Hydraulic Conductivity Tensor (for anisotropic media)

Darcy's Law can be expressed as: $\vec{q} = -\vec{K} \bullet \nabla h$ where $\vec{q}$ is the specific discharge vector with components q_x, q_y, q_z in the directions of the Cartesian x, y, and z coordinates, and $-\nabla h$ is the hydraulic gradient with components $-\frac{\partial h}{\partial x}$, $-\frac{\partial h}{\partial y}$, and $-\frac{\partial h}{\partial z}$, in the x, y, and z directions.

When the flow is through a homogeneous, isotropic medium, K is a constant scalar. However, in an anisotropic porous medium, K is a symmetric second rank tensor of the form

$$|\vec{K}| = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix} \text{ for three-dimensional spaces,}$$

and $K_{xy} = K_{yx}$, $K_{xz} = K_{zx}$, and $K_{yz} = K_{zy}$.

For this case, Darcy's Law takes the form

$$q_x = -\left(K_{xx}\frac{\partial h}{\partial x} + K_{xy}\frac{\partial h}{\partial y} + K_{xz}\frac{\partial h}{\partial z} \right),$$

$$q_y = -\left(K_{yx}\frac{\partial h}{\partial x} + K_{yy}\frac{\partial h}{\partial y} + K_{yz}\frac{\partial h}{\partial z} \right),$$

$$q_z = -\left(K_{zx}\frac{\partial h}{\partial x} + K_{zy}\frac{\partial h}{\partial y} + K_{zz}\frac{\partial h}{\partial z} \right).$$

From tensor analysis, it is always possible to find three mutually orthogonal directions in space such that when these directions are chosen as the coordinate system for expressing the components K_{ij}, $K_{ij} = 0$ for all $i \neq j$ and $K_{ij} \neq 0$ for $i = j$. These direction in space are called the principal directions of the permeability of the anisotropic porous medium.

When the principal directions are used as the coordinate system, $|\vec{K}| = \begin{bmatrix} K_{xx} & 0 & 0 \\ 0 & K_{yy} & 0 \\ 0 & 0 & K_{zz} \end{bmatrix}$,

and Darcy's Law reduces to

$$q_x = -K_x \frac{\partial h}{\partial x},$$

$$q_y = -K_y \frac{\partial h}{\partial y},$$

$$q_z = -K_z \frac{\partial h}{\partial z}$$

where $K_{xx} = K_x$, $K_{yy} = K_y$, and $K_{zz} = K_z$.

References:

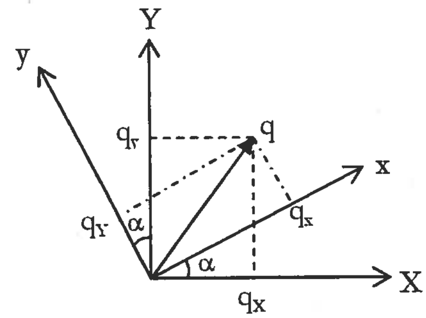
Bear, J. and Verruijt, A., *Modeling Groundwater Flow and Pollution*, Dordrecht, Reidel, 1987.

Domenico, P.A., and Schwartz, F.W., *Physical and Chemical Hydrogeology*, 2nd ed., New York, Wiley, 1998.

Fetter, C.W., *Contaminant Hydrogeology*, 2nd ed., Upper Saddle River, Prentice Hall, 1999.

Rotation of Axes in Anisotropic Flow Fields

Since K is a second rank tensor, the transformation of its components from the coordinate system with principal axes (X,Y,Z) into components in a coordinate system with axes (x,y,z) is obtained by rotation.



Principal Axes (X,Y)

$$q_X = -K_{XX}\frac{\partial h}{\partial X}$$

$$q_Y = -K_{YY}\frac{\partial h}{\partial Y}$$

Rotate axes by angle α to get axes (x,y)

$x = X\cos\alpha + Y\sin\alpha$

$y = Y\cos\alpha - X\sin\alpha$

Also, from the diagram,

$$q_x = q_X\cos\alpha + q_Y\sin\alpha = -K_{XX}\frac{\partial h}{\partial X}\cos\alpha - K_{YY}\frac{\partial h}{\partial Y}\sin\alpha$$

$$q_y = q_Y\cos\alpha - q_X\sin\alpha = -K_{YY}\frac{\partial h}{\partial Y}\cos\alpha + K_{XX}\frac{\partial h}{\partial X}\sin\alpha$$

Now, using the chain rule,

$$\frac{\partial h}{\partial X} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial X} = \frac{\partial h}{\partial x}\cos\alpha + \frac{\partial h}{\partial y}(-\sin\alpha)$$

$$\frac{\partial h}{\partial Y} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial Y} = \frac{\partial h}{\partial x}\sin\alpha + \frac{\partial h}{\partial y}\cos\alpha$$

where the values of $\frac{\partial x}{\partial X}, \frac{\partial y}{\partial X}, \frac{\partial x}{\partial Y}$, and $\frac{\partial y}{\partial Y}$ were obtained from the above rotation equations for x and y.

