CHAPTER 2

GROUND WATER HYDROLOGY

2.1 INTRODUCTION

This chapter is devoted to the properties of ground water including the definition of aquifer systems and the parameters that can be used to characterize them. Porosity, hydraulic conductivity, storage coefficient, and hydraulic gradient are all important in determining the rate and direction of ground water flow. Governing equations of ground water flow are introduced and solved for simple flow systems in confined and unconfined aquifers. Flow nets are useful representations of streamlines and equipotential lines in two dimensions, and provide a graphical picture of ground water heads and gradients of flow. The chapter ends with an introduction to the unsaturated zone, which lies above the water table up to the soil root zone.

2.2 PROPERTIES OF GROUND WATER

2.2.1 Vertical Distribution of Ground Water

Ground water can be characterized according to vertical distribution as defined by Todd (1980). Figure 2.1 indicates the main divisions of subsurface water, and the **water table** generally divides the unsaturated zone from the saturated ground water zone. The **soil water zone**, which extends from the ground surface down through the major root zone, varies with soil type and vegetation, but is usually a few feet in thickness. The amount of water present in the soil water zone depends primarily on recent exposure to rainfall, infiltration, and vegetation. **Hygroscopic** water is a film of water tightly held by surface forces, and remains adsorbed to the surface of soil grains, while gravitational water drains through the soil under the influence of gravity.

The **unsaturated zone (vadose zone)** extends from the surface to the water table through the root zone, intermediate zone, and the capillary zone (see Figure 2.1). **Capillary water** is held by surface tension forces just above the **water table**, which is defined as the level to which water will rise in a well drilled into the saturated zone. Thickness of the unsaturated zone may vary from a few feet for high water table conditions to hundreds of feet (meters) in arid regions of the country, such as Arizona or New Mexico. Unsaturated zone water is held in place by surface forces, and infiltrating water passes downward towards the water table as gravitational flow, subject to retardation by capillary forces.

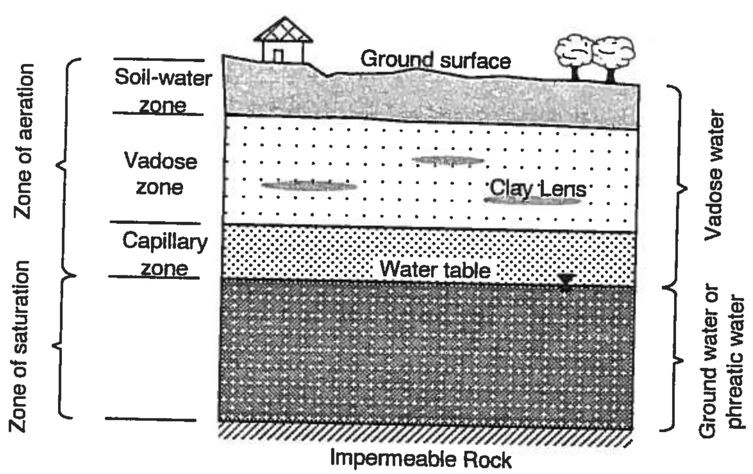

Figure 2.1 Vertical zones of subsurface water.

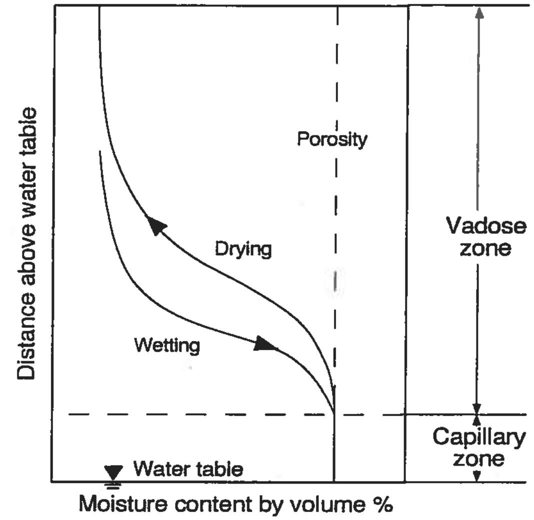

Figure 2.2 Typical soil-moisture relationship.

The capillary zone, or fringe, extends from the water table up to the limit of capillary rise, which varies inversely with the pore size of the soil and directly with the surface tension. The capillary forces at work are related to the meniscus between the gas phase and the liquid phase, as in the case of a small straw inserted into a pan of water. Capillary rise can range from 2.5 cm for fine gravel to more than 200 cm for silt (Todd, 1980). Just above the water table almost all pores contain capillary water, and the water content decreases with height depending on the type of soil. A typical soil moisture curve is shown in Figure 2.2, where the amount of moisture in the vadose zone generally decreases with vertical distance above the water table. Soil moisture curves vary with soil type and with the wetting cycle; more details on the unsaturated zone are contained in Section 2.7 and Chapter 9. A more detailed treatment is available from Guymon (1994).

In the **saturated zone**, which occurs beneath the water table, the **porosity** is a direct measure of the water contained per unit volume, expressed as the ratio of the volume of voids to the total volume. Porosity averages about 25% to 35% for most aquifer systems. Also, ground water flows according to Darcy's Law, which relates velocity to gradient and hydraulic conductivity, in the saturated zone (Section 2.3). Only a portion of the water can be removed from the saturated zone by drainage or by pumping from a well. **Specific yield** is defined as the volume of water released from an unconfined aquifer per unit surface area per unit head decline in the water table. Fine-grained materials yield little water whereas coarse-grained materials provide significant water and thus serve as aquifers. In general, specific yields for unconsolidated formations fall in the range of 7% to 25%.

2.2.2 Aquifer Systems

An **aquifer** can be defined as a formation that contains sufficient saturated permeable material to yield significant quantities of water to wells or springs. Aquifers are generally areally extensive and may be overlain or underlain by a confining bed, defined as a relatively impermeable material. Figure 2.3 shows some typical examples of confined aquifers, which have relatively impermeable confining units above, such as clay or silt, and are under pressure. An **aquitard** is a low permeability stratum, such as a sandy clay unit, that may leak water vertically to adjacent aquifers.

Aquifers can be characterized by the porosity of a rock or soil, expressed as the ratio of the volume of voids V_v to the total volume V. Porosity may also be expressed by

$$n = \frac{V_v}{V} = 1 - \frac{\rho_b}{\rho_m} \tag{2.1}$$

where ρ_m is the density of the grains and ρ_b is the **bulk density**, defined as the oven-dried mass of the sample divided by its original volume. Table 2.1 shows a range of porosities (6% to 46%) for a number of aquifer materials. In practice, the porosity of an aquifer is usually assumed to be about 30%, assuming that there are no fractures present. Fractured rock such as limestone is more complex and can have much lower porosities in the range of 1% to 10%. Figure 2.4 shows the theoretical maximum porosity of 47.65% associated with cubic packing of spheres, and the lowering effect of adding smaller grains into the void space.

Unconsolidated geologic materials are normally classified according to their size and distribution. Standard soil classification based on particle size is shown in Table 2.2. Particle sizes are measured by mechanically sieving grain sizes larger than 0.05 mm and measuring rates of settlement for smaller particles in suspension. A typical particle size distribution graph is plotted in Figure 2.5, where a fine sand has a more uniform distribution than an alluvium which includes a range of particle sizes. The value of the **uniformity coefficient**, defined as D_{60}/D_{10}, indicates the relative sorting of the material, and takes on a low value for the fine sand compared to the alluvium. The texture of a soil is defined by the relative proportions of sand, silt and clay present in the particle size analysis and can be expressed most easily on a triangle diagram of soil textures (see any soils textbook). For example, a soil with 30% clay, 60% silt, and 10% sand is referred to as a silty clay loam.

