Review of Fluid Mechanics Principles Useful in Environmental Flow & Transport Models

Properties

Physical condition of a fluid described by its properties

extensive & intensive properties

extensive properties relate to a system, a defined quantity of mass

intensive properties relate to components of a system.

W - weight is an extensive property of a system

γ - specific weight (weight per unit volume) is an intensive property

Properties involving mass

ρ - mass density; mass per unit volume

γ - specific weight; weight per unit volume; $\gamma = \rho g$

In many fluids ρ is strongly dependent on applied normal stress (pressure)

$\rho = \rho(p)$

These fluids are called "compressible" (gasses, atmosphere).

When the bulk compressibility is small

$\Rightarrow \frac{d\rho}{dp} \approx \text{small} \; (10^{-6} g/atm)$ the fluid is called "incompressible" (water, hydraulic oil)

S.G. - specific gravity; ratio of the specific weight of a liquid to the specific weight of water. (eg. S.G. = 13.6 for elemental Hg)

Properties involving heat (energy)

c - specific heat; amount of heat that must be transferred to a unit mass of material to raise its temperature by one degree ($\frac{1°C}{1g}$ = calorie ; $\frac{1°F}{1b}$ = Btu)

u - specific internal energy; energy a substance possesses because of molecular activity

h – specific enthalpy; $h = u + \frac{p}{\rho}$; energy in a substance because of molecular activity and applied pressure.

Gas properties

Equation of state $\quad p\forall = \frac{m}{M} RT \qquad \left(\frac{m}{M} = n \leftarrow \#\text{moles}\right)$ (m – mass of gass; M – MW of gas)

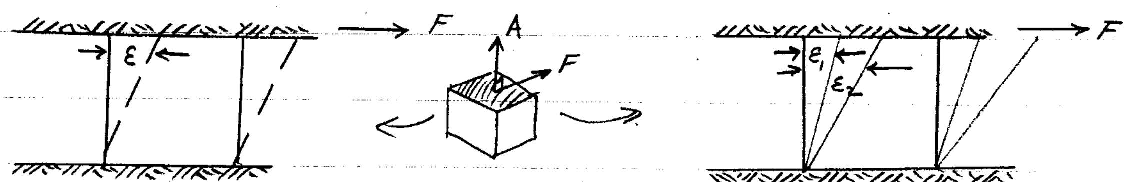
c_v – constant volume specific heat.

c_p – constant pressure specific heat.

Viscosity

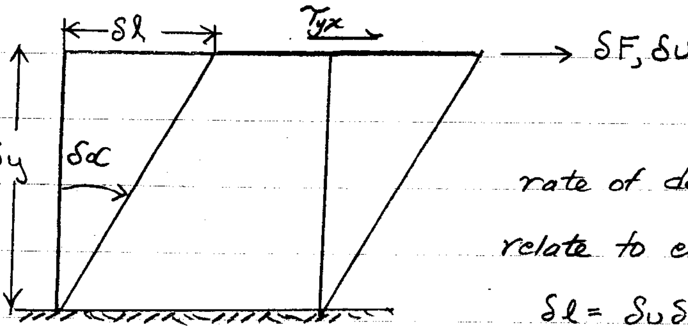
A fluid is a substance that deforms continuously under application of shear stress

solid $\qquad\qquad\qquad\qquad$ fluid



solid	fluid
$\varepsilon \propto \frac{F}{A} = \tau$	$\varepsilon \propto F/A = \tau$
$\frac{d\varepsilon}{dt} = 0$	$\frac{d\varepsilon}{dt} \neq 0 \qquad \varepsilon_1 = \varepsilon(t_1),\ \varepsilon_2 = \varepsilon(t_2),\ \varepsilon_1 \neq \varepsilon_2$
$\varepsilon = f(\tau)$	$\varepsilon = f(\tau, t)$

The rate of deformation is used to define viscosity



$$\tau_{yx} = \lim_{\delta A \to 0} \frac{\delta F}{\delta A} = \frac{dF}{dA}$$

rate of deformation: $\lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} = \frac{d\alpha}{dt}$

relate to element geometry

$\delta l = \delta u\,\delta t \quad$ (displacement = velocity * time)

$\delta l = \delta y\,\delta\alpha \quad \left(\tan(\delta\alpha) = \frac{\delta l}{\delta y};\ \tan(\alpha) = \alpha \text{ for small } \alpha\right)$

$\therefore \frac{\delta\alpha}{\delta t} = \frac{\delta u}{\delta y}$; in the limit: $\frac{d\alpha}{dt} = \frac{du}{dy}$

If the rate of deformation is proportional to the stress the fluid is called a Newtonian fluid, and the constant of proportionality is called the absolute viscosity, μ.

$$\tau_{yx} = \mu \frac{du}{dy}$$

If the rate of deformation is not linearily propochonal, the fluid is called a non-Newtonian fluid, one model used is a power-law model

$$\tau_{yx} = \underbrace{k\left(\frac{du}{dy}\right)^{n-1}}_{\text{this group is called the apparent viscosity}} \frac{du}{dy}$$

Elasticity or bulk compressibility is the volume change of a fluid element for a given change in applied pressure

$$E_V = -\frac{dp}{dV} V \qquad \text{or} \qquad E_V = -\frac{dp}{d\rho} \rho$$

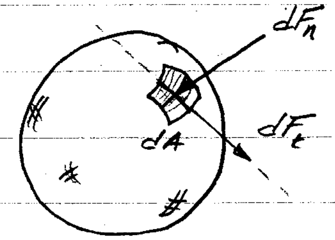
Surface tension, σ, is the work per unit area required to separate two fluids. (Dimensions of σ are force per unit length) Surface tension is one reason why water can rise up into capillary tubes or porous materials

Fluid Statics

Body forces are developed without contact and are distributed over the entire volume of a fluid. Weight is a body force.
Surface forces act at boundaries of a medium through contact

Stress

Consider the surface of a bubble, with some small area defined on the surface:

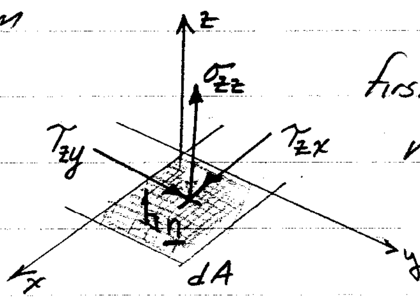


Stress is the limiting value of $\frac{dF}{dA}$.

