x}{\rho g} - \frac{c\tau\Delta x}{\rho g} - \Delta z = \frac{1}{g}\frac{\partial u}{\partial t} + \frac{1}{g}\left(\frac{\partial u^2}{\partial x}\Delta x\right)$$

laminar flow occurs at low Re, $\therefore$ $u^2 \ll u$, neglect inertial term

steady flow $\frac{\partial u}{\partial t} = 0$

$$- \frac{\partial p}{\partial x}\frac{\Delta x}{\rho g} - \Delta z = \frac{c\tau\Delta x}{\rho g}$$

$\tau = \mu\frac{du}{dr}$ (in a pipe) in laminar flow $\tau = \mu\bar{U}c_2$

$$-\frac{\partial p}{\partial x}\frac{\Delta x}{\rho g} - \Delta z = \frac{c \nu \bar{U} c_2 \Delta x}{\rho g}$$

Now divide by Δx

$$-\frac{\partial p}{\partial x}\cdot\frac{1}{\rho g} - \frac{\Delta z}{\Delta x} = \frac{\nu \bar{U} c c_2}{\rho g}$$

$\lim \Delta x \to 0$

$$-\frac{\partial p}{\partial x}\left(\frac{1}{\rho g}\right) - \frac{\partial z}{\partial x} = \frac{c c_2 \nu \bar{U}}{\rho g}$$

Solve for $\bar{U}$

$$\bar{U} = -\frac{\rho g}{c c_2 \nu}\frac{\partial}{\partial x}\left(\frac{p}{\rho g} + z\right)$$

Now let $k = \frac{1}{c c_2}$ and we have $\bar{U} = -\frac{k \rho g}{\nu}\frac{\partial}{\partial x}\left(\frac{p}{\rho g} + z\right)$

k is called the intrinsic permeability

$\frac{k \rho g}{\nu}$ is called the hydraulic conductivity

$\frac{p}{\rho g} + z$ is called the head (static head)

U is called the seepage velocity or specific discharge

$U = -K \frac{\partial h}{\partial x}$ is called Darcy's Law.

In porous media flow, Darcy's Law is almost always used to determine velocity, regardless of whether flow is steady or not. The usual justification is that the inertial terms dissipate rapidly and flow is dominated by pressure, gravity, and friction.