Most aquifers can be considered as underground storage reservoirs that receive recharge from rainfall or from a river system. Water flows out of an aquifer due to gravity drainage according to water level gradients or to pumping from extraction wells. An aquifer may be classified as **unconfined** if the water table exists under atmospheric pressure, as defined by levels in shallow wells. A **confined** aquifer is one that is overlain by a relatively impermeable unit such that the aquifer is under pressure and the pressure level rises above the confined unit. A **leaky confined** aquifer represents a stratum that allows water to flow

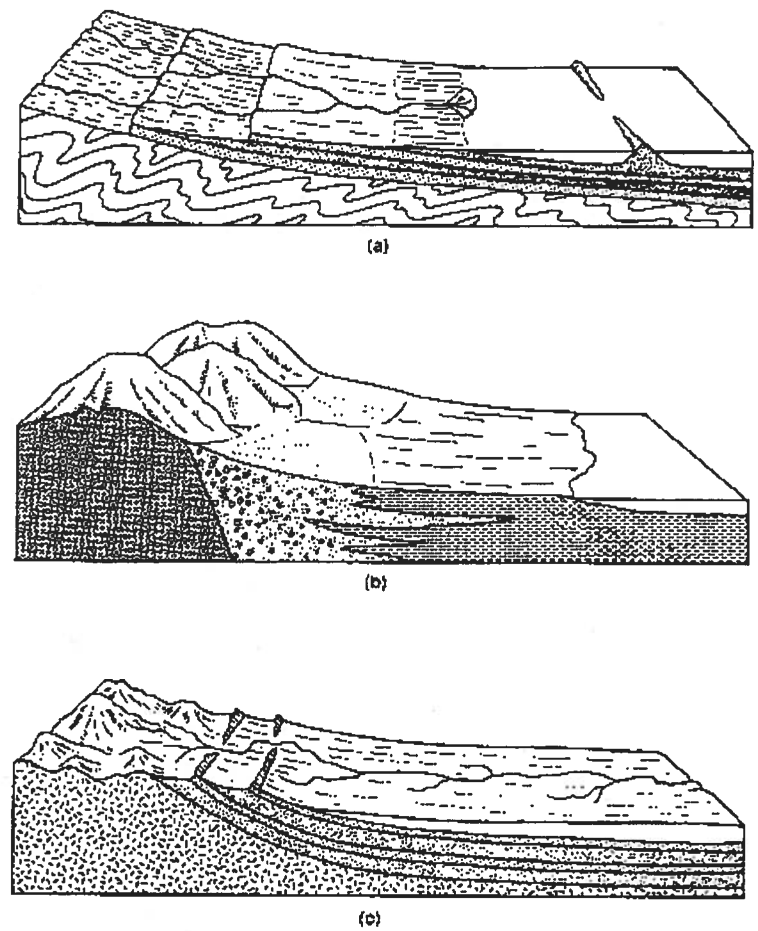

Figure 2.3 Confined aquifers are created when aquifers are overlain by confining beds. (a) Confined aquifers created by alternating aquifers and confining units deposited on regional dip. (b) Confined aquifers created by deposition of alternating layers of permeable sand and gravel and impermeable silts and clays deposited in intermontane basins. (c) Confined aquifer created by upwarping of beds by intrusions. Source: C.W. Fetter, Applied Hydrology, 2/E. Reprinted with the permission of Macmillan College Publishing © 1988.

TABLE 2.1 Representative values of porosity

Material	Porosity (%)	Material	Porosity (%)	Material	Porosity (%)
Gravel, coarse	28†	Sandstone, medium	37	Claystone	43
Gravel, medium	32†	Limestone	30	Shale	6
Gravel, fine	34†	Dolomite	26	Till, predominantly silt	34
Sand, coarse	39	Dune sand	45	Till, predominantly sand	31
Sand, medium	39	Loess	49	Tuff	41
Sand, fine	43	Peat	92	Basalt	17
Silt	46	Schist	38	Gabbro, weathered	43
Clay	42	Siltstone	35	Granite, weathered	45
Sandstone, fine	33				

† These values are for repacked samples, all others are undisturbed. Source: Morris and Johnson, 1967.

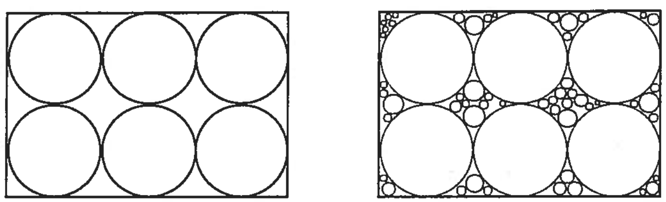

Figure 2.4 (a) Cubic packing of spheres of equal diameter with a porosity of 47.65%. (b) Cubic packing of spheres with void spaces occupied by grains of smaller diameter, resulting in a much lower overall porosity.

from above through a confining zone into the underlying aquifer. A **perched aquifer** is where an unconfined water zone sits on top of a clay lens, separated from the main aquifer below.

Figure 2.6 shows a vertical cross section illustrating typical characteristics of unconfined and confined aquifers. An unconfined aquifer is usually identified at a field site by drilling bore holes and wells to shallow depths, as described below and in Chapter 5. A confined aquifer (artesian) is one that is identified by a confining unit at depth, and the water level (elevation plus pressure head) that is under pressure, and rises above the confining unit. If the water level rises above the land surface, a flowing well or spring results and is referred to as an **artesian well** or spring.

TABLE 2.2 Soil classification based on particle size

Material	Particle Size (mm)
Clay	<0.004
Silt	0.004-0.062
Very fine sand	0.062-0.125
Fine sand	0.125-0.25
Medium sand	0.25-0.5
Coarse sand	0.5-1.0
Very coarse sand	1.0-2.0
Very fine gravel	2.0-4.0
Fine gravel	4.0-8.0
Medium gravel	8.0-16.0
Coarse gravel	16.0-32.0
Very coarse gravel	32.0-64.0

Source: Morris and Johnson, 1967

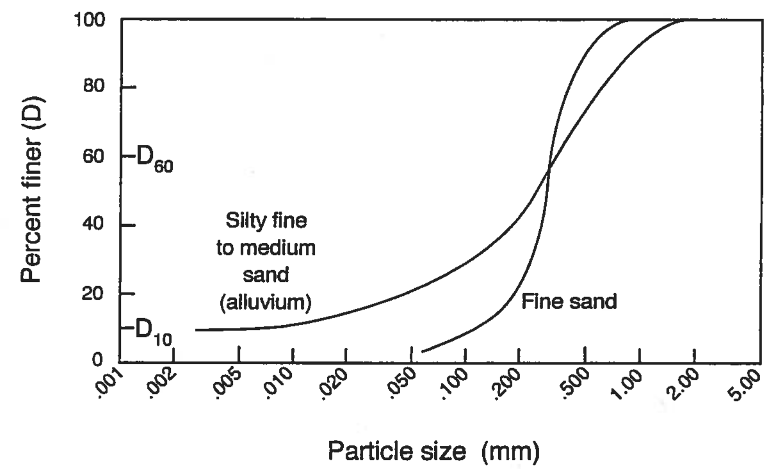

Figure 2.5 Particle-size distribution for two geologic samples.