Two kinds of stress are defined:
normal (pressure)
shear (tangential to surface)

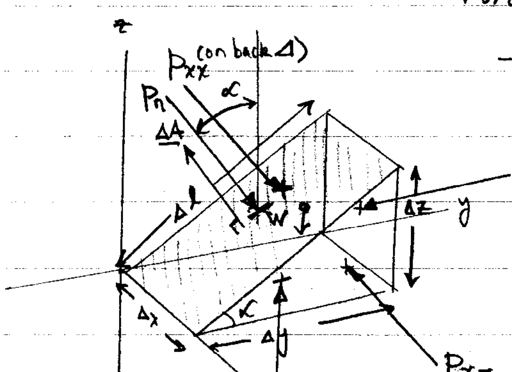
The typical notation is
$$\sigma = \lim_{dA\to 0} \frac{dF_n}{dA} \quad \text{(normal)}$$
$$\tau = \lim_{dA\to 0} \frac{dF_t}{dA} \quad \text{(shear)}$$

When applied to an area in 3-dimensions there will be one normal stress and two shear stress terms in an orthogonal coordinate system



first subscript is direction of normal vector wrt. the plane ($\underline{n}$)
second subscript is direction of stress application.

Fluid pressure is the normal stress applied by/to a fluid element. Because pressure is a normal force per unit area at any <u>point</u> in a fluid, it is treated as a scalar quantity:



force balances: (y-direction)

$-p_{yy}\Delta x\Delta z + p_n \Delta l \sin\alpha \Delta x = 0$; $\Delta z = \Delta l \sin\alpha$

$\therefore p_{yy} = p_n$

(x-direction)

$\frac{1}{2} p_{xx}\Delta y \Delta z - \frac{1}{2} p_{xx}\Delta y\Delta z = 0 \quad \therefore p_{xx} = p_{xx}$

(z-direction)

$p_{zz}\Delta x\Delta y - p_n \Delta x \underbrace{\Delta l \cos\alpha}_{=\Delta y} - \frac{1}{2}\rho g \Delta y \Delta z \Delta x = 0$

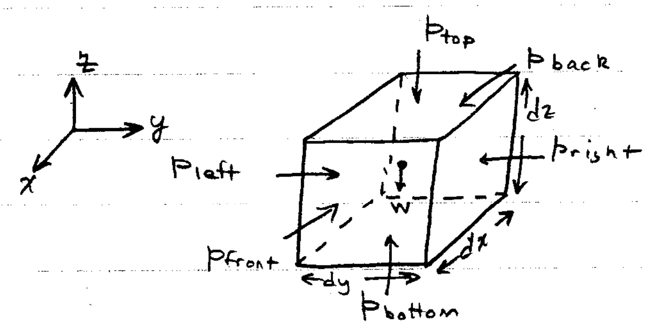
$\therefore p_{zz} - p_n - \frac{1}{2}\rho g \Delta z = 0$; $\lim_{\Delta z \to 0} \Rightarrow p_{zz} = p_n$

Thus $p_n = p_{yy} = p_{zz}$ and because orientation is arbitrary $p_n = p$.

From this analysis one can conclude that pressure in a static fluid at a point is a single value, independent of direction — pressure is a scalar.

Fluid statics means that a fluid is free of relative motion – the entire fluid behaves as a "rigid body". The absence of angular deformation implies an absence of shear stress – static fluids can only sustain normal stress. Often fluid statics principles are sucessfully applied to study the behavior of moving fluids.

Fluid statics means $\frac{du}{dy} = 0$, it does not require $\frac{du}{dt} = 0$, so static balances can be used to study certain kinds of motion.



Force balance on a fluid element

$$\Sigma dF = d\underline{F}_{body} + d\underline{F}_{surf} = dm\,\underline{a} \qquad (F = ma)$$

$$dm = \rho\, dx\, dy\, dz$$

$$d\underline{F}_{body} = \rho \underline{g}\, dx\,dy\,dz \qquad d\underline{F}_{surf} = (p_{back} - p_{front})\,dz\,dy\,\underline{i}$$
$$(p_{left} - p_{right})\,dx\,dz\,\underline{j}$$
$$(p_{bottom} - p_{top})\,dx\,dy\,\underline{k}$$

Taylor series expansion about centroid is used to express the spatial pressure variation

$p_{back} = p_0 - \frac{\partial p}{\partial x}\frac{dx}{2}$ $p_{front} = p_0 + \frac{\partial p}{\partial x}\frac{dx}{2}$

$p_{left} = p_0 - \frac{\partial p}{\partial y}\frac{dy}{2}$ $p_{right} = p_0 + \frac{\partial p}{\partial y}\frac{dy}{2}$

$p_{bottom} = p_0 - \frac{\partial p}{\partial z}\frac{dz}{2}$ $p_{top} = p_0 + \frac{\partial p}{\partial z}\frac{dz}{2}$

Substitute into force balance expression & simplify

$$\rho g\,dx\,dy\,dz - \underbrace{\left(\frac{\partial p}{\partial x}\underline{i} + \frac{\partial p}{\partial y}\underline{j} + \frac{\partial p}{\partial z}\underline{k}\right)}_{\text{gradient of pressure}}dx\,dy\,dz = \rho\,\underline{a}\,dx\,dy\,dz$$

$\therefore$ $\rho g - \nabla p = \rho \underline{a}$ (Euler's equation of motion for a fluid – applies to flows where only forces are pressure and gravity; non-viscous flow

When the fluid is not accelerating $\underline{a} = 0$ and a special case is $\underbrace{\rho g}_{\text{body force/volume}} - \underbrace{\nabla p}_{\text{pressure force/volume}} = 0$. This case is called "hydrostatic"

Written as component equations

$$\rho g_x - \frac{\partial p}{\partial x} = 0\ ;\ \rho g_y - \frac{\partial p}{\partial y} = 0\ ;\ \rho g_z - \frac{\partial p}{\partial z} = 0.$$

Often g is aligned with the $-z$ axis so we obtain

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \quad \text{and} \quad \underbrace{\frac{\partial p}{\partial z} = -\rho g = -\gamma}_{\text{typical form of hydrostatic fluid}}$$

The typ. form displays the fundamental relationship between pressure variation and depth in a static fluid. In practice, this relationship is often used with moving fluids as a first approximation.

