CHAPTER 1

The Process

After the completion of this chapter, the student should:

1. Understand the basic systems-analysis approach to formulating and solving engineering problems with mathematical models.
2. Appreciate how labor power, physical, and financial limitations as well as institutional requirements such as indicated in building and design codes affect engineering design, planning, and management processes.
3. Understand the advantages and limitations of systems analysis for solving practical engineering problems.

Systems analysis is a coordinated set of procedures that can be used to address issues of project planning, engineering design, and management. Systems analysis is a decision-making tool. An engineer can use it for determining how resources can be used most efficiently and most effectively to achieve a specified goal or objective. For successful decision making, *both technological* and *economic* considerations must be employed in the analysis. The premise is followed throughout this textbook.

Since systems analysis can be applied to a broad range of decision-making and engineering problems, we shall illustrate its application to problems in structural, geotechnical, environmental, transportation, water resources, and construction engineering. In each illustration we attempt to show how the principles of engineering can be combined with the principles of economics to achieve an optimum solution.

In the fields of economics, mathematics, and business, systems analysis is commonly referred to as *operations research*. In this textbook, we are concerned with the application of these principles to the solution of design, planning, and management problems in civil engineering.

1.1 RESOURCE ALLOCATION AND MATHEMATICAL MODELS

Systems analysis is an approach for allocating *resources* in an effective manner. Resources can be broadly classified as: *labor power*, *money*, and *materials*. Since resources have market value, that is, can be bought and sold, and since money is generally in short supply, the allocation of resources is extremely important. This is especially true when civil engineers are involved in large-scale public-works projects that cost millions of dollars.

The Goal and Objective

In order to allocate resources efficiently and effectively, we must have a clearly established *goal*, or *objective*. Generally, our goal will be to allocate resources in a manner that will maximize profit for a firm or maximize a societal benefit for the public. The resources of labor power, money, and materials are assumed to be used for producing goods and services. Since social benefits are difficult to measure and will only tend to confuse our introductory remarks, we utilize the profit motive and other simple measures to illustrate fundamental concepts of systems analysis. For example, the profit P is defined as the difference between the revenue or monetary benefit R received for a good or service and the cost C to provide the good or service. The following mathematical expression summarizes this definition.

$$P = R - C$$

If the revenue is assumed to be fixed, the greatest profit is obtained by minimizing the cost of production. Typically, the role of the engineer is to achieve this goal.

Constraints

Finding ways to minimize cost for the purpose of maximizing profit is easier said than done. *Financial*, *physical*, and *institutional constraints* must be considered. Financial constraints are brought about, generally, by the limited supply of money and the costs of borrowing it. This money may be used to obtain resources, such as by buying material or hiring workers. Physical constraints refer generally to the limitations of the properties of materials. A material has certain properties that can be measured, such as strength, elasticity, and other engineering characteristics. Institutional constraints are generally rules, laws, or guidelines specified by society, government, and the engineering profession. The rules as specified in design and building codes are examples of institutional constraints that must be considered by the engineer. Clearly, if an engineering design satisfies all constraints at the same time that it satisfies the given objective to maximize profit or minimize cost, the design can be considered an *optimum solution*. The purpose in this textbook is to discuss the ways of achieving this end.

The Optimum Solution

In systems analysis, a *mathematical model* is an important element of the decision-making process. The mathematical model is an exact and explicit statement of the objective, or goal, to be achieved. In addition, it consists of a set of financial, physical, and institutional constraint conditions that must be satisfied. The solution to a problem, the *optimum solution*, is a statement of how resources should be used in the most efficient and effective manner.

In this textbook we investigate problems from many areas of civil engineering. The goal, or objective, will vary from problem to problem. For example, our objective may be to minimize the cost of production, minimize the weight of a structure or choose the best alternative design that satisfies a public need. All these problems may be structured as systems-analysis problems using mathematical models. The important point to realize is that a systems-analysis model, regardless of the particular discipline within civil engineering concerned or the goal of the project, is always formulated using the same approach. The resulting mathematical model can be presented in a standard mathematical form with an objective function and a set of constraint equations.

In this chapter we are primarily concerned with the statement of the project goal, the statement of constraint conditions, and the formulation of mathematical models. The solution of systems-analysis or optimization problems is the subject of the following chapters of the book. We shall see that there are different algorithms for solving different types of mathematical models. We shall use graphical methods and methods of calculus to illustrate basic principles for determining the optimum solution.