A recharge area supplies water to a confined aquifer, and such an aquifer can convey water from the recharge area to locations of natural or artificial discharge. The **piezometric surface** (or **potentiometric surface**) of a confined aquifer is the hydrostatic pressure level of water in the aquifer, defined by the water level that occurs in a lined penetrating well. It should be noted that a confined aquifer will become unconfined when the piezometric sur-

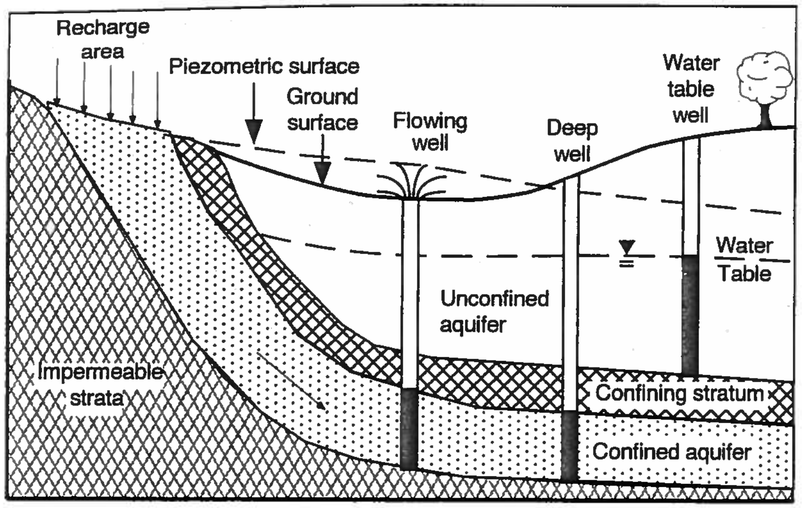

Figure 2.6 Schematic cross-section of unconfined and confined aquifers. Black areas indicate screened zones.

face falls below the bottom of the upper confining bed. Contour maps and profiles can be prepared for the water table elevation for an unconfined aquifer, or the piezometric surface for a confined aquifer. These **equipotential lines** are described in more detail in Section 2.6. Once determined from a series of wells in an aquifer, orthogonal lines can be drawn to indicate the general direction of ground water flow, in the direction of decreasing head (Figure 2.7).

A parameter of some importance relates to the water-yielding capacity of an aquifer. The **storage coefficient** S is defined as the volume of water that an aquifer releases from or takes into storage per unit surface area per unit change in piezometric head. For a confined aquifer, values of S fall in the range of 0.00005 to 0.005, indicating that large pressure changes produce small changes in the storage volume. For unconfined aquifers, a change in storage volume is expressed simply by the product of the volume of aquifer lying between the water table at the beginning and end of a period of head change, and the average specific yield of the formation. Thus, the storage coefficient for an unconfined aquifer is approximately equal to the specific yield, typically in the range of 0.07 to 0.25.

A vertical hole, which is extended into an aquifer at depth, is referred to as a **well**, and the steel or PVC plastic pipe which extends from the surface to the screened zone is called the **casing**. Wells are either drilled or pushed into the subsurface with specialized GeoProbe equipment (Chapter 5). Wells are used for pumping of water and contaminants, injection of water or disposal of chemicals, water level observation, and water quality sampling. The portion of the well that is open to the aquifer is **screened** to prevent aquifer material from entering the well. The annular space around the screen is often filled with sand or gravel to minimize hydraulic resistance to flow. The annulus is usually cemented up to the surface with a material such as bentonite clay to protect against contamination of the well from surface leakage. Details on well construction and development methods are presented in Chapter 5.

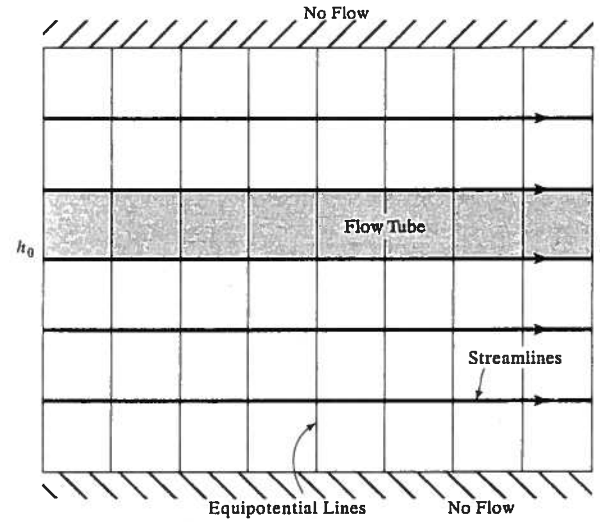

Figure 2.7 Simple flow net.

2.3 GROUND WATER MOVEMENT

2.3.1 Darcy's Law

The movement of ground water is well understood by hydraulic principles reported in 1856 by Henri Darcy, who investigated the flow of water through beds of permeable sand. Darcy advanced one of the most important laws in hydrology—that the flow rate through porous media is proportional to the head loss and inversely proportional to the length of the flow path. Darcy's Law serves as the basis for present-day knowledge of ground water flow and well hydraulics, and for the derivation of governing ground water flow shown in the equations in Section 2.4.

Figure 2.8 depicts the experimental setup for determining head loss through a sand column, with **piezometers** located a distance L apart. Total energy for this system can be expressed by the Bernoulli equation

$$\frac{p_1}{\gamma}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{v_2^2}{2g}+z_2+h_l \tag{2.2}$$

where

p = pressure
γ = specific weight of water = ρg
v = velocity
z = elevation
h_l = head loss

Because velocities are very small in porous media, velocity heads may be neglected, allowing head loss to be expressed as

$$h_l=\left(\frac{p_1}{\gamma}+z_1\right)-\left(\frac{p_2}{\gamma}+z_2\right) \tag{2.3}$$

It follows that the head loss is independent of the inclination of the column. Darcy related flow rate to head loss and length of column through a proportionality constant referred to as K, the **hydraulic conductivity**, a measure of the ability of the porous media to transmit water. Darcy's Law can be stated thus:

$$v=-\frac{Q}{A}=-K\frac{dh}{dL} \tag{2.4}$$

The negative sign indicates that flow of water is in the direction of decreasing head. The Darcy velocity that results from Eq. (2.4) is an average discharge velocity, v, through the entire cross section of the column. The actual flow is limited to the pore space only, so that the seepage velocity v_s is equal to the Darcy velocity divided by porosity:

$$v_s=\frac{Q}{nA} \tag{2.5}$$

Thus, actual seepage velocities are usually much higher (by a factor of 3 or 4) than the Darcy velocities. Seepage velocity is used later in the text for all transport calculations.

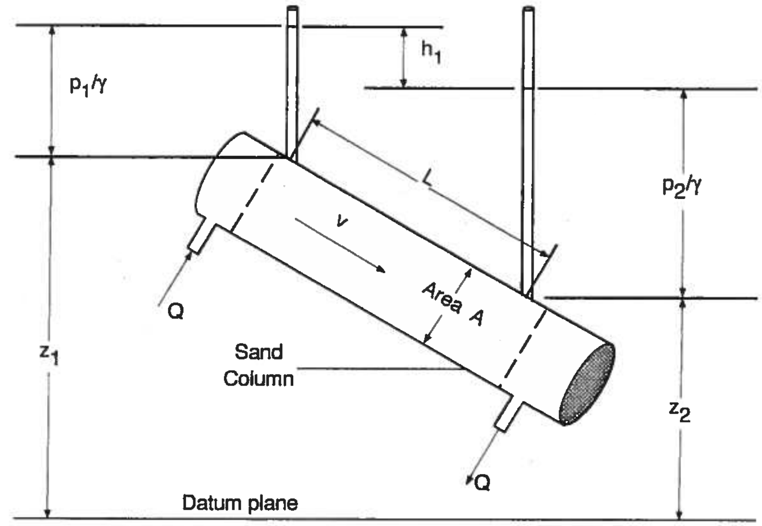

Figure 2.8 Head loss through a sand column.

It should be pointed out that Darcy's law applies to laminar flow in porous media, and experiments indicate that Darcy's law is valid for Reynolds numbers ($R = \rho vd/\mu$) less than 1 and perhaps as high as 10. This represents an upper limit to the validity of Darcy's law, which turns out to be applicable in most ground water systems. Deviations can occur near heavily pumped wells and in fractured aquifer systems, such as limestone, where flow is primarily along the fractures.

2.3.2 Hydraulic Conductivity

The hydraulic conductivity (K) of a soil or rock depends on a variety of physical factors and is an indication of an aquifer's ability to transmit water. Thus, sand aquifers have K values many orders of magnitude larger than clay units. Typical values for aquifers would be 10^{-2} cm/sec for sand, 10^{-4} cm/sec for silt, and 10^{-7} cm/sec for clay. Table 2.3 indicates representative values of hydraulic conductivity for a variety of materials. As can be seen, K can vary over many orders of magnitude in an aquifer system that contains several different types of material. Thus, velocities and flow rates can also vary over the same range, as expressed by Darcy's law.