<u>Velocity field</u> (moving fluids)

Lagrangian approach – choose an individual fluid particle (parcel)

Eulerian approach – choose a particular location in space

Fluid particle kinematics (Lagrangian approach) position $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$

Velocity $\underline{V}$ or $\underline{U}(t) = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k} = U(t)\underline{i} + V(t)\underline{j} + Z(t)\underline{k}$

$u(t) = \frac{dx}{dt}$ at current position of particle

$v(t) = \frac{dy}{dt}$ at current position of particle

$w(t) = \frac{dz}{dt}$ at current position of particle

acceleration $\underline{a}(t) = \frac{du}{dt}\underline{i} + \frac{dw}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$

reference is <u>always</u> <u>current</u> <u>particle</u> <u>position</u>

Fluid Element kinematics (Eulerian approach)

position: $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ (some fixed point in space)

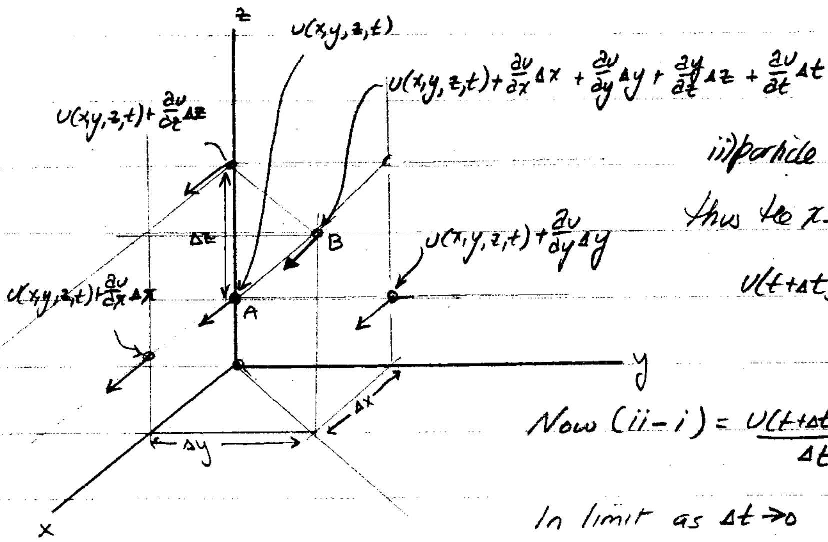
Velocity: $\underline{U}(x,y,z,t) = \frac{dx}{dt}\Big|_{x,y,z,t}\underline{i} + \frac{dy}{dt}\Big|_{x,y,z,t}\underline{j} + \frac{dz}{dt}\Big|_{x,y,z,t}\underline{k} = U(x,y,z,t)\underline{i} + v(x,y,z,t)\underline{j} + w(x,y,z,t)\underline{k}$

at current position in space.

acceleration:
$$\underline{a}(x,y,z,t) = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\underline{i} + \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\underline{j} + \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\underline{k}$$

Consider just x-direction $a_x = \underbrace{\frac{\partial u}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{convective acceleration}}$

$\therefore$ It is possible to have convective acceleration in steady $\left(\frac{\partial(\;)}{\partial t} = 0\right)$ flow!



i) (x-direction) Velocity at (A) at time t is $U(t) = U(x,y,z,t)$

ii) particle moves from (A) to (B) over an interval of Δt, thus the x-comp. of velocity is

$$U(t+\Delta t) = \underbrace{U(x,y,z,t)}_{U(t)} + \underbrace{\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z}_{\text{translation in space } \Delta x, \Delta y, \Delta z \text{ units}} + \underbrace{\frac{\partial u}{\partial t}\Delta t}_{\text{translation in time } \Delta t \text{ units}}$$

Now (ii − i) $= \frac{U(t+\Delta t) - U(t)}{\Delta t} = \frac{\partial u}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial u}{\partial z}\frac{\Delta z}{\Delta t} + \frac{\partial u}{\partial t}\frac{\Delta t}{\Delta t}$

In limit as $\Delta t \to 0$ $\frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$

Eulerian acceleration expressions are important in contaminant transport models that use particle tracking principles. They are also important in 3-d unsteady flows where viscosity and turbulence is important.

Flow patterns

Timeline is a line formed by marking adjacent particles at same instant.

pathline is the trajectory of a particular fluid particle

streakline is the trajectory of many particles that all pass through a common point.

* Streamline is a line in a flow field that is tangent to the velocity field everywhere. No flow occurs across a streamline.

uniform flow is a flow field where velocity does not change along a streamline

nonuniform flow is where velocity does vary with position

Steady flow is a flow field where the velocity at a point is constant (in time)

unsteady flow is a flow field where velocity at a point varies in time.

Flow dimensions are classified by how many spatial coordinates are required to specify the velocity field. All real flows are 3-dimensional but many useful engineering analyses are possible using 2D and 1D approximations. Most real flows are unsteady but many useful engineering analyses are possible using steady flow approximations

Control volume analysis

Control volume is a fundamental tool in flow mechanics - it is the equivalent of the free-body diagram in particle/rigid body mechanics. The cell balance method is an applied control volume analysis.

The control volume is the basis of the Reynold's transport theorem that allows analysis from a Eulerian perspective rather than tracking individual particles. The general idea is to express fundamental processes (physics, chemistry) in an integral form.

* 1) conservation of mass $\left.\frac{dm}{dt}\right|_{system} = 0$

* 2) conservation of linear momentum $\left. m\frac{d\underline{U}}{dt}\right|_{system} = \Sigma \underline{F}$

3) conservation of angular momentum $\left. m\frac{d\omega}{dt}\right|_{system} = \Sigma \underline{r} \times \underline{F}$

* 4) Conservation of energy $\frac{dQ}{dt} - \frac{dW}{dt} = \left.\frac{dE}{dt}\right|_{system}$ Q - heat into system, W - work done by system

5) Entropy $\left.\frac{dS}{dt}\right|_{system} \geq \frac{1}{T}\frac{dQ}{dt}$ T - abs. temperature, S - entropy

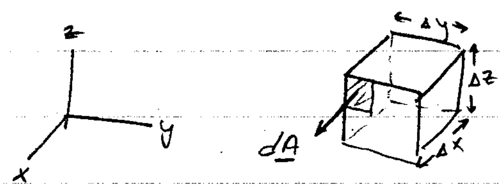
The most useful processes in environmental flow and transport modeling are conservation of mass, (linear) momentum, and energy