Substituting $\frac{\partial h}{\partial X}, \frac{\partial h}{\partial Y}$ into q_x, q_y:

$$q_x = -K_{XX}\left[\frac{\partial h}{\partial x}\cos\alpha - \frac{\partial h}{\partial y}\sin\alpha\right]\cos\alpha - K_{YY}\left[\frac{\partial h}{\partial x}\sin\alpha + \frac{\partial h}{\partial y}\cos\alpha\right]\sin\alpha$$

$$q_x = -\left[K_{XX}\cos^2\alpha + K_{YY}\sin^2\alpha\right]\frac{\partial h}{\partial x} - \left[(K_{YY} - K_{XX})\sin\alpha\cos\alpha\right]\frac{\partial h}{\partial y}$$

and

$$q_y = -K_{YY}\left[\frac{\partial h}{\partial x}\sin\alpha + \frac{\partial h}{\partial y}\cos\alpha\right]\cos\alpha + K_{XX}\left[\frac{\partial h}{\partial x}\cos\alpha - \frac{\partial h}{\partial y}\sin\alpha\right]\sin\alpha$$

$$q_y = \left[(K_{XX} - K_{YY})\sin\alpha\cos\alpha\right]\frac{\partial h}{\partial x} - \left[K_{XX}\sin^2\alpha + K_{YY}\cos^2\alpha\right]\frac{\partial h}{\partial y}$$

By definition,

$$q_x = -K_{xx}\frac{\partial h}{\partial x} - K_{xy}\frac{\partial h}{\partial y}$$

So,

$$K_{xx} = K_{XX}\cos^2\alpha + K_{YY}\sin^2\alpha$$

using the trigonometric expression

$$a\cos^2\alpha + b\sin^2\alpha = \frac{1}{2}(a+b) + \frac{1}{2}(a-b)\cos 2\alpha$$

$$K_{xx} = \frac{1}{2}(K_{XX} + K_{YY}) + \frac{1}{2}(K_{XX} - K_{YY})\cos 2\alpha$$

$$K_{xy} = \frac{1}{2}(K_{YY} - K_{XX})\sin\alpha\cos\alpha$$

using the trigonometric expression

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$$

$$K_{xy} = \frac{1}{2}(K_{YY} - K_{XX})\sin 2\alpha$$

Similarly,

$$q_y = -K_{yx}\frac{\partial h}{\partial x} - K_{yy}\frac{\partial h}{\partial y}$$

$$K_{yx} = K_{xy}$$

$$K_{yy} = K_{YY}\cos^2\alpha + K_{XX}\sin^2\alpha$$

$$K_{yy} = \frac{1}{2}(K_{XX} + K_{YY}) - \frac{1}{2}(K_{XX} - K_{YY})\cos 2\alpha$$

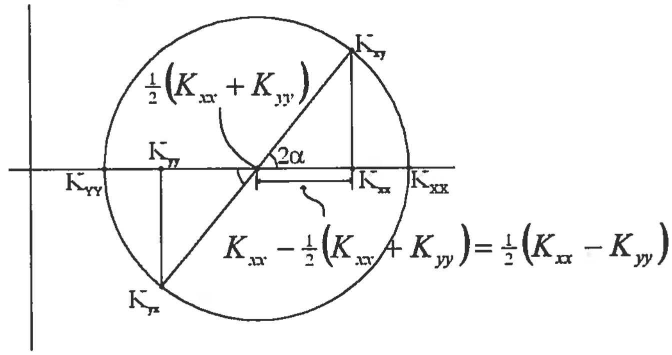
In summary,
For principal axes

$$|\vec{K}| = \begin{bmatrix} K_{XX} & 0 \\ 0 & K_{YY} \end{bmatrix}$$

Rotate by angle α

$$|\vec{K}| = \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix}$$ related to K_{XX}, K_{YY} by above equations.

For the opposite situation, to find the components of K for the principal axes (X,Y) given the (x,y) system, Mohr's Circle can be used.



So, $\tan 2\alpha = \dfrac{K_{xy}}{\frac{1}{2}(K_{xx} - K_{yy})}$

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2K_{xy}}{K_{xx} - K_{yy}}\right)$$

Can show that

$$K_{XX} = \frac{K_{xx} + K_{yy}}{2} + \left[\left(\frac{K_{xx} - K_{yy}}{2}\right)^2 + K_{xy}^{\ 2}\right]^{\frac{1}{2}}$$

$$K_{YY} = \frac{K_{xx} + K_{yy}}{2} - \left[\left(\frac{K_{xx} - K_{yy}}{2}\right)^2 + K_{xy}^{\ 2}\right]^{\frac{1}{2}}$$

$$K_{XY} = 0$$