The term **transmissivity** is often used in the description of ground water hydraulics applied to confined aquifers. It is defined as the product of K and the **saturated thickness** of the aquifer b. Hydraulic conductivity is usually expressed in m/day (ft/day) and transmissivity T is expressed in m²/day (ft²/day). An older unit for T that is still reported in some applications is gal/day/ft. The **intrinsic permeability** (k) of a rock or soil is a property of the medium only, independent of fluid properties. Intrinsic permeability can be related to hydraulic conductivity K by

$$K = k\left(\frac{\rho g}{\mu}\right) \tag{2.6}$$

where

μ = dynamic viscosity

ρ = fluid density

g = gravitational constant.

Intrinsic permeability k has units of m² or darcy (where 1 darcy = 0.987 μm²); k is often used in the petroleum industry, whereas K is primarily used in ground water hydrology for categorizing aquifer systems.

TABLE 2.3 Representative values of hydraulic conductivity

Material	Hydraulic Conductivity (cm/sec)
UNCONSOLIDATED SEDIMENTARY DEPOSITS	
Gravel	3 to 3×10^{-2}
Coarse sand	6×10^{-1} to 9×10^{-5}
Medium sand	5×10^{-2} to 9×10^{-5}
Fine sand	2×10^{-2} to 2×10^{-5}
Silt, loess	2×10^{-3} to 1×10^{-7}
Till	2×10^{-4} to 1×10^{-10}
Clay	5×10^{-7} to 1×10^{-9}
Unweathered marine clay	2×10^{-7} to 8×10^{-11}
SEDIMENTARY ROCKS	
Karst limestone	2 to 1×10^{-4}
Limestone and dolomite	6×10^{-4} to 1×10^{-7}
Sandstone	6×10^{-4} to 3×10^{-8}
Shale	2×10^{-7} to 1×10^{-11}
CRYSTALLINE ROCKS	
Permeable basalt	2 to 4×10^{-5}
Fractured igneous and metamorphic	3×10^{-2} to 8×10^{-7}
Basalt	4×10^{-5} to 2×10^{-9}
Unfractured igneous and metamorphic	2×10^{-8} to 3×10^{-12}
Weathered granite	3×10^{-4} to 5×10^{-3}

Note on units: 1 m/sec = 1×10^2 cm/sec = 1.04×10^5 darcy

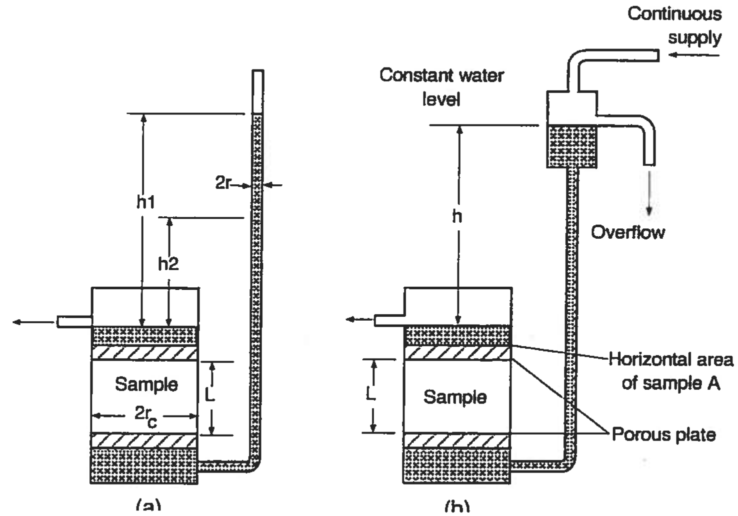

Figure 2.9 Permeameters for measuring hydraulic conductivity of geologic samples. (a) falling head. (b) constant head.

2.3.3 Determination of Hydraulic Conductivity

Hydraulic conductivity in saturated zones can be determined by a number of techniques in the laboratory as well as in the field. **Constant head** and **falling head permeameters** are used in the laboratory for measuring K and are described in detail below. They provide only a rough indication of field values since only small core samples are usually collected and analyzed from a limited number of locations at a site.

A permeameter (Figure 2.9) is used in the laboratory to measure K by maintaining flow through a small column of material and measuring flow rate and head loss. For a constant head permeameter, Darcy's law can be directly applied to find K, where V is volume flowing in time t through a sample of area A, length L, and with constant head h:

$$K = \frac{VL}{Ath} \tag{2.7}$$

The falling head permeameter test consists of measuring the rate of fall of the water level in the tube or column and noting

$$Q = \pi r^2 \frac{dh}{dt} \tag{2.8}$$

Darcy's law can be written for the sample as

$$Q = \pi r_c^2 K \frac{h}{L} \tag{2.9}$$

After equating and integrating,

$$K = \frac{r^2 L}{r_c^2 t} \ln\left(\frac{h_1}{h_2}\right) \tag{2.10}$$

where L, r, and r_c are as shown in Figure 2.9 and t is the time interval for water to fall from h_1 to h_2.

At a typical field site, **pump tests**, **slug tests**, and **tracer tests** are the preferred methods for determination of K. Pump tests and tracer tests provide estimates of average K over a large area of aquifer, based on the size of the test. Pump tests use the constant removal of water and the measurement of head change to estimate K. The slug test for shallow wells involves the measurement of decline or recovery of the water level in the well through time. Slug tests are used to estimate K around an individual well or boring. These tests are described in more detail in Chapter 3 under the general heading of well hydraulics.

Tracer tests involve controlled injection of inorganic and organic tracers (chloride, bromide, and selected organics) and the temporal measurement of concentrations and water levels in wells that are positioned in the direction of ground water flow. Average seepage velocities can be determined by analyzing the resulting curves in space and time, and K values can be computed from Darcy's Law (Figure 2.10). Major tracer tests were implemented at a number of sites in the 1980s and are described in Chapter 6.

The pump test involves the constant removal of water from a single well and observations of water level declines at several adjacent wells. In this way, an integrated K value for a portion of the aquifer is obtained. Field methods generally yield significantly different values of K from those in corresponding permeameter tests performed on cores removed from the aquifer. Thus, field tests are preferable for the accurate determination of aquifer parameters (Chapter 3).

2.3.4 Anisotropic Aquifers

Most real geologic systems tend to have variations in one or more directions due to the processes of deposition and layering that can occur. In the typical field situation in alluvial deposits, the hydraulic conductivity in the vertical direction K_z is often less than the value in

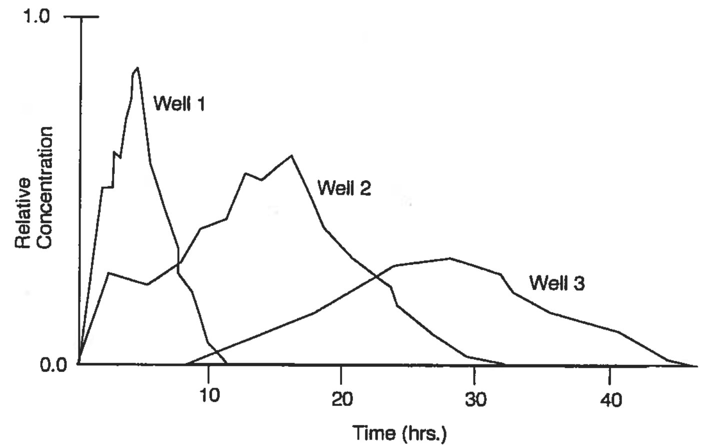

Figure 2.10 Tracer test results.