Extensive quantity - a property that extends throughout a given <u>mass</u> of fluid

Intensive quantity - amount of extensive quantity per unit mass

Extensive		Intensive
$m = \int \frac{m}{m}\, dm$		$= \int \rho\, d\forall$
$B = \int \beta\, dm$	$\therefore$ for mass $\beta = 1$	$= \int \beta \rho\, d\forall$
$m\underline{V} = \int \frac{m}{m} \bar{V}\, dm$	$\therefore$ momentum $\beta = \underline{V}$	$= \int \underline{V} \rho\, d\forall$
$\underline{r} \times m\underline{V} = \int \frac{\underline{r} \times m\underline{V}}{m}\, dm$	$\therefore$ angular momentum $\beta = \underline{r} \times \underline{V}$	$= \int \underline{r} \times \underline{V}\, \rho\, d\forall$
$me = \int \frac{me}{m}\, dm$	$\therefore$ energy $\beta = e$ (sp. energy)	$= \int e \rho\, d\forall$
$ms = \int \frac{ms}{m}\, dm$	$\therefore$ entropy $\beta = s$ (sp. entropy)	$= \int s \rho\, d\forall$

Control volume is a defined volume in space



the bounding surface is called the control surface.

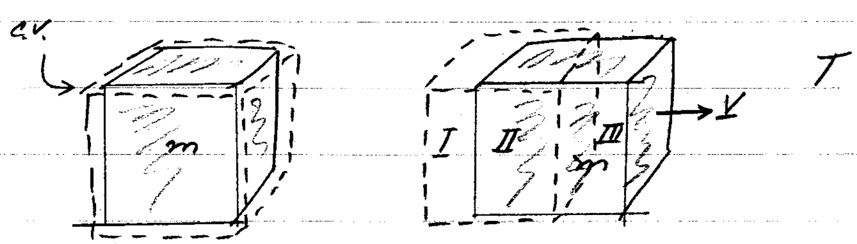
$d\underline{A}$ is the outward pointing area vector

$d\underline{A}_{back} = -\Delta y \Delta z\, \underline{i}$ $\qquad$ $d\underline{A}_{front} = \Delta y \Delta z\, \underline{i}$

$d\underline{A}_{left} = -\Delta x \Delta z\, \underline{j}$ $\qquad$ $d\underline{A}_{Right} = \Delta x \Delta z\, \underline{j}$

$d\underline{A}_{bottom} = -\Delta x \Delta y\, \underline{k}$ $\qquad$ $d\underline{A}_{top} = \Delta x \Delta y\, \underline{k}$

Fundamental relationship between extensive and intensive properties is $B_{system} = \int \beta\, dm = \int \beta \rho\, d\forall$. The system equations are written as $\left.\frac{dB}{dt}\right|_{system} = rhs$. Reynolds transport theorem is used to change rhs into volume based terms.



time = t $\qquad$ time = t + Δt

c.v. is fixed in space. Mass m moves in a Eulerian velocity field $\underline{V} = \underline{V}(x, y, z, t)$.

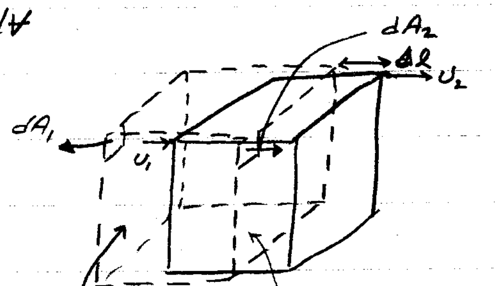
$$\left.\frac{dB}{dt}\right|_{sys.} = \lim_{\Delta t \to 0} \frac{B_{t+\Delta t} - B_t}{\Delta t} \; ; \quad B_t = B_{cv_t}$$

$$B_{t+\Delta t} = (B_{II} + B_{III})_{t+\Delta t} = (B_{cv} - B_I + B_{III})_{t+\Delta t}$$

In terms of intensive properties

$$\frac{dB}{dt} = \lim_{\Delta t \to 0} \frac{\int_{cv} \beta\varphi d\forall|_{t+\Delta t} - \int_I \beta\varphi d\forall|_{t+\Delta t} + \int_{III} \beta\varphi d\forall|_{t+\Delta t} - \int_{cv} \beta\varphi d\forall|_t}{\Delta t}$$

$$= \underbrace{\lim_{\Delta t \to 0} \frac{\int_{cv} \beta\varphi d\forall|_{t+\Delta t} - \int_{cv} \beta\varphi d\forall|_t}{\Delta t}}_{\frac{d}{dt}\int_{cv} \beta\varphi d\forall} + \lim_{\Delta t \to 0} \frac{\int_{III} \beta\varphi d\forall|_{t+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{\int_I \beta\varphi d\forall|_{t+\Delta t}}{\Delta t}$$



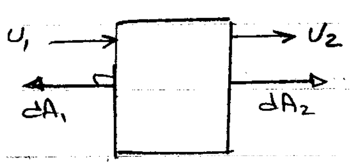
$$\underbrace{-\int_I \beta\varphi d\forall|_{t+\Delta t} = -\int_{cs} \beta\varphi \Delta l \, dA_1}_{\text{Gauss Divergence Theorem}} \qquad \underbrace{\int_{III} \beta\varphi d\forall|_{t+\Delta t} = \int_{cs} \beta\varphi \Delta l \, dA_2}_{\text{Gauss Divergence Theorem}}$$

$\therefore$

$$\lim_{\Delta t \to 0} \frac{\int_{III} \beta\varphi d\forall|_{t+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \int_{C.S.} \beta\varphi \frac{\Delta l}{\Delta t} dA_2 = \int_{C.S.} \beta\varphi U dA_2$$

likewise for the left face we have $-\int_{C.S.} \beta\varphi U_1 dA_1$

Now study relationship between dA & $\underline{V}$



$$-U_1 dA_1 = \underline{V} \cdot d\underline{A}|_{①}$$

$$U_2 dA_2 = \underbrace{\underline{V} \cdot d\underline{A}|_{②}}_{\text{Vector calculus}}$$

U_1 & dA_1 are in opposite direc

U_2 & dA_2 are in same direct

U_1 & U_2 are in same directio

inner product preserves correct sign relation

$$\therefore \left.\frac{dB}{dt}\right|_{sys.} = \frac{d}{dt}\int_{cv} \beta\varphi d\forall + \int_{C.S.} \beta\varphi (\underline{V} \cdot d\underline{A})$$