The graphical method is a very powerful means of solving a certain class of optimization problems. Generally, it gives better insight into understanding the problem and evaluating alternative solutions than do mathematical algorithms. Whenever possible, graphical methods will be utilized to complement the discussion of mathematical approaches.

Mathematical Models

A typical systems analysis model will consist of a single objective function and a set of constraint equations. Models consisting of multiple objective functions are discussed in Section 1.4. Our discussion, for the most part, will be restricted to models with a single objective function.

The objective function is assumed to be a function of a set of *design*, *decision*, or *control variables*. In this textbook, the term "control variable" is generally used. The objective function may be expressed as a mathematical relationship.

$$z = f(x_1, x_2, \ldots, x_n)$$

where $x_1, x_2, \ldots, x_n$ are designated as a set of n control variables. The control variables, which may typically represent the assignment of the number of workers, an amount of money, and a volume of material, are all nonnegative values; they are introduced into the mathematical model as

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ &\vdots \\ x_n &\geq 0 \end{aligned}$$

The financial, physical, and institutional limitations are represented by a set of m constraint equations.

$$\begin{aligned} g_1(x_1, x_2, \ldots, x_n)&\{=, \leq, \geq\}b_1 \\ g_2(x_1, x_2, \ldots, x_n)&\{=, \leq, \geq\}b_2 \\ &\vdots \\ g_m(x_1, x_2, \ldots, x_n)&\{=, \leq, \geq\}b_m \end{aligned}$$

The set $\{=, \leq, \geq\}$ represents the possibility of having an equal to, less than or equal to, or greater than or equal to constraint condition. For example, in the construction of a steel building, the right-hand side of the equation set may consist of: $b_1 =$ \$10,000, the amount of money that may be expended on fabrication; $b_2 = 500$ ft, the number of linear feet of steel available for this project; and $b_3 = 20{,}000$ psi, an allowable stress limitation permitted in structural support members. By utilizing vector notation, we may simplify the model into a compact form:

$$z = f(\boldsymbol{x})$$

subject to

$$g_i(\boldsymbol{x})\{=, \leq, \geq\}b_i$$

where

$$i = 1, 2, \ldots, m \qquad \text{and} \qquad \boldsymbol{x} \geq \boldsymbol{0}$$

In vector notation the set of control variables $x_1, x_2, \ldots, x_n$ is represented by the *control vector* $\boldsymbol{x}$ or

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The constraint set may be represented as a vector also. Let

$$\boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ g_m(x_1, x_2, \ldots, x_n) \end{bmatrix} = \begin{bmatrix} g_1(\boldsymbol{x}) \\ g_2(\boldsymbol{x}) \\ \vdots \\ g_m(\boldsymbol{x}) \end{bmatrix}$$

and

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The model becomes

$$\begin{aligned} z &= f(\boldsymbol{x}) \\ \boldsymbol{g}(\boldsymbol{x})&\{=, \leq, \geq\}\boldsymbol{b} \\ \boldsymbol{x} &\geq 0 \end{aligned}$$

The function $f(\boldsymbol{x})$ and the set of functions represented by $\boldsymbol{g}(\boldsymbol{x})$ may be either linear or nonlinear set of functions of $\boldsymbol{x}$.

Control Variables and Vectors

A control variable is a term used to designate any parameter that may vary in the design, planning, or management process. A control variable may be either a discrete or continuous variable. In this textbook, the letter x_i, where the subscript $i = 1, 2, \ldots, n$, is generally used to identify the particular control variable.

An example of a discrete control variable is the number of workers assigned to a particular task. Let x_1 be equal to the number of assigned workers, where $x_1 = 1, 2, 3, \ldots$. Since the number of workers is always a positive number or equal to zero, it is represented as

$$x_1 \geq 0$$

We shall see that in most formulations of physical systems the control variables are nonnegative values. This simple constraint will have an important impact in our search for an optimum solution.

An example of the use of continuous control variables is illustrated with the use of Figure 1.1. In this case, the beam length is fixed and the beam width and height are considered the design variables. Since the length is a constant, it is not designated as a control variable. We define x_1 and x_2 as control variables, where x_1 is the beam width b and x_2 is the beam height h. In this case x_1 and x_2 are nonnegative values.

