CE 5361 Surface Water Hydrology
Unit Hydrographs - II
Analysis and Applications

Contents

1 Unit Hydrographs - Analysis Principles 1

2 Unit Hydrographs How To Construct from Observations 2
2.1 Back-substitution 2
2.2 Optimization Approach - Solving the Normal Equations 4

3 Using the UH 6
3.1 Convolution/De-Convolution 7

4 References 11

1 Unit Hydrographs - Analysis Principles

A unit hydrograph (UH) is the DRH that results from one unit of excess precipitation depth uniformity applied over the watershed over some specified duration of time. Typically the depth excess is either 1.0 inches or 1.0 cm. The idea of a unit hydrograph is that if one can determine an average unit hydro- graph for a watershed then one can predict the DRH for any rainfall excess pattern over the same unit time by scaling the unit hydrograph response with the arbitrary excess precipitation depth. For instance, if a UH for a 1.0 cm, 1.0 hr. excess precipitation depth is available, then the DRH for a 2.0 cm precipitation depth would look like the UH with the vertical scale multiplied by a factor of 2.0. The method was originally applied to relatively large watersheds, 3000-4000 square kilometers but seems to work acceptably for watersheds as small as 100 hectares.

Developing a UH requires several procedural steps. First is to obtain a precipitation time series and runoff time series for the target watershed. Obviously, if there are no data, then some other methods must be applied. Assuming a single intensity event can be found then the hydrograph is separated into base flow and DRH components. The total volume in the DRH is determined by accumulating the incremental (instantaneous) discharges then dividing by the watershed area to obtain the cumulative DRH depth. The precipitation event is scaled by an appropriate method (Loss model) so its cumulative depth is the same as the DRH cumulative depth. The resulting DRH is a unit hydrograph for the scaled precipitation depth. Next, the DRH and precipitation are rescaled to a meaningful unit precipitation depth , say 1.0 cm.

Multiple peaked storms are treated using a convolution-deconvolution approach (discussed later).

2 Unit Hydrographs How To Construct from Observations

Naturally we usually dont know the underlying response functions in advance of an analysis, and usually have to infer the results from actual data. The classical approach is illustrated in the following discussion.

Consider the following observed data for some watershed in Figure 1.

The discrete equation of the DRH is expressed as

$$Q_n = \sum_{n=1}^{m<M} P_m U_{n-m+1} \tag{1}$$

This equation states that the observed discharge at time increment n (Q_n) is a linear combination of all precipitation pulses (P_m) prior to and including the time increment n. The weights (U) are called the discrete unit hydrograph (analogous to the kernel function). This equation is exactly analogous to a discrete convolution equation. Using the example data, we would write 11 equations for the discharge function. Note the equation structure and the dependence on prior precipitation as depicted in Figure 2

Solution of this system for the unknown weights provides the unitgraph. It is a little easier to express as a vector-matrix equation (because the solution becomes obvious)

Now the objective is to find solutions to this linear system (of equations). We need to specify [U] in a fashion that best explains the data. Various methods include simple back-substitution (can produce negative weights), linear programming, non-linear programming, least-squares and successive iteration. We will illustrate with back-substitution and least squares matrix algebra.

2.1 Back-substitution

Straightforward. Solve each equation successively (back substitute) for U .
$U_1 = Q_1/P_1 = 428/1.06 = 404 cfs/in.$

Unit Hydrograph (Classical Example)				
0.5	1	1.06	428	
1	2	1.93	1923	
1.5	3	1.81	5297	
2	4		9131	
2.5	5		10625	
3	6		7834	
3.5	7		3921	
4	8		1846	
4.5	9		1402	
5	10		830	
5.5	11		313	

Figure 1: Data for UH application example

$U2 = (Q2 - P_2U_1)/P1 = (1923 - 1.93 * 404)/1.06 = 1079cfs/in.$
And so on . . . .