the horizontal direction K_x. For the case of a two-layered aquifer of different K in each layer and different thicknesses, we can apply Darcy's law to horizontal flow to show

$$K_x = \frac{K_1 z_1 + K_2 z_2}{z_1 + z_2} \tag{2.11}$$

or, in general,

$$K_x = \frac{\Sigma K_i z_i}{\Sigma z_i} \tag{2.12}$$

where

$K_i = K$ in layer i

z_i = thickness of layer i

For the case of vertical flow through two layers, q_z is the same flow per unit horizontal area in each layer:

$$dh_1 + dh_2 = \left(\frac{z_1}{K_1} + \frac{z_2}{K_2}\right) q_z \tag{2.13}$$

but

$$dh_1 + dh_2 = \left(\frac{z_1 + z_2}{K_z}\right) q_z \tag{2.14}$$

where K_z is the hydraulic conductivity for the entire system. Setting Eqs. (2.13) and (2.14) equal to each other, we have

$$K_z = \frac{z_1 + z_2}{(z_1 / K_1) + (z_2 / K_2)} \tag{2.15}$$

or, in general,

$$K_z = \frac{\Sigma z_i}{\Sigma z_i / K_i} \tag{2.16}$$

Ratios of K_x to K_z usually fall in the range of 2 to 10 for alluvium, with values up to 100 where clay layers exist. Many real field sites have spatial and vertical variations in K that can be addressed only by statistical methods or modeling methods. In actual application to layered systems, it is usually necessary to apply ground water flow models that can properly handle complex geologic strata through numerical simulation. Various flow and transport modeling techniques are described in Chapters 6 and 10.

2.3.5 Flow Nets

Darcy's law was originally derived in one dimension, but because many ground water problems are really two- or three-dimensional, graphical methods are available for the determination of flow rate and direction. A specified set of **streamlines** and **equipotential lines** can be constructed for a given set of boundary conditions to form a **flow net** (Figures 2.7 and 2.11) in two dimensions.

Equipotential lines are prepared based on observed water levels in wells penetrating an **isotropic** aquifer. Flow lines are then drawn orthogonally to indicate the direction of flow. For the flow net of Figure 2.11, the hydraulic gradient i is given by

$$i = \frac{dh}{ds} \tag{2.17}$$

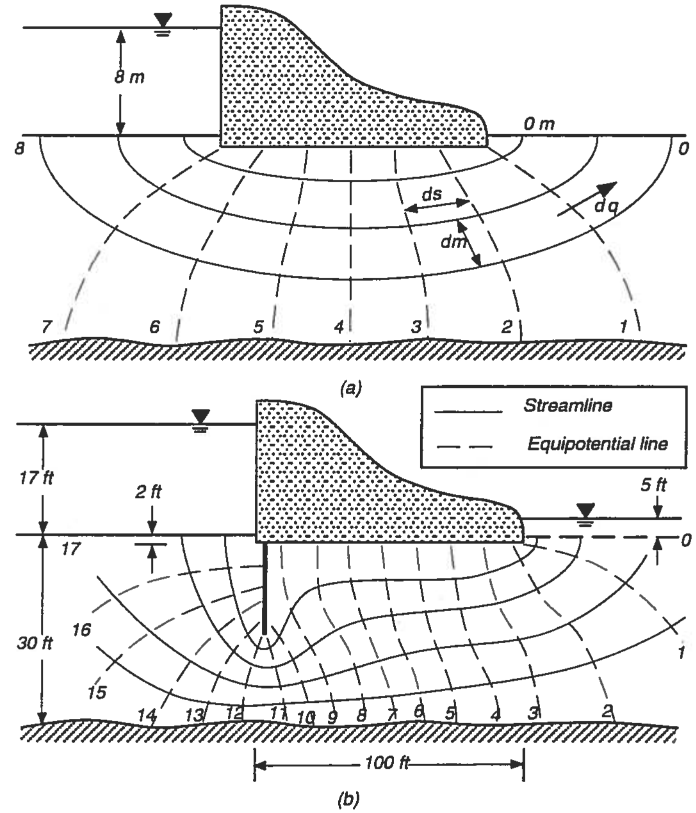

Figure 2.11 (a) Flow net for a simple dam section. (b) Flow net for a complex dam section.

and constant flow q per unit thickness between two adjacent flow lines is

$$q = K\frac{dh}{ds}dm \tag{2.18}$$

If we assume $ds = dm$ for a square net, then for n squares between two flow lines over which total head is divided $(h = H/n)$ and for m divided flow channels,

$$Q' = mq = \frac{KmH}{n} \tag{2.19}$$

where

Q' = flow per unit width,
K = hydraulic conductivity of the aquifer,
M = number of flow channels,
n = number of squares over the direction of flow, and
H = Total head loss in direction of flow.

Flow nets are useful graphical methods to display streamlines and equipotential lines. Since no flow can cross an impermeable boundary, streamlines must parallel it. Also, streamlines are usually horizontal through high K material and vertical through low K material because of refraction of lines across a boundary between different K media. It can be shown that

$$\frac{K_1}{K_2} = \frac{\tan\theta_1}{\tan\theta_2} \tag{2.20}$$

where θ_1 and θ_2 are angles that the velocity vectors make with the normal to the boundary between two materials.

Flow nets can be used to evaluate the directions of flow as a function of different boundary conditions, and the effects of pumping on ground water levels and directions of flow. Figure 2.12 depicts an x–y contour map of water levels resulting from heavy pumping near a source zone in the Atlantic coastal plain. Directions of flow are perpendicular to the equipotential lines, which are lines of constant head. Figure 2.13 shows a flow net for a single well pumping from a uniform flow field.

Example 2.1 FLOW NET COMPUTATION

Compute the total flow seeping under the dam in Figure 2.11a, where width is 20 m and K = 10^{-5} m/sec. Equation (2.19) is used to provide flow per unit width. From the figure $m = 4$, $n = 8$, $H = 8$m, and K is given above.

Solution

$$Q' = \frac{(10^{-5}\,\text{m/sec})(4)(8\text{m})}{8}$$
$$= 4\times10^{-5}\,\text{m}^2/\text{sec} = 3.46\text{m}^2/\text{day}$$

Total flow $Q = 3.46$ m²/day (20m) = 69.1 m³/day

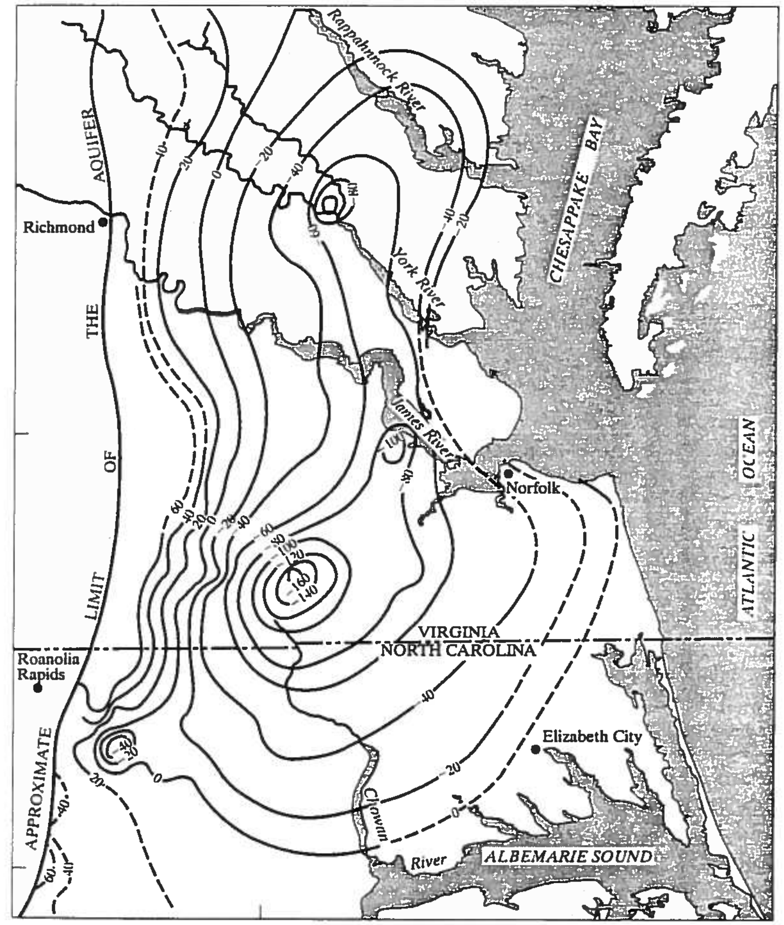

Figure 2.12 Potentiometric surface of lower aquifer in the Atlantic Coastal Plain.