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Utilizing vector notation, this set of variables can be written as

$$\boldsymbol{x} \geq \boldsymbol{0}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \\ h \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \boldsymbol{0}$$

The Objective Function

The objective function $f(\boldsymbol{x})$ is a single-valued function of the set of control variables or the control vector $\boldsymbol{x}$. The objective function is a mathematical statement of the goal and a measure of how effectively the goal is met. The selection of the measure

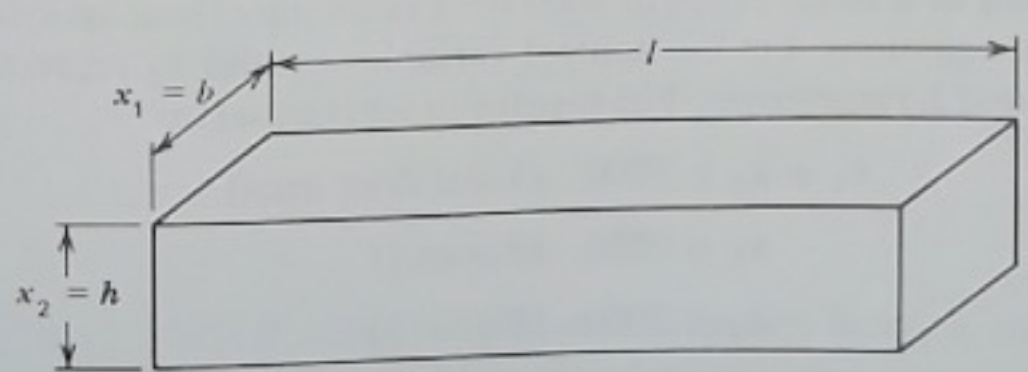


Figure 1.1 Rectangular beam.

of effectiveness will have an important effect on the outcome or solution. In some problems the selection of the measure of effectiveness and $f(x)$ is obvious. In other situations the choice may be extremely difficult. More will be said about measures of effectiveness in Section 1.2.

The objective function may either be maximized or minimized. For example, in a problem to maximize profits P, the objective function, is written as

$$\text{Maximize } z = \text{Maximize } P = f(x)$$

The symbol z represents the scalar quantity of the function $f(x)$. In this case, z is a scalar measure of profit P expressed in dollars and is assumed to be a function of the control variables $x_1, x_2, \ldots$.

The objective function may also be a minimization of z. For instance, the goal of a structural designer of aircraft is to minimize total weight W. In this case, the measure of effectiveness is a nonmonetary measure in units of pounds. The objective function for this case is

$$\text{Minimize } z = \text{Minimize } W = f(x)$$

Constraint Equations

A constraint equation is a mathematical equation expressing a financial, physical, or institutional limitation placed upon the problem. Generally, it is derived from fundamental principles of engineering or economics. A constraint equation may be stated as an equation with a *strict equality* condition

$$g(x) = b$$

with a *less than or equal* condition

$$g(x) \leq b$$

or a *greater than or equal* condition

$$g(x) \geq b$$

For instance, in the planning of a two-room building as shown in Figure 1.2, the contract states that the total floor area of the building must have a minimum area of 5000 ft^2 and that each room of the building must have a maximum specified area of 5000 and 3000 ft^2, respectively. With these specifications, it is possible that the optimum solution will be a single room of 5000 ft^2. Let us state these specifications as a set of constraint equations. Let the control variables x_1 and x_2 represent the floor area of rooms 1 and 2, respectively. The constraint set is equal to

$$\begin{aligned} x_1 + x_2 &\geq 5000 \quad \text{(Total floor area)} \\ x_1 &\leq 5000 \quad \text{(Room 1)} \\ x_2 &\leq 3000 \quad \text{(Room 2)} \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