Illustrated in a spreadsheet environment the arithmetic is relatively simple as in Figure 4. The spreadsheet in Figure 13 is the result of successive back-substitution. This method does not always work, as the linear system is over-determined (more equations than unknowns).

$$
\begin{aligned}
Q_1 &= P_1U_1 \\
Q_2 &= P_2U_1 + P_1U_2 \\
Q_3 &= P_3U_1 + P_2U_2 + P_1U_3 \\
Q_4 &= + P_3U_2 + P_2U_3 + P_1U_4 \\
Q_5 &= + P_3U_3 + P_2U_4 + P_1U_5 \\
Q_6 &= + P_3U_4 + P_2U_5 + P_1U_6 \\
Q_7 &= + P_3U_5 + P_2U_6 + P_1U_7 \\
Q_8 &= + P_3U_6 + P_2U_7 + P_1U_8 \\
Q_9 &= + P_3U_7 + P_2U_8 + P_1U_9 \\
Q_{10} &= + P_3U_8 + P_2U_9 + P_1U_{10} \\
Q_{11} &= + P_3U_9 + P_2U_{10} + P_1U_{11}
\end{aligned}
$$

Figure 2: UH Equation Array (Discrete Convolution)

$$
\begin{bmatrix} P_1 & & & \\ P_2 & P_1 & & \\ & & \ddots & \\ & & & P_M \end{bmatrix} \bullet \begin{bmatrix} U_1 \\ U_2 \\ \\ U_{N-M+1} \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ \vdots \\ Q_N \end{bmatrix}
$$

Figure 3: UH Equation Array (Vector-Matrix)

2.2 Optimization Approach - Solving the Normal Equations

The least squares solution to the matrix equation is
, $[\mathbf{U}] = [[\mathbf{P}^T][\mathbf{P}]]^{-1}[\mathbf{P}^T][\mathbf{Q}]$

Again using a spreadsheet the result is displayed in Figure 5. This method also sometimes fails, but it can be completely automated (no brains required - unless it fails then a lot of brains are required to figure out what went wrong).

Three other approaches in common practice are optimization using linear programming (Danzig's algorithm) - excess and deficits are summed and minimized; non-linear programming (essentially a variation of the least-squares, but can constrain solution space); and

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
1	Unit Hydrograph (Back-Substitute)																					
2	Observations						[P]												[U]	[Q*]	[Q]-[Q*]	
3	Time(hrs)	Time (increment)	Excess Rain (in)	Direct Runoff (cfs)			1	2	3	4	5	6	7	8	9	10	11					
4	0.5	1	1.06	428		1	1.06	0	0	0	0	0	0	0	0	0	0		403.77	428	-0.000228	403.774
5	1	2	1.93	1923		2	1.93	1.06	0	0	0	0	0	0	0	0	0		1079	1923.02	-0.023434	1078.98
6	1.5	3	1.81	5297		3	1.81	1.93	1.06	0	0	0	0	0	0	0	0		2343.1	5297	0	2343.11
7	2	4	0	9131		4	0	1.81	1.93	1.06	0	0	0	0	0	0	0		2505	9130.5	0.5025618	2505.5
8	2.5	5	0	10625		5	0	0	1.81	1.93	1.06	0	0	0	0	0	0		1461	10624.3	0.6560813	1461.66
9	3	6	0	7834		6	0	0	0	1.81	1.93	1.06	0	0	0	0	0		453	7833.96	0.04	453.04
10	3.5	7	0	3921		7	0	0	0	0	1.81	1.93	1.06	0	0	0	0		379.5	3920.97	0.03	379.53
11	4	8	0	1846		8	0	0	0	0	0	1.81	1.93	1.06	0	0	0		276.9	1845.88	0.121	277.021
12	4.5	9	0	1402		9	0	0	0	0	0	0	1.81	1.93	1.06	0	0		170.5	1402.04	-0.042	170.458
13	5	10	0	830		10	0	0	0	0	0	0	0	1.81	1.93	1.06	0		-0.47	829.756	0.2442	-0.2258
14	5.5	11	0	313		11	0	0	0	0	0	0	0	0	1.81	1.93	1.06		5.32	313.337	-0.3371	4.9829
15																						
16										Back substitute (lazy way)												
17										(1) Guess U												
18										(2) Compute Q*												
19										(3) Compute difference												
20										(4) Adjust U one-by-one to get difference small												
21										(5) Stop when difference is small percent of actual Q												
22										(6) If SOLVER works on machine - can automate - crashes on Mac (Microsoft hates Mac!)												