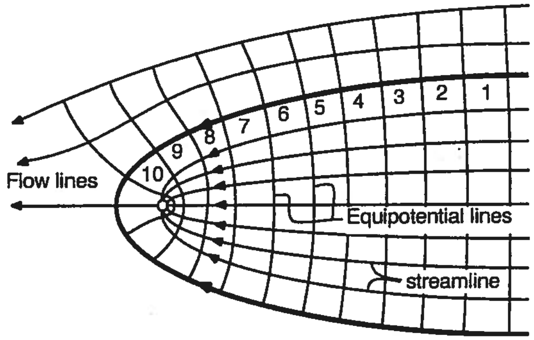

Figure 2.13 Flow net.

2.4 GENERAL FLOW EQUATIONS

The governing flow equations for ground water are derived in most of the standard texts in the field (Bear, 1979; Todd, 1980; Domenico and Schwartz, 1998). The equation of continuity from fluid mechanics is combined with Darcy's law in three dimensions to yield a partial differential equation of flow in porous media, as shown in the next section. Both steady-state and transient flow equations can be derived. Mathematical solutions for specific boundary conditions are well known for the governing ground water flow equation. For complex boundaries and heterogeneous systems, numerical computer solutions must be used (Chapters 9 and 10).

2.4.1 Steady-State Saturated Flow

Consider a unit volume of porous media called an elemental control volume. The law of conservation of mass requires that

(Mass flux in – Mass flux out) = (change in Mass per time)

For steady-state conditions, the right-hand side is zero, and the equation of continuity becomes

$$-\frac{\partial}{\partial x}(\rho v_x)-\frac{\partial}{\partial y}(\rho v_y)-\frac{\partial}{\partial z}(\rho v_z)=0 \qquad (2.21)$$

The units of ρv are mass/area/time as required. For an incompressible fluid, $\rho(x,y,z)$ = constant, and ρ can be divided out of Eq. (2.21). Substitution of Darcy's law for v_x, v_y, and v_z yields

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right)=0 \qquad (2.22)$$

For an isotropic, homogeneous medium, $K_x = K_y = K_z = K$ and can be divided out of the equation to yield

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2}=0 \qquad (2.23)$$

Equation (2.23) is called Laplace's equation and is one of the best-understood partial differential equations. The solution is $h = h(x,y,z)$, the hydraulic head at any point in the flow domain. In two dimensions, the solution is equivalent to the graphical flow nets described in Section 2.3.5. If there were no variation of h with z, then the equation would reduce to two terms on the left-hand side of Eq. (2.23).

2.4.2 Transient Saturated Flow

The transient equation of continuity for a confined aquifer becomes

$$-\frac{\partial}{\partial x}(\rho v_x)-\frac{\partial}{\partial y}(\rho v_y)-\frac{\partial}{\partial z}(\rho v_z)=\frac{\partial}{\partial t}(\rho n)=n\frac{\partial \rho}{\partial t}+\rho\frac{\partial n}{\partial t} \qquad (2.24)$$

The first term on the right-hand side of Eq. (2.24) is the mass rate of water produced by an expansion of water under a change in ρ. The second term is the mass rate of water produced by compaction of the porous media (change in n). The first term relates to the compressibility of the fluid β and the second term to the aquifer compressibility α.

Compressibility and effective stress are discussed in detail in Freeze and Cherry (1979) and Domenico and Schwartz (1998), and will only be briefly reviewed here. According to Terzaghi (1925), the total stress acting on a plane in a saturated porous media is due to the sum of the weight of overlying rock and fluid pressure. The portion of the total stress not borne by the fluid is the effective stress σ_e. Since total stress can be considered constant in most problems, the change in effective stress is equal to the negative of the pressure change in the media, which is related to head change by $dp = \rho\, gdh$. Thus, a decrease in hydraulic head or pressure results in an increase in effective stress, since $d\sigma_e = -dp$.

The compressibility of water β implies that a change in volume occurs for a given change in stress or pressure, and is defined as $(-dV/V)/dp$, where dV is volume change of a given mass of water under a pressure change of dp. The compressibility is approximately constant at $4.4 \times 10^{-10}\,m^2/N$ for water at usual ground water temperatures.

The compressibility of the porous media or aquifer, α, is related to vertical consolidation for a given change in effective stress, or $\alpha = (db/b)/d\sigma_e$, where b is the vertical dimension. From laboratory studies, α is a function of the applied stress and is dependent on previous loading history. Clays respond differently than sands in this regard, and compaction of clays is largely irreversible for a reduced pressure in the aquifer compared to the response in sands. Land surface subsidence is a good example of aquifer compressibility on a regional scale where clays have been depressured over time.

Freeze and Cherry (1979) indicate that a change in head will produce a change in ρ and n in Eq. (2.24), and the volume of water produced for a unit head decline is S_s, the specific storage. Theoretically, one can show that specific storage is related to aquifer compressibility and the compressibility of water by

$$S_s = \rho g(\alpha + n\beta), \tag{2.25}$$

and the mass rate of water produced (right-hand side of Eq. (2.24) is $S_s(\partial h/\partial t)$. Equation (2.24) becomes, after substituting Eq. (2.25) and Darcy's law,

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z}\right) = S_s \frac{\partial h}{\partial t} \tag{2.26}$$

For homogeneous and isotropic media,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K}\frac{\partial h}{\partial t} \tag{2.27}$$

For the special case of a horizontal confined aquifer of thickness b,

$S = S_s b$ where S is the storativity or storage coefficient

$T = Kb$

and

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} \quad \text{in two dimensions.} \tag{2.28}$$

Solution of Eq. (2.28) requires knowledge of S and T to produce $h(x,y)$ over the flow domain. The classical development of Eq. (2.28) was first advanced by Jacob (1940) along with considerations of storage concepts. More advanced treatments consider the problems of a fixed elemental control volume in a deforming media (Cooper, 1966), but these are unnecessary considerations for most practical problems.

2.5 DUPUIT EQUATION

For the case of unconfined ground water flow, Dupuit (1863) developed a theory that allows for a simple solution based on several important assumptions:

1. The water table or free surface is only slightly inclined.
2. Streamlines may be considered horizontal and equipotential lines, vertical.
3. Slopes of the free surface and hydraulic gradient are equal.

Figure 2.14 shows the graphical example of Dupuit's assumptions for essentially one-dimensional flow. The free-surface from $x = 0$ to $x = L$ can be derived by considering Darcy's law and the governing one-dimensional equation. The Dupuit approach neglects any vertical components of flow and reduces a complex 2-D problem to a one-dimensional problem which can be easily solved. For steady state flow, the discharge through the system must be constant and requires the free surface to be a parabola. Example 2.2 demonstrates the derivation of the Dupuit equations.

Example 2.2 DUPUIT EQUATION

Derive the equation for one-dimensional flow in an unconfined aquifer using the Dupuit assumptions (Figure 2.14).