Result is called "Reynolds' Transport Theo

Apply to fundamental conservation laws

Mass

$$0 = \frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{C.S.} \rho(\underline{V}\cdot d\underline{A})$$

Momentum

$$\Sigma \underline{F} = \frac{d}{dt}\int_{CV} \rho \underline{V} \, d\forall + \int_{CS} \rho \underline{V}(\underline{V}\cdot d\underline{A})$$

Angular Momentum

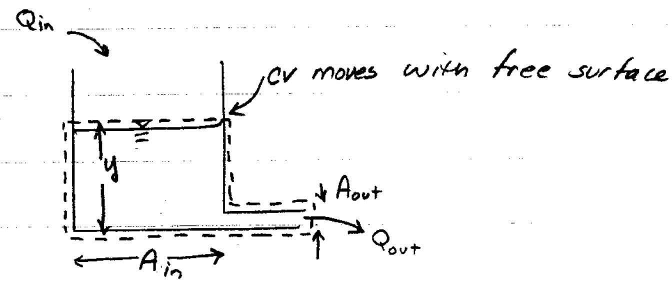
$$\Sigma \underline{r}\times\underline{F} = \frac{d}{dt}\int_{CV} \rho(\underline{r}\times\underline{V})\,d\forall + \int_{CS}\rho(\underline{r}\times\underline{V})(\underline{V}\cdot d\underline{A})$$

Energy

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{d}{dt}\int_{CV} \rho e \, d\forall + \int_{CS} \rho e(\underline{V}\cdot d\underline{A})$$

Alternative View

Consider a reactor vessel



$$0 = \frac{d}{dt}\int_{CV} \rho \, d\forall + \int_{C.S.} \rho(\underline{V}\cdot d\underline{A})$$

$$\frac{d}{dt}\int_{CV} \rho \, d\forall = \rho A_{IN}\frac{dy}{dt}$$

$$\int_{C.S.} \rho(\underline{V}\cdot d\underline{A}) = -\rho \underbrace{\left(\frac{Q_{IN}}{A_{IN}}\right)}_{V_{IN}}(A_{IN}) + \rho\underbrace{\left(\frac{Q_{OUT}}{A_{OUT}}\right)}_{V_{OUT}}(A_{OUT})$$

$$\therefore\ 0 = \rho A_{IN}\frac{dy}{dt} + \rho Q_{OUT} - \rho Q_{IN} \longrightarrow \underbrace{\rho Q_{IN}}_{\text{rate of mass into vessel}} - \underbrace{\rho Q_{OUT}}_{\text{rate of mass out of vessel}} = \overbrace{\rho A_{IN}\frac{dy}{dt}}^{\text{rate of mass stored in vessel}}$$

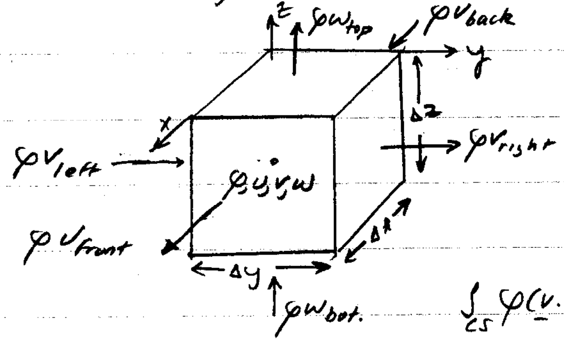
$\therefore$ The classic environmental engineering approach to mass balance is simply a restatement of the principles of Reynolds' transport theorem.

We can treat momentum, energy etc. the same way

forces on element + rate of momentum in − rate of momentum out = rate of momentum sto

rate of heat in − rate of work out + rate of energy in − rate of energy out = rate of energy accumulated

Continuity at a point (Critical for all types of models involving flow)



Velocity field $\underline{V} = u\underline{i} + v\underline{j} + w\underline{k}$

Conservation of mass

$$0 = \frac{d}{dt}\int_{cv} \rho \, d\forall + \int_{cs} \rho(\underline{v}\cdot d\underline{A})$$

$$\int_{cs} \rho(\underline{v}\cdot d\underline{A}) = \rho u_{back}\,\underline{i}\cdot(-\Delta y \Delta z\,\underline{i}) + \rho u_{front}\,\underline{i}\cdot(\Delta y \Delta z\,\underline{i})$$
$$+ \rho v_{left}\,\underline{j}\cdot(-\Delta x \Delta z\,\underline{j}) + \rho v_{right}\,\underline{j}\cdot(\Delta x \Delta z\,\underline{j})$$
$$+ \rho w_{bot}\,\underline{k}\cdot(-\Delta x \Delta y\,\underline{k}) + \rho w_{top}\,\underline{k}\cdot(\Delta x \Delta y\,\underline{k})$$

Using a Taylor series expansion about x, y, z for the ρu, ρv & ρw terms

$$\left.\begin{aligned} \rho u_{back} &= \rho u - \frac{\partial \rho u}{\partial x}\frac{\Delta x}{2}, & \rho u_{front} &= \rho u + \frac{\partial \rho u}{\partial x}\frac{\Delta x}{2} \\ \rho v_{left} &= \rho v - \frac{\partial \rho v}{\partial y}\frac{\Delta y}{2}, & \rho v_{right} &= \rho v + \frac{\partial \rho v}{\partial y}\frac{\Delta y}{2} \\ \rho w_{bot} &= \rho w - \frac{\partial \rho w}{\partial z}\frac{\Delta z}{2}, & \rho w_{top} &= \rho w + \frac{\partial \rho w}{\partial z}\frac{\Delta z}{2} \end{aligned}\right\} \text{substitute into above integral}$$

$$\int_{cs} \rho(\underline{v}\cdot d\underline{A}) = \frac{\partial \rho u}{\partial x}\Delta x \Delta y \Delta z + \frac{\partial \rho v}{\partial y}\Delta x \Delta y \Delta z + \frac{\partial \rho w}{\partial z}\Delta x \Delta y \Delta z \,.$$

The volume integral is

$$\frac{d}{dt}\int_{cv} \rho \, d\forall = \int_{cv} \frac{d\rho}{dt} d\forall + \int_{cs} \rho(\underset{\uparrow}{\underline{V}_s}\cdot d\underline{A})$$