Figure 4: Backsubstitution in a spreadsheet

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1	Unit Hydrograph (Least Squares Example)																		
2	Observations						[P]												[U]
3	Time(hrs)	Time (increment)	Excess Rain (in)	Direct Runoff (cfs)	=MMULT(MINVERSE(MMULT(G18:Q28,G4:Q14)),MMULT(G18:Q28,D4:D14))		1	2	3	4	5	6	7	8	9	10	11		
4	0.5	1	1.06	428		1	1.06	0	0	0	0	0	0	0	0	0	0		403.774
5	1	2	1.93	1923		2	1.93	1.06	0	0	0	0	0	0	0	0	0		1078.98
6	1.5	3	1.81	5297		3	1.81	1.93	1.06	0	0	0	0	0	0	0	0		2343.15
7	2	4	0	9131		4	0	1.81	1.93	1.06	0	0	0	0	0	0	0		2505.44
8	2.5	5	0	10625		5	0	0	1.81	1.93	1.06	0	0	0	0	0	0		1460.75
9	3	6	0	7834		6	0	0	0	1.81	1.93	1.06	0	0	0	0	0		452.74
10	3.5	7	0	3921		7	0	0	0	0	1.81	1.93	1.06	0	0	0	0		380.425
11	4	8	0	1846		8	0	0	0	0	0	1.81	1.93	1.06	0	0	0		275.774
12	4.5	9	0	1402		9	0	0	0	0	0	0	1.81	1.93	1.06	0	0		170.931
13	5	10	0	830		10	0	0	0	0	0	0	0	1.81	1.93	1.06	0		0.89846
14	5.5	11	0	313		11	0	0	0	0	0	0	0	0	1.81	1.93	1.06		1.77518
15																			
16							[P]-transpose												
17																			
18						1	1.06	1.93	1.81	0	0	0	0	0	0	0	0		
19						2	0	1.06	1.93	1.81	0	0	0	0	0	0	0		
20						3	0	0	1.06	1.93	1.81	0	0	0	0	0	0		
21					=TRANSPOSE(G4:	4	0	0	0	1.06	1.93	1.81	0	0	0	0	0		
22						5	0	0	0	0	1.06	1.93	1.81	0	0	0	0		
23						6	0	0	0	0	0	1.06	1.93	1.81	0	0	0		
24						7	0	0	0	0	0	0	1.06	1.93	1.81	0	0		
25						8	0	0	0	0	0	0	0	1.06	1.93	1.81	0		
26						9	0	0	0	0	0	0	0	0	1.06	1.93	1.81		
27						10	0	0	0	0	0	0	0	0	0	1.06	1.93		
28						11	0	0	0	0	0	0	0	0	0	0	1.06		

Figure 5: Least-Squares Minimization (by Normal Equations) in a spreadsheet

pattern searching (also a constrained approach).

3 Using the UH

Now once the unitgraph is determined any future precipitation signal can be passed through the system (of equations) to predict the DRH. First, we will pass the original precipitation signal (the three pulses) through the two unitgraphs to see how they differ in predicting watershed response. Recall that this rainfall signal was used to generate the unit graph, thus the response should be close to the observed response.