Solution. Darcy's law gives the one-dimensional flow per unit width as

$$q = -Kh\frac{dh}{dx}$$

where h and x are as defined in Figure 2.14. At steady state, the rate of change of q with distance is zero, or

$$\frac{d}{dx}\left(-Kh\frac{dh}{dx}\right) = 0$$

$$-\frac{K}{2}\frac{d^2h^2}{dx^2} = 0$$

which implies that,

$$\frac{d^2h^2}{dx^2} = 0$$

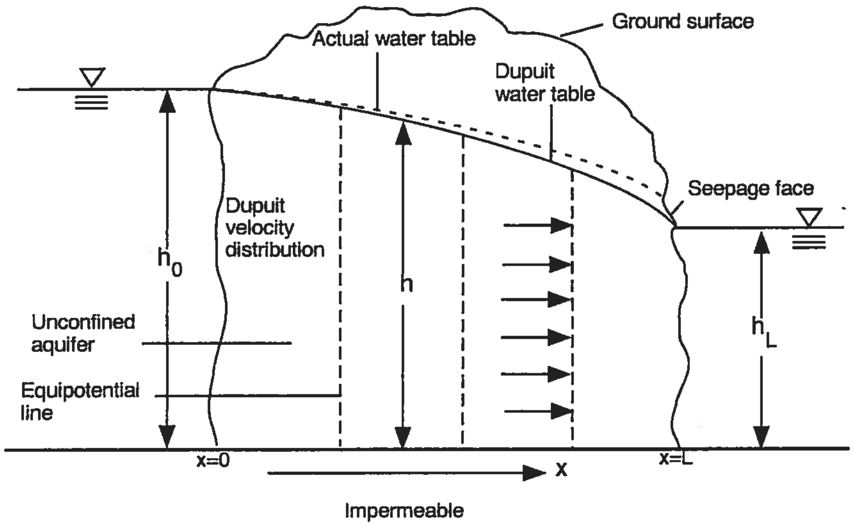

Figure 2.14 Steady flow in an unconfined aquifer between two water bodies with vertical boundaries.

Integration yields

$$h^2 = ax + b$$

where a and b are constants. Setting the boundary condition $h = h_0$ at $x = 0$, we can solve for b,

$$b = h_0^2$$

Differentiation of $h^2 = ax + b$ allows us to solve for a,

$$a = 2h\frac{dh}{dx}$$

From Darcy's law,

$$h\frac{dh}{dx} = -\frac{q}{K}$$

so, by substitution

$$h^2 = h_0^2 - \frac{2qx}{K}$$

Setting $h = h_L$ at $x = L$ and neglecting flow across the seepage face yields

$$h_L^2 = h_0^2 - \frac{2qL}{K}$$

Rearrangement gives

$$q = \frac{K}{2L}\left(h_0^2 - h_L^2\right) \qquad \text{Dupuit Equation}$$

Then the general equation for the shape of the parabola is

$$h^2 = h_0^2 - \frac{x}{L}\left(h_0^2 - h_L^2\right) \qquad \text{Dupuit Parabola}$$

The derivation of the Dupuit equations in Example 2.2 does not consider recharge to the aquifer. For the case of a system with recharge, the **Dupuit parabola** will take the mounded shape shown in Figure 2.15. The point where $h = h_{max}$ is known as the **water divide.** At the water divide, $q = 0$ since the gradient is zero. Example 2.3 derives the Dupuit equation for recharge and illustrates the use of the water divide concept.

Example 2.3 DUPUIT EQUATION WITH RECHARGE W [L/T]

Two rivers located 1000 m apart fully penetrate an aquifer (Figure 2.15). The aquifer has a K value of 0.5 m/day. The region receives an average rainfall of 15 cm/yr and evaporation is about 10 cm/yr. Assume that the water elevation in River 1 is 20 m and the water elevation in River 2 is 18 m. Determine the daily discharge per meter width into each river.

The Dupuit equations with recharge W becomes:

$$h^2 = h_0^2 + \frac{\left(h_L^2 - h_0^2\right)}{L}x + \frac{Wx}{K}(L - x) \qquad \text{Dupuit Parabola.}$$

This equation will give the shape of the Dupuit parabola shown in Figure 2.15. If $W =$ 0, this equation will reduce to the parabolic equation found in Example 2.2, and

$$q = \frac{K}{2L}\left(h_0^2 - h_L^2\right) + W\left(x - \frac{L}{2}\right)$$

This equation will give the shape of the Dupuit parabola shown in Figure 2.15. If $W = 0$, this equation will reduce to the parabolic equation found in Example 2.2, and

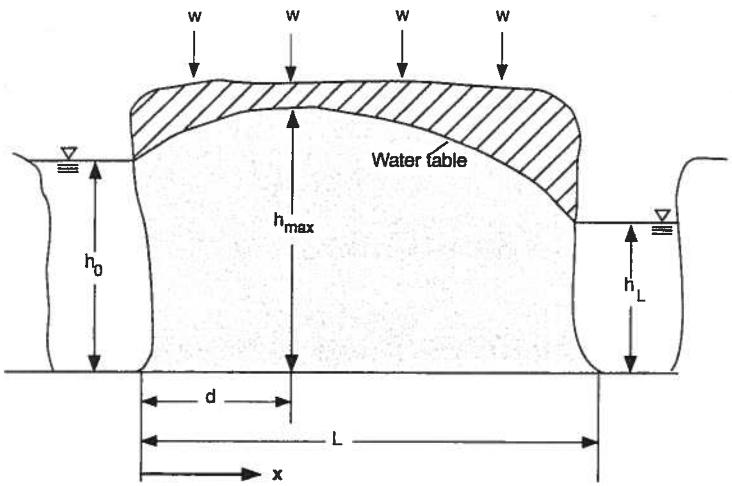

Figure 2.15 Dupuit parabola with recharge.

$$q = \frac{K}{2L}\left(h_0^2 - h_L^2\right) + W\left(x - \frac{L}{2}\right)$$

Given

$L = 1000$ m,
$K = 0.5$ m/day,
$h_0 = 20$ m,
$h_L = 18$ m, and
$W = 5$ cm/yr $= 1.369 \times 10^{-4}$ m/day.

For discharge into River 1, set $x = 0$ m:

$$q = \frac{K}{2L}\left(h_0^2 - h_L^2\right) + W\left(0 - \frac{L}{2}\right)$$

$$= \frac{0.5\text{m/day}}{(2)(1000\text{m})}\left(20^2\text{m}^2 - 18^2\text{m}^2\right) + \left(1.369\times10^{-4}\text{m/day}\right)(-1000\text{m}/2)$$

$$q = -0.05 \text{ m}^2/\text{day}$$

The negative sign indicates that flow is in the opposite direction from the x direction. Therefore,

$$q = 0.05 \text{ m}^2/\text{day into river 1}$$

For discharge into River 2, set $x = L = 1000$ m:

$$q = \frac{K}{2L}\left(h_o^2 - h_L^2\right) + W\left(L - \frac{L}{2}\right)$$

$$= \frac{0.5\text{m/day}}{(2)(1000\text{m})}\left(20^2\text{m}^2 - 18^2\text{m}^2\right) + \left(1.369\times10^{-4}\text{m/day}\right)\left(1000\text{m} - \frac{1000\text{m}}{2}\right)$$

$$q = 0.087 \text{ m}^2/\text{day into river 2}$$

By setting $q = 0$ at the divide and solving for x_d, the water divide is located 361.2 m from the edge of River 1 and is 20.9 m high.

2.6 STREAMLINES AND EQUIPOTENTIAL LINES

The formal mathematical definition of the flow net can be derived using the equation of continuity for steady, incompressible, isotropic flow in two dimensions. The concept of velocity potential and a stream function was presented concisely by De Wiest (1965) in a classic text on geohydrology. The continuity equation states

$$\frac{\partial u}{dx} + \frac{\partial v}{\partial y} = 0$$

The governing steady-state flow equation is

$$\nabla^2 h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{2.29}$$

The **velocity potential** ϕ is a scalar function and can be written

$$\phi(x, y) = -K(z + p/\gamma) + c \tag{2.30}$$

where K and c are assumed constant. From Darcy's law in two dimensions,

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \tag{2.31}$$

Using Eq. (2.29) in two dimensions, we have

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2.32}$$

where $\phi(x,y) =$ constant represents a family of equipotential curves on a two-dimensional surface. It can be shown that the **stream function** $\psi(x,y) =$ constant is orthogonal to $\phi(x,y)$ = constant and that both satisfy the equation of continuity and Laplace's equation. The stream function $\psi(x,y)$ is defined by

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \tag{2.33}$$

Combining Eqs. (2.31) and (2.33), the Cauchy-Riemann equations become

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{2.34}$$

It can further be shown that ψ also satisfies Laplace's equation

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{2.35}$$

Example 2.4 STREAMLINES AND EQUIPOTENTIAL LINES

Prove that ψ and ϕ are orthogonal for isotropic flow, given Figure 2.16, where $V = (u,v)$ is a velocity vector tangent to ψ_2. Show that flow between two streamlines is constant.