Velocity of c.s. relative to Eulerian reference frame (in this case $\underline{V}_s = \underline{0}$)

$\therefore \frac{d}{dt}\int_{cv} \rho \, d\forall = \int_{cv} \frac{d\rho}{dt} d\forall$. If volume is not deforming then: $= \frac{d\rho}{dt}\Delta x \Delta y \Delta z$

Substitute the volume & flux integrals into conservation of mass equation to obtain

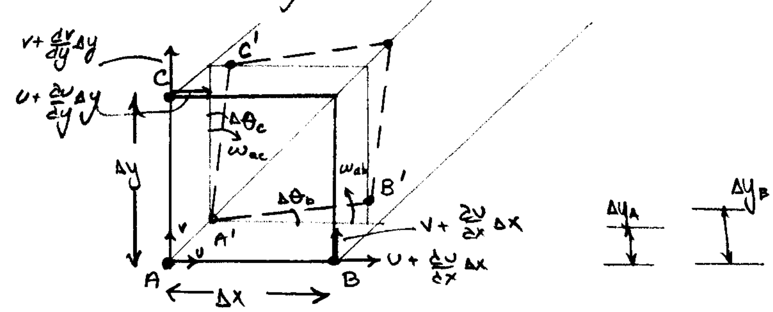
$$0 = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z + \frac{\partial \rho u}{\partial x}\Delta x \Delta y \Delta z + \frac{\partial \rho v}{\partial y}\Delta x \Delta y \Delta z + \frac{\partial \rho w}{\partial z}\Delta x \Delta y \Delta z$$

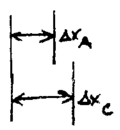
Now divide by element volume to obtain

$$\frac{\partial \rho}{\partial t} + \underbrace{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}}_{\text{divergence of mass flux}} = 0$$

Usually one sees this expression as: $\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho \underline{V})$ or $-\nabla\cdot(\rho\underline{V})$

Rotation & Vorticity





segments AB & AC are initially orthogonal. At time later the element has moved as shown.

Expected position of all vertices in translation only is $\Delta x_A, \Delta y_A$
Points B' & C' show a little extra translation, $\Delta x_c, \Delta y_B$ because of slight deformation (rotation) of the element.

Rate of rotation of AB is $\omega_{ab} = \lim_{\Delta t \to 0} \frac{\Delta\theta_B}{\Delta t}$

$$\tan(\Delta\theta_B) = \frac{\Delta y_B - \Delta y_A}{\Delta x} \quad (\text{for small angles } \tan(\theta) = \theta)$$

$$\therefore \Delta\theta_B \approx \frac{\Delta y_B - \Delta y_A}{\Delta x} \quad \text{thus} \quad \Delta\theta_B = \frac{\frac{\partial v}{\partial x}\Delta x \Delta t}{\Delta x} = \frac{\partial v}{\partial x}\Delta t$$

$$\omega_{ab} = \lim_{\Delta t \to 0} \frac{\Delta\theta_B}{\Delta t} = \lim_{\Delta t \to 0} \frac{\partial v}{\partial x}\frac{\cancel{\Delta t}}{\cancel{\Delta t}} = \frac{\partial v}{\partial x}$$

Rate of rotation of AC is $\omega_{ac} = \lim_{\Delta t \to 0} \frac{\Delta\theta_c}{\Delta t}$ which by similar reasoning is

$$\omega_{ac} = -\frac{\partial u}{\partial y}$$

Average rate of rotation is

$$\omega_z = \frac{\omega_{ab} + \omega_{ac}}{2} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Extension of these concepts to 3D produces the angular velocity vector.

$$\underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$$

Where

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The vorticity vector is defined as twice the angular velocity vector

$$\underline{\Omega} = \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\underline{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\underline{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{k}}_{\text{curl of the velocity field}}$$

$\underline{\Omega} = \text{curl}(\underline{V})$ or $\nabla \times \underline{V}$.

In a case where vorticity vanishes, the flow is called irrotational.

Many practical cases can be treated as irrotational, which is a tremendously useful simplification. However, in some problems of practical importance, rotation is important and the vorticity cannot be neglected.

Summary of important mechanics relationships

hydrostatic pressure: $\rho \underline{a} = \rho g - \nabla p$

continuity : $\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \underline{V}$ $\quad$ $\underline{V}$ - velocity vector in Eulerian sense.

vorticity : $\underline{\Omega} = \nabla \times \underline{V}$

<u>Bernoulli's Equation</u> (Fluid mechanics to hydraulics)

Assume non-viscous (inviscid flow)

$\rho \underline{a} = \rho g - \nabla p$.

Require $g = -g\underline{k}$; then $\rho \underline{a} = -\nabla(p + \rho g z)$

Require ρg = constant (incompressible) then $\frac{\underline{a}}{g} = -\nabla\left(\frac{p}{\rho g} + z\right)$

Write $\underline{a}$ in differential form

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Require steady flow $\left(\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0\right)$

Then

$$\frac{1}{g}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho g}+z\right)$$

$$\frac{1}{g}\left(u\frac{\partial v}{dx}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial}{\partial y}\left(\frac{p}{\rho g}+z\right)$$

$$\frac{1}{g}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial}{\partial z}\left(\frac{p}{\rho g}+z\right)$$

Require $\nabla\times\underline{V}=\underline{0}$ (Irrotational flow), then $\frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}$; $\frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}$; $\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$

Substitute:

$$\frac{1}{g}\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho g}+z\right)$$

$$\frac{1}{g}\left(u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial y}\right)=-\frac{\partial}{\partial y}\left(\frac{p}{\rho g}+z\right)$$

$$\frac{1}{g}\left(u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial}{\partial z}\left(\frac{p}{\rho g}+z\right)$$

Recall $y\frac{dy}{dx}=\frac{1}{2}\frac{dy^2}{dx}$ so

$$\frac{1}{g}\left(\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\gamma}+z\right) \qquad \gamma=\rho g$$

$$\frac{1}{g}\left(\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)\right)=-\frac{\partial}{\partial y}\left(\frac{p}{\gamma}+z\right) \qquad \text{observe } \underline{V}\cdot\underline{V}=u^2+v^2+w^2$$

$$\frac{1}{g}\left(\frac{\partial}{\partial z}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)\right)=-\frac{\partial}{\partial z}\left(\frac{p}{\gamma}+z\right)$$

Rearrange

$$0=-\frac{\partial}{\partial x}\left(\frac{p}{\gamma}+z+\frac{u^2+v^2+w^2}{2g}\right)=-\frac{\partial}{\partial y}\left(\frac{p}{\gamma}+z+\frac{u^2+v^2+w^2}{2g}\right)=-\frac{\partial}{\partial z}\left(\frac{p}{\gamma}+z+\frac{u^2+v^2+w^2}{2g}\right),$$

If $\frac{dy}{dx}=0$, then $y=$const. wrt x. All three equations must equal the same constant for equality to be satisfied.

$$\therefore \quad \frac{p}{\gamma}+z+\frac{\underline{V}\cdot\underline{V}}{2g}=C$$

This result is called Bernoulli's equation; it is valid for <u>steady</u>, <u>incompressible</u>, <u>irrotational</u>, <u>inviscid</u> flow.