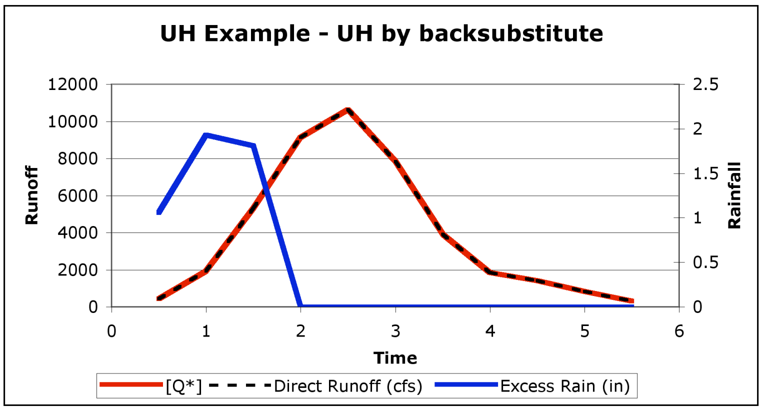


Figure 6: UH Comparison Backsubstitute

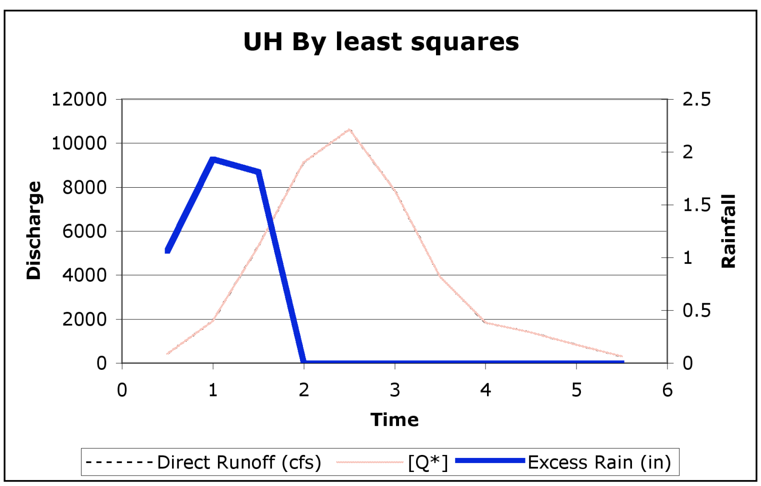


Figure 7: UH Comparison Least Squares

The markers are the original runoff hydrograph while the lines are the DRH predicted by the two unitgraphs. In this example the least-squares approach makes a faithful reconstruction.

Just getting the unit graph is meaningless unless we intend to use it. In the present example, if we use the unitgraph for different rainfall signals we can predict the direct runoff hydrograph for these events.

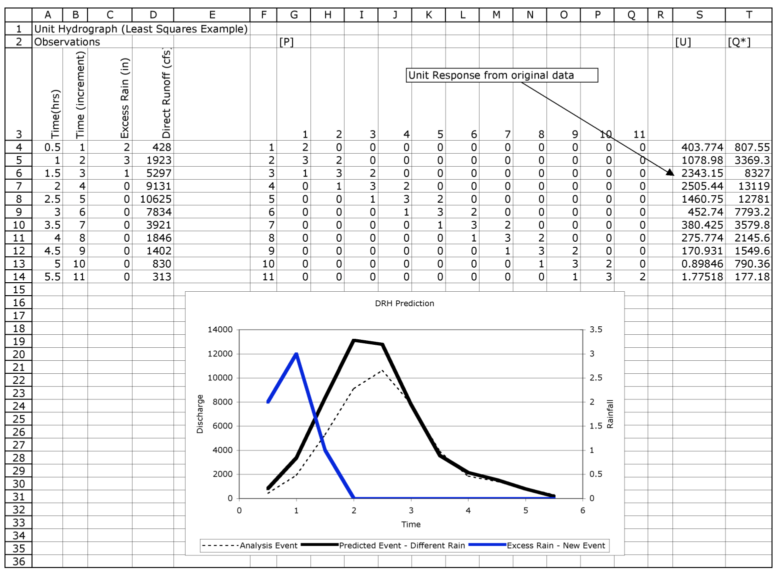
For example suppose we wish to evaluate the DRH for

P=[2.00,3.00,1.00]

P=[5.00,0.00,0.00]

P=[0.00,0.00,5.00]

Then we simply evaluate the matrix equation [Q]=[P][U] with different [P] matrices. (Results in figures)