Solution. We can write for the slope of the streamline

$$\frac{v}{u} = \frac{dy}{dx} = \tan \alpha$$

and

$$vdx - udy = 0$$

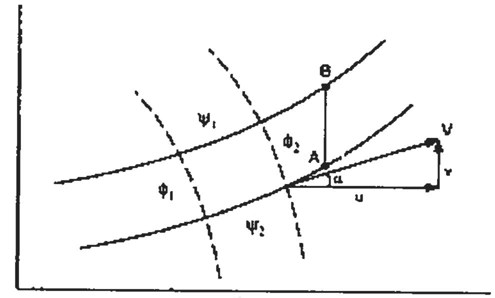

Figure 2.16 Streamlines and equipotential lines.

Since

$$u = \frac{d\psi}{dy} \qquad v = -\frac{d\psi}{dx}$$

then

$$\frac{d\psi}{dx}dx + \frac{d\psi}{dy}dy = 0$$

or $d\psi(x,y) = 0$. The total differential equals zero, and $\psi(x,y)$ = constant, as required by the stream function. The ϕ_1 and ϕ_2 lines represent equipotential lines and can be represented by the total differential

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$

Substituting for $\partial\phi/\partial x$ and $\partial\phi/\partial y$ from Darcy's law produces $udx + vdy = 0$, which can be solved for

$$\frac{dy}{dx} = -\frac{u}{v}, \qquad \text{the slope of the equipotential line.}$$

Thus, since the two slopes are negative inverses of one another, equipotential lines are normal to streamlines for a homogeneous isotropic system. The system of orthogonal lines forms a flow net, as described earlier in this chapter.

Consider the flow crossing a vertical section AB between streamlines ψ_1 and ψ_2. The discharge across the section is designated Q, and it is apparent from fluid mechanics that

$$Q = \int_{\psi_2}^{\psi_1} u\,dy$$
$$= \int_{\psi_2}^{\psi_1} d\psi$$
$$= \psi_1 - \psi_2$$

Thus, the flow between streamlines is constant, and the spacing between streamlines reveals the relative magnitude of flow velocities between them. Once either streamlines or equipotential lines are determined in a domain, the other can be evaluated from the Cauchy-Riemann equations Eq. (2.34). Thus,

$$\psi = \int \left(\frac{\partial \phi}{\partial x} dy - \frac{\partial \phi}{\partial y} dx \right)$$

and

$$\phi = \int \left(\frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy \right)$$

Example 2.5 FLOW FIELD CALCULATION

A flow field is defined by $u = 2x$ and $v = -2y$. Find the stream function and potential function for this flow and sketch the flow net.

Solution. For this example define

$$d\psi = -v dx + u dy$$
$$d\phi = -u dx - v dy$$

Continuity requires that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2 = 0$$

The stream function becomes

$$d\psi = -v dx + u dy = 2y\,dx + 2x\,dy$$

or

$$\psi = 2xy + C_1$$

where C_1 is a constant. The velocity potential ϕ exists if flow is irrotational and $\partial v/\partial x - \partial u/\partial y = 0$, which is satisfied in this case. Thus,

$$d\phi = -u dx - v dy = -2x dx + 2y dy$$

$$\boxed{\phi = -(x^2 - y^2) + C_2}$$

One can sketch the flow field by substituting values of ψ into the expression $\psi = 2xy$. For $\psi = 60$, $x = 30/y$ and for $\phi = 60$ we have $x = \pm\sqrt{y^2 - 60}$. Figure 2.17 shows the flow field for the case of flow in a corner.

2.7 UNSATURATED FLOW AND THE WATER TABLE

Hydraulic conductivity $K(\theta)$ in the unsaturated zone above the water table relates velocity and hydraulic gradient in Darcy's law. **Moisture content** θ is defined as the ratio of the volume of water to the total volume of a unit of porous media. To complicate the analysis of unsaturated flow, the moisture content θ and the hydraulic conductivity K are functions of the capillary suction ψ. Also, it has been observed experimentally that the θ–ψ relationships differ significantly for different types of soil. Figure 2.2 shows the characteristic drying and wetting curves that occur in soils which are draining water or receiving infiltration of water. The nonlinear nature of these curves greatly complicates analyses in the unsaturated zone (see Figure 9.1).

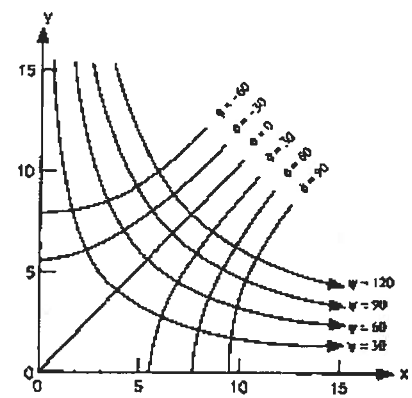

Figure 2.17 Flow net for flow in a corner.

The **water table** defines the boundary between the unsaturated and saturated zones and is defined by the surface on which the fluid pressure P is exactly atmospheric, or $P = 0$. Hence, the total hydraulic head $h = \psi + z$, where $\psi = P/\rho g$, the pressure head. For saturated ground water flow, θ equals the **porosity** n of the sample, defined as the ratio of volume of voids to total volume of sample; for unsaturated flow above a water table, $\theta < n$.

Darcy's law is used with the unsaturated value for K and can be written

$$v = -K(\theta)\frac{\partial h}{\partial z} \tag{2.36}$$

where

v = Darcy velocity,
z = depth below surface,
h = potential or head = $z + \psi$,
ψ = tension or suction,
$K(\theta)$ = unsaturated hydraulic conductivity, and
θ = volumetric moisture content.

Near the water table, a capillary fringe can occur where ψ is a small negative pressure corresponding to the air entry pressure. This capillary zone is small for sandy soils but can be up to two meters in depth for fine grained soils. By definition, pressure head is negative (under tension) at all points above the water table and is positive for points below the water table. The value of ψ is greater than zero in the saturated zone below the water table and equals zero at the water table. Soil physicists refer to $\psi < 0$ as the tension head or capillary suction head, and it can be measured in the laboratory or field by an instrument called a tensiometer.

To summarize the properties of the unsaturated zone as compared to the saturated zone, Freeze and Cherry (1979) state that for the unsaturated zone:

1. It occurs above the water table and above the capillary fringe.
2. The soil pores are only partially filled with water; the moisture content θ is less than the porosity n.
3. The fluid pressure P is less than atmospheric; the pressure head ψ is less than zero.
4. The hydraulic head h must be measured with a tensiometer.
5. The hydraulic conductivity K and the moisture content θ are both functions of the pressure head ψ.

More details on the unsaturated zone can be found in Chapter 9, where both flow and transport in the unsaturated zone are described along with applications of analytical and numerical methods, and in Guymon (1994).

SUMMARY

Chapter 2 has presented mechanisms of ground water flow in the subsurface. Aquifer characteristics such as hydraulic conductivity, hydraulic gradient and porosity are defined and used in governing equations of flow. Both steady-state and transient saturated flow equations are derived for confined and unconfined aquifers. The Dupuit equation is derived and applied to seepage examples. Flow net theory is derived and several applications are presented for local ground water problems. The chapter ends with a brief introduction to the unsaturated zone.

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