Usually written as $\frac{p}{\gamma}+z+\frac{V^2}{2g}=C$ $\quad V^2=$ magnitude of

Energy Equation

Reynold's transport theorem applied to the energy principle gives

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{d}{dt}\int_{CV} \rho\left(u + \frac{V^2}{2} + gz\right) d\forall + \int_{C.S.} \rho\left(u + \frac{V^2}{2} + gz\right) \underline{V} \cdot d\underline{A}$$

u - specific internal energy (note how it appears in same location as pressure in the momentum equation).

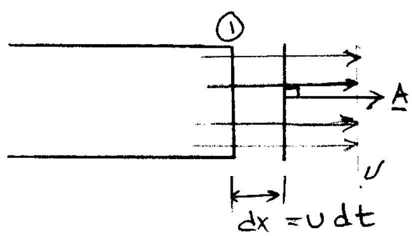
$\frac{dQ}{dt}$ = heat into C$\forall$.

$\frac{dW}{dt}$ = work done by C$\forall$.

Flow Work

Work = force · distance : $W = F\,dx$

$\frac{dW}{dt} = \frac{d}{dt}(F\,dx) = F\frac{dx}{dt} = FU$ (for constant force, such as pressure)



Momentum at ①

$F_p = \int_{CS} \rho \underline{V}(\underline{V} \cdot d\underline{A})$

$pA = \rho U^2 A$

Distance through which this force is applied is $dx = U\,dt$

$\therefore \quad F_p\,dx = pAU\,dt = \rho U^2 A U\,dt$

So $\quad \frac{dW}{dt} = F_p \frac{dx}{dt} = p\underbrace{AU}_{Q \text{ (discharge)}} = \rho U^2 \underbrace{AU}_{Q}$ Usually flow work is incorporated into the flux integral as

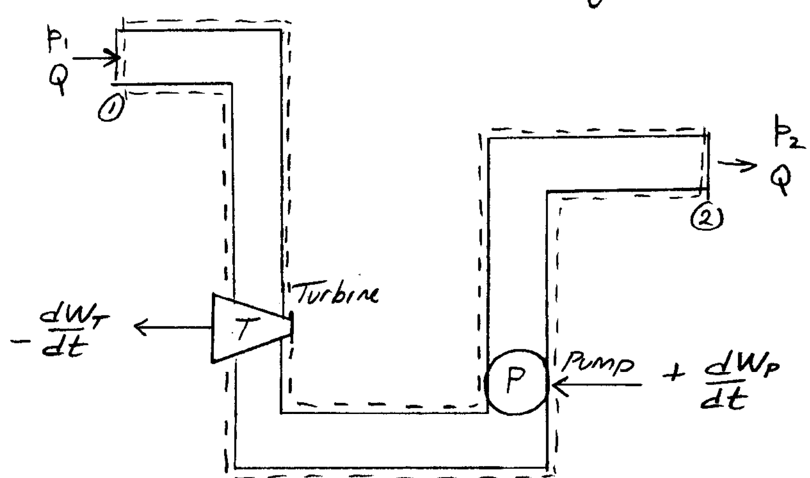
$$\int_{cs} \rho\left(\frac{p}{\rho} + \frac{V^2}{2} + gz + u\right) \underline{V} \cdot d\underline{A}$$

and the $\frac{dW}{dt}$ is called shaft work (excludes flow work)

Now notice the remarkable similarity to Bernoulli's form.

Flow work is the energy required for the fluid to flow against pressure forces. Shaft work is the energy that can be removed by mechanical devices from the flow field and put to use.

As an illustration consider steady flow in a pipeline



$$\frac{dQ_{heat}}{dt} - \frac{dW_s}{dt} = \int_{①}\left(\frac{p_1}{\rho} + gz_1 + \frac{U_1^2}{2} + u_1\right)\rho \underline{V}\cdot d\underline{A} + \int_{②}\left(\frac{p_2}{\rho} + gz_2 + \frac{U_2^2}{2} + u_2\right)\rho \underline{V}\cdot d\underline{A}$$

$$+ \cancel{\frac{d}{dt}\int_{cv}\left(\frac{U^2}{2} + gz + u\right)\rho\, d\forall}$$

steady flow, fixed c.v.

$$-\frac{dW_s}{dt} = -\frac{dW_T}{dt} + \frac{dW_P}{dt}$$

flux integrals evaluate as

$$\frac{dQ_{heat}}{dt} - \frac{dW_T}{dt} + \frac{dW_P}{dt} = -\frac{p_1}{\rho}\rho Q - gz_1\rho Q - \frac{\alpha_1 U_1^2}{2}\rho Q - u_1\rho Q$$

$$+ \frac{p_2}{\rho}\rho Q - gz_2\rho Q - \frac{\alpha_2 U_2^2}{2}\rho Q - u_2\rho Q$$

α_1 & α_2 are kinetic energy correction coefficients : $\alpha = \dfrac{\int U_1^3 dA}{\int \bar{U}^3 dA}$

where $\bar{U} = \dfrac{\int u\, dA}{\int dA}$

$1 \leq \alpha \leq 2$ ($\alpha \approx 1$ for turbulent flow $\to 2$ for laminar flows) (usual practice is to assume $\alpha = 1$)

Result (after rearrangement & simplification is

$$\underbrace{\left(\frac{dW_P}{dt}\right)\left(\frac{1}{\rho g Q}\right)}_{\text{added head from pump}} + \frac{p_1}{\rho g} + z_1 + \frac{\alpha_1 U_1^2}{2g} = \frac{p_2}{\rho g} + z_2 + \frac{\alpha_2 U_2^2}{2g} + \overbrace{\left(\frac{dW_T}{dt}\right)\left(\frac{1}{\rho g Q}\right)}^{\text{head removed by turbine}}$$

$$+ \underbrace{\frac{u_2 - u_1}{\rho g Q} - \left(\frac{dQ_{HEAT}}{dt}\right)\left(\frac{1}{\rho g Q}\right)}_{\text{head loss}}$$

The dimensions are energy per unit weight of fluid, which is a length. These energy terms are called "head"

The head loss term includes frictional loss (the $\frac{dQ_{HEAT}}{dt}$) and losses from changes in internal energy. If the liquid does not change phase (flash to a gas) the internal energy changes are simply expressed as a change in temperature.