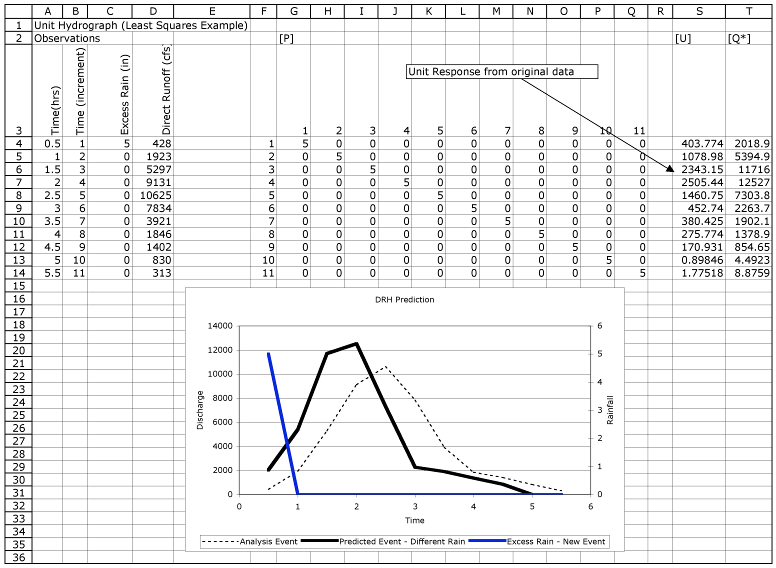


	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	Unit Hydrograph (Least Squares Example)																			
2	Observations						[P]												[U]	[Q*]
3	Time(hrs)	Time (increment)	Excess Rain (in)	Direct Runoff (cfs)			1	2	3	4	5	6	7	8	9	10	11			
4	0.5	1	2	428		1	2	0	0	0	0	0	0	0	0	0	0		403.774	807.55
5	1	2	3	1923		2	3	2	0	0	0	0	0	0	0	0	0		1078.98	3369.3
6	1.5	3	1	5297		3	1	3	2	0	0	0	0	0	0	0	0		2343.15	8327
7	2	4	0	9131		4	0	1	3	2	0	0	0	0	0	0	0		2505.44	13119
8	2.5	5	0	10625		5	0	0	1	3	2	0	0	0	0	0	0		1460.75	12781
9	3	6	0	7834		6	0	0	0	1	3	2	0	0	0	0	0		452.74	7793.2
10	3.5	7	0	3921		7	0	0	0	0	1	3	2	0	0	0	0		380.425	3579.8
11	4	8	0	1846		8	0	0	0	0	0	1	3	2	0	0	0		275.774	2145.6
12	4.5	9	0	1402		9	0	0	0	0	0	0	1	3	2	0	0		170.931	1549.6
13	5	10	0	830		10	0	0	0	0	0	0	0	1	3	2	0		0.89846	790.36
14	5.5	11	0	313		11	0	0	0	0	0	0	0	0	1	3	2		1.77518	177.18

Figure 8: DRH using P=[2.00,3.00,1.00]

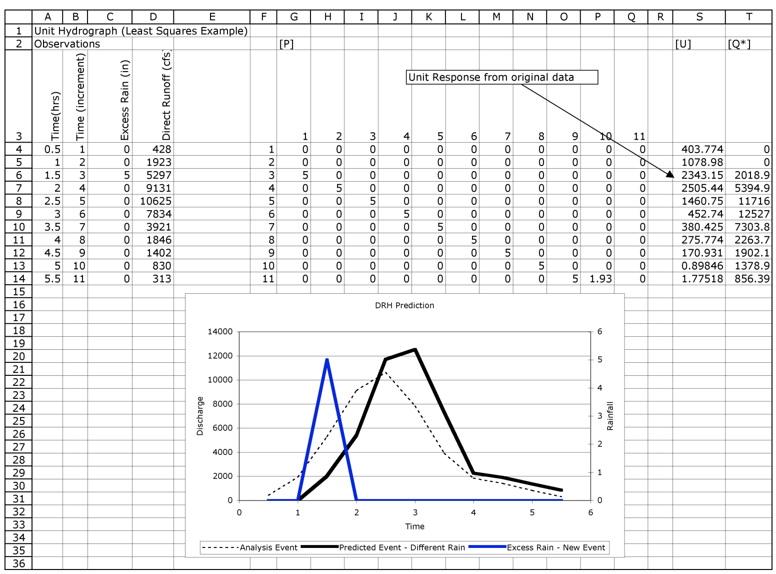
3.1 Convolution/De-Convolution

Complicated storms can be analyzed as a sequence of unit precipitation events each event producing its own unit hydrograph response. These sequences, each displaced in time corresponding to the event that produced the response, are summed together to produce the response for the complex event. This response-summation procedure is called convolution. (In the classic examples above, the summation of impulses is in fact convolution). The reverse of the process is called de-convolution (again in the example above, the determination of the unit weights is de-convolution). It is implicitly assumed that each event in the sequence has the same duration (time base); it is also assumed that the response can