In water systems Δu is usually small and the head loss term is mostly friction loss.

The energy equation is usually written as

$$\underbrace{\frac{p_1}{\gamma}}_{\text{pressure head}} + \underbrace{\frac{\alpha_1 V_1^2}{2g}}_{\text{velocity head}} + \underbrace{z_1}_{\text{elevation head}} + h_P = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_T - h_L$$

Note the remarkable similarity to Bernoulli's equation.

$\underbrace{\quad}$ + $\underbrace{\quad}_{\text{elevation head}}$ = static head

$\underbrace{\qquad\qquad}$ total dynamic head

This energy equation is valid for steady, one-dimensional flow.

Bernoulli's equation
Energy equation
continuity

are used extensively in environmental flow modeling, even in flows where some assumptions are violated, they produce good approximations.

Angular Momentum

Reynold's transport theorem applied to the conservation of angular momentum gives

$$\sum(\underline{r}\times\underline{F}) = \frac{d}{dt}\int_{CV} \underline{r}\times\underline{V}\,\rho\,d\forall + \int_{C.S.} (\underline{r}\times\underline{V})\rho(\underline{V}\cdot d\underline{A})$$

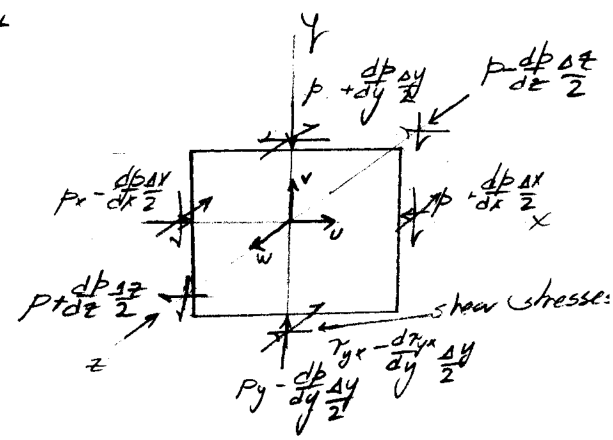
Its application often involves both continuity and the Bernoulli equation or energy equation

Navier Stokes Equations

Apply force balance to a fluid element
(Use x-direction as illustration

$$\sum F_x = \frac{d}{dt}\int_{CV} \rho u\,d\forall + \int_{C.S.} u\rho(\underline{V}\cdot d\underline{A})$$

inertial reference frame (C.V. is fixed in space)



Forces

gravity : $g_x \rho \Delta x \Delta y \Delta z$

pressure : $-\frac{dp}{dx}\Delta x \Delta y \Delta z$

shear : $\frac{\partial}{\partial y}\tau_{yx}\Delta x\Delta y\Delta z + \frac{\partial}{\partial z}\tau_{zx}\Delta x\Delta y\Delta z + \underbrace{\frac{\partial}{\partial x}\tau_{xx}\Delta x\Delta y\Delta z}$

normal stress other than pressure — proportional to shear rate; think of it as a "dynamic" added pressure that vanishes when fluid is stationary

∴ Momentum becomes (in 3D)

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} + \rho g_x$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy} + \rho g_y$$

$$\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} + \rho g_z$$

To "complete" these equations constitutive equations that relate the shear stresses to the velocity field are used

For a Newtonian, incompressible fluid

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Inserting these constitutive relations into the momentum equation, simplifying by grouping of like terms produces a set of equations known as the Navier-Stokes equations:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\left(\frac{\partial^2 u}{\partial x\partial x} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right)$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \mu\left(\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 v}{\partial y\partial y} + \frac{\partial^2 w}{\partial y\partial z}\right)$$

$$\underbrace{\rho\frac{\partial w}{\partial t}}_{\text{local acceleration}} + \underbrace{\rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z}}_{\text{convective acceleration}} = \underbrace{-\frac{\partial p}{\partial z}}_{\text{pressure gradient}} + \underbrace{\rho g_z}_{\text{body (gravity) force}} + \underbrace{\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)}_{\substack{\text{normal momentum diffusion (inertia)} \\ \mu\nabla\cdot(\nabla\underline{V})^T}} + \underbrace{\mu\left(\frac{\partial^2 u}{\partial z\partial x} + \frac{\partial^2 v}{\partial z\partial y} + \frac{\partial^2 w}{\partial z\partial z}\right)}_{\substack{\text{tangential momentum diffusion} \\ \mu\nabla\cdot\nabla\underline{V}}}$$

In inviscid flow the viscosity terms are negligible so the Navier-Stokes equations reduce to

$\rho \frac{D\underline{V}}{Dt} = -\nabla p + \rho g$ Which is identical to Euler's equation of fluid flow.

In compact notation the Navier-Stokes equations displayed above are written as:

$$\rho\frac{D\underline{V}}{Dt} = -\nabla p + \rho g + \mu\nabla\cdot(\nabla\underline{V})^T + \mu\nabla\cdot(\nabla\underline{V})$$

The kinds of problems where solution of Navier-Stokes equations are used include:

(1) Inviscid compressible flow

continuity, momentum & energy NS equations

rocket nozzle flow, aircraft inlet flows, re-entry & rocket aerodynamic flows, blast field (explosion) flows.

(2) Turbulent flows

A turbulence closure model supplies a "stress" term. so that mean flow calculations are made on "averaged" NS equations

Applications: estuary and lake flows, atmospheric flows

(3) Incompressible flows:

Open channel flow, potential flow (porous media)

Most models in Environmental Flows are extreme simplifications of a full NS model, but underlying all the models is the theoretical background of the Navier-Stokes equation.