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	Unit Hydrograph (Least Squares Example)																			
2	Observations						[P]												[U]	[Q*]
3	Time(hrs)	Time (increment)	Excess Rain (in)	Direct Runoff (cfs)			1	2	3	4	5	6	7	8	9	10	11			
4	0.5	1	5	428		1	5	0	0	0	0	0	0	0	0	0	0		403.774	2018.9
5	1	2	0	1923		2	0	5	0	0	0	0	0	0	0	0	0		1078.98	5394.9
6	1.5	3	0	5297		3	0	0	5	0	0	0	0	0	0	0	0		2343.15	11716
7	2	4	0	9131		4	0	0	0	5	0	0	0	0	0	0	0		2505.44	12527
8	2.5	5	0	10625		5	0	0	0	0	5	0	0	0	0	0	0		1460.75	7303.8
9	3	6	0	7834		6	0	0	0	0	0	5	0	0	0	0	0		452.74	2263.7
10	3.5	7	0	3921		7	0	0	0	0	0	0	5	0	0	0	0		380.425	1902.1
11	4	8	0	1846		8	0	0	0	0	0	0	0	5	0	0	0		275.774	1378.9
12	4.5	9	0	1402		9	0	0	0	0	0	0	0	0	5	0	0		170.931	854.65
13	5	10	0	830		10	0	0	0	0	0	0	0	0	0	5	0		0.89846	4.4923
14	5.5	11	0	313		11	0	0	0	0	0	0	0	0	0	0	5		1.77518	8.8759

Figure 9: DRH using P=[5.00,0.00,0.00]



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	Unit Hydrograph (Least Squares Example)																			
2	Observations						[P]												[U]	[Q*]
3	Time(hrs)	Time (increment)	Excess Rain (in)	Direct Runoff (cfs)			1	2	3	4	5	6	7	8	9	10	11			
4	0.5	1	0	428		1	0	0	0	0	0	0	0	0	0	0	0		403.774	0
5	1	2	0	1923		2	0	0	0	0	0	0	0	0	0	0	0		1078.98	0
6	1.5	3	5	5297		3	5	0	0	0	0	0	0	0	0	0	0		2343.15	2018.9
7	2	4	0	9131		4	0	5	0	0	0	0	0	0	0	0	0		2505.44	5394.9
8	2.5	5	0	10625		5	0	0	5	0	0	0	0	0	0	0	0		1460.75	11716
9	3	6	0	7834		6	0	0	0	5	0	0	0	0	0	0	0		452.74	12527
10	3.5	7	0	3921		7	0	0	0	0	5	0	0	0	0	0	0		380.425	7303.8
11	4	8	0	1846		8	0	0	0	0	0	5	0	0	0	0	0		275.774	2263.7
12	4.5	9	0	1402		9	0	0	0	0	0	0	5	0	0	0	0		170.931	1902.1
13	5	10	0	830		10	0	0	0	0	0	0	0	5	0	0	0		0.89846	1378.9
14	5.5	11	0	313		11	0	0	0	0	0	0	0	0	5	1.93	0		1.77518	856.39

Figure 10: DRH using P=[0.00,0.00,5.00]

be obtained by super-imposing one hydrograph over another (with proper time lag). These assumptions are an implicit acceptance that the watershed can be approximated by linear systems theory.

As an illustration, we will repeat the classic analysis, but use functions instead of a linear combination. The results will be roughly the same, except that a functional representation can be used over many time bases.

A simple transfer function used in this example is $U(t) = K\ (\frac{t}{T})^N\ exp(-\frac{t}{T})$. The unknown parameters are K,N, and T. The parameter T is a timing parameter and essentially locates the peak of the discharge it is analogous in concept to T_p or T_c or T_L depending on how the equation is constructed. Parameters K is related to the drainage area, and N is related basin

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
1	Unit Hydrograph (Transfer Function)																	K	31.931			
2												U(t)=K*(t/T)^N*exp(-(t/T))						N	5.7815			
3																		T	0.3052			
4																						
5																						
6	Observations						[P]												[U]	[Q*]	[Q]-[Q*]	Bias
7	Time(hrs)	Time (increment)	Excess Rain (in)	Direct Runoff (cfs)			1	2	3	4	5	6	7	8	9	10	11					
8	0.5	1	1.06	428		1	1.06	0	0	0	0	0	0	0	0	0	0		107.63	114.084	313.916	0.73345
9	1	2	1.93	1923		2	1.93	1.06	0	0	0	0	0	0	0	0	0		1150.7	1427.44	495.5613	0.2577
10	1.5	3	1.81	5297		3	1.81	1.93	1.06	0	0	0	0	0	0	0	0		2331.6	4887.1	409.902	0.07738
11	2	4	0	9131		4	0	1.81	1.93	1.06	0	0	0	0	0	0	0		2391.2	9117.33	13.67158	0.0015
12	2.5	5	0	10625		5	0	0	1.81	1.93	1.06	0	0	0	0	0	0		1688.6	10625	6.73E-11	6.3E-15
13	3	6	0	7834		6	0	0	0	1.81	1.93	1.06	0	0	0	0	0		941.73	8585.17	-751.1711	-0.0959
14	3.5	7	0	3921		7	0	0	0	0	1.81	1.93	1.06	0	0	0	0		446.27	5346.89	-1425.892	-0.3637
15	4	8	0	1846		8	0	0	0	0	0	1.81	1.93	1.06	0	0	0		187.72	2764.82	-918.8209	-0.4977
16	4.5	9	0	1402		9	0	0	0	0	0	0	1.81	1.93	1.06	0	0		72.088	1246.46	155.5369	0.11094
17	5	10	0	830		10	0	0	0	0	0	0	0	1.81	1.93	1.06	0		25.765	506.209	323.7912	0.39011
18	5.5	11	0	313		11	0	0	0	0	0	0	0	0	1.81	1.93	1.06		8.6889	189.416	123.5837	0.39484
19																						
20							(1) Guess K,N,T													Average Bias		0.09169
21							(2) Compute Q*															
22							(3) Compute difference															
23							(4) K,N,T to get difference small															
24							(5) Stop when average difference is small percent of actual Q															
25							(6) If SOLVER works on machine - can automate - crashes on Mac (Microsoft hates Mac!)															

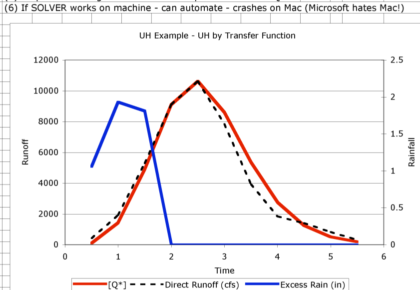


Figure 11: Unit hydrograph as a transfer function

shape, but like timing they cannot be easily be inferred from maps etc. The main difference here, is that the unit weights are values of the transfer function at different locations in time, so instead of just 9 unit weights we actually have as many as needed for a given case, and this information makes changing the time base simpler.

To estimate K,N, and T we simply construct the [Q]=[P][U] model where [U] is given by the above function at the correct times, then adjust K,N, and T to minimize the differences between the observed [Q] and the model [Q]. Figure 11 is an example of the approach. I choose to minimize the sum of squared differences at the peak as the merit function, but one could choose others. The process of using a minimization procedure to estimate the parameters (and ultimately the unit weights) is often called de-convolution. The value of this approach is when we want to examine different time bases or different rainfall patterns. As an example three patterns are presented, first the original pattern then two patterns with the same total depth but different time distributions. Figure 12 is an illustration; the principal value is that the simulation time now extends for 24 intervals instead of the original

11, and using transfer functions very complicated storms can be analyzed.

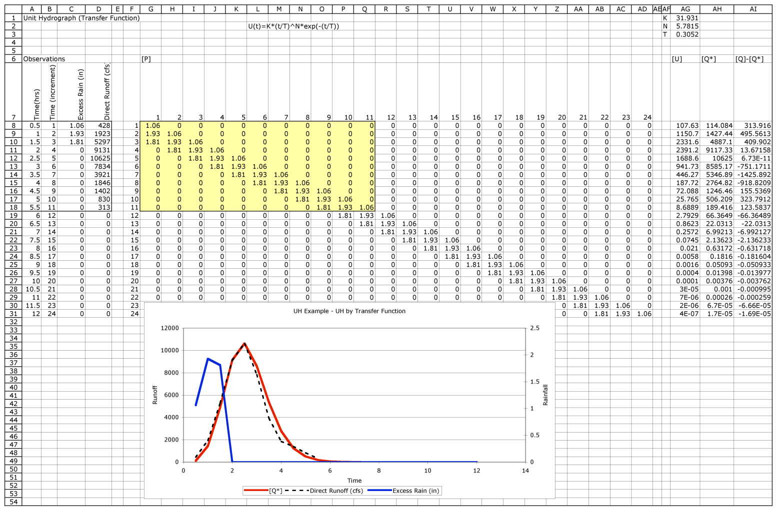


Figure 12: Extended time-base using transfer function based unit hydrograph. Shaded portion is original partition - note extension is everything else.

Once the time base is extended different storm patterns can be studied. As an example a storm with longer duration, but same depth as the calibration storm is presented in Figure 13.The storms can also be time-shifted. Time-shifting is technically trivial, but it is important when comparing many different storms and greatly improves analyst understanding. Figure 14 is an illustration of the previous storm, occurring slightly later in time.

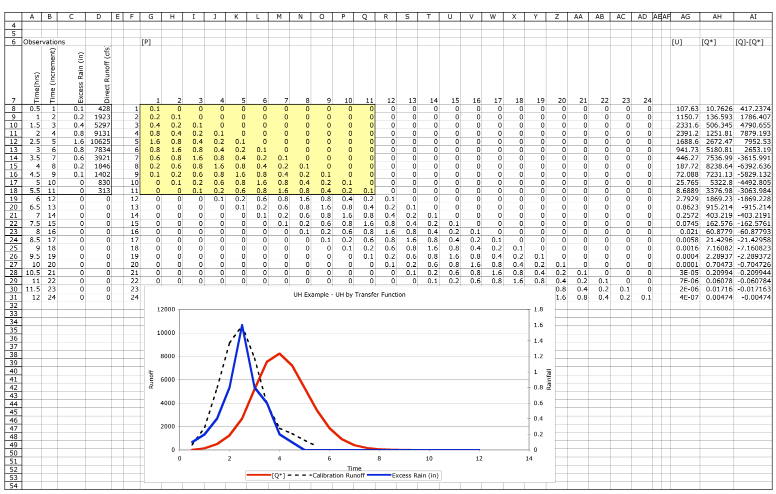


Figure 13: Extended time-base using transfer function based unit hydrograph - new rainfall pattern, same depth as original

These extended patterns are all based on the same values of K,N, and T. An alternative approach using the S-curve hydrograph works with conventional unit hydrographs (this approach is left as an exercise). With the functional approach (not the S-curve), however, the pattern can be moved, or even extended over a longer durations and the time changes need not be integer multiples of the original duration. This transfer function approach is essentially a technique to infer an IUH from the data. Complicated storms require far more difficult computations, but use essentially the same methods.

In this discussion I have ignored changing time bases, which in the classical method involves quite a bit of manipulation, but is quite easy with transfer functions. Most current hydrology computer programs use some version of classical methodology and to my knowledge do not incorporate functional models (at least not directly, but the user could create a tabulation of the transfer function and use that in the program).

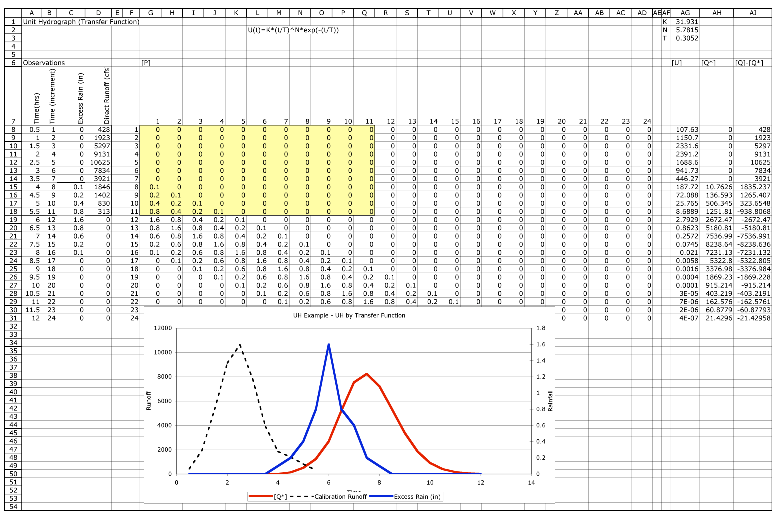


Figure 14: Extended time-base using transfer function based unit hydrograph - new rainfall pattern, same depth, but time shifted.

4 References

Chow, V. T., Maidment, D. R., and Mays, L. W., 1988. Applied Hydrology, McGraw-Hill, 572pp.

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