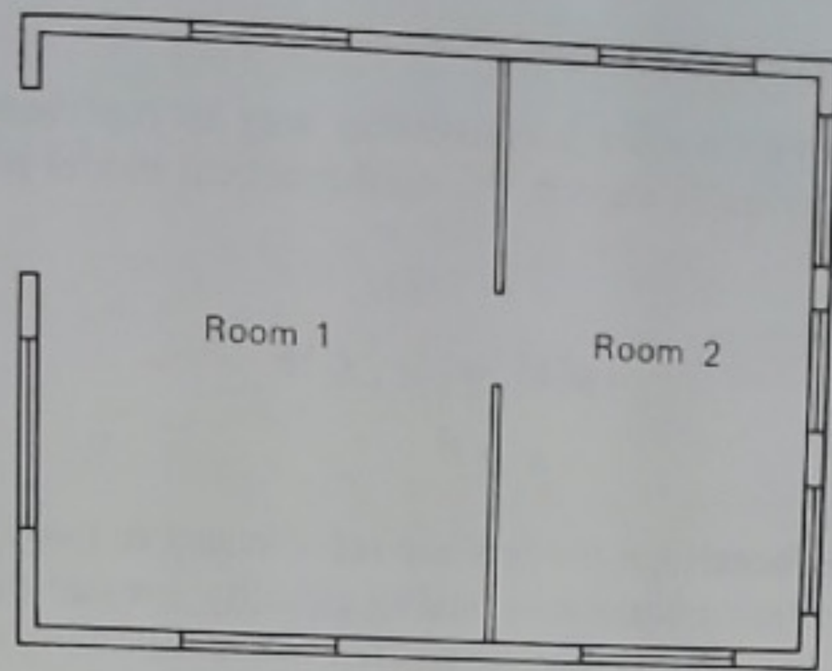


Figure 1.2 Floor plan.

The total floor area is equal to $x_1 + x_2$. According to specifications, it must be equal to or greater than 5000 ft^2. The other constraints, called *side constraints*, limit the room sizes, 1 and 2, to be less than or equal to 5000 and 3000 ft^2, respectively. Since the room sizes cannot be negative in value, we state that x_1 and x_2 are restricted to be positive numbers.

Feasible and Optimum Solutions

Any combination of control variables that satisfies the set of constraint conditions is called a *feasible solution*. A solution that does not satisfy all constraint equation conditions is called an *infeasible* solution. An *optimum solution* is a feasible solution that satisfies the goal of the objective function as well.

In the floor plan example the goal is to maximize total revenue. Assume that \$50/ft^2 and \$60/ft^2 are the unit revenues received for rooms 1 and 2 respectively. The maximum total revenue received, R, is a function of the floor area x_1 and x_2 and can be represented by the following objective function and constraint set.

$$\text{Maximize } R = \$50x_1 + \$60x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 5000 \\ x_1 &\leq 5000 \\ x_2 &\leq 3000 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

This mathematical model consists of a linear objective function and a set of linear constraint equations. A problem with this mathematical form is called a *linear mathematical model*. A model with either a nonlinear objective function or one or more nonlinear constraint equations is called a *nonlinear mathematical model*.

Linear Mathematical Models

Vector and matrix notation offer a convenient way to represent classes of mathematical models. As previously shown, the mathematical model is represented as

$$z = f(\boldsymbol{x})$$
$$g(\boldsymbol{x})\{=, \geq, \leq\}\boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

In this text linear mathematical models are represented in two ways. A model consisting of a set of constraint equations having equality, greater than or equal to, and less than or equal to constraints may be written in the following manner.

$$z = f(\boldsymbol{x}): \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$g(\boldsymbol{x})\{=, \leq, \geq\}\boldsymbol{b}: \quad \begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\{=, \leq, \geq\}b_1 \\ &a_{21}x_2 + a_{22}x_2 + \cdots + a_{2n}x_n\{=, \leq, \geq\}b_2 \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\{=, \leq, \geq\}b_m \end{aligned}$$

$$\boldsymbol{x} \geq 0: \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $c_j = c_1, \ldots, c_n$, and $b_i = b_1, b_2, \ldots, b_m$ and $a_{ij} = a_{11}, a_{12}, \ldots, a_{mn}$ are constant parameters. Using matrix notation, linear mathematical models will have the following compact form.

$$z = \boldsymbol{c}'\boldsymbol{x}$$
$$\boldsymbol{a}_i'\boldsymbol{x}\{=, \leq, \geq\}b_i \qquad \text{where } i = 1, 2, \ldots, m$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

and $\boldsymbol{a}_i$, $\mathbf{c}$, and b_i are vectors of the elements of the objective and constraint equations. The elements of c_j and b_i are generally called the unit cost and resource parameters, respectively. The elements of $\boldsymbol{a}_i$ are derived from technological and economic considerations; therefore, they are called technological parameters. The objective function may be represented as

$$z = [c_1 \ c_2 \cdots c_n] \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Likewise, the ith constraint equation may be represented as

$$[a_{i1}\ a_{i2} \cdots a_{in}] \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \{=, \leq, \geq\} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

where $i = 1, 2, \ldots, m$.

In Chapter 6 the linear model will be written in the so-called *standard form*, the second way of representing linear models:

$$\text{Minimize } z = \boldsymbol{c}'\boldsymbol{x}$$

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{x} \geq \boldsymbol{0}$$

All elements of $\boldsymbol{b}$ are positive values, $b_i \geq 0$, where $i = 1, 2, \ldots, m$. All contraint equations are strict equality constraints. The constraint set in expanded matrix form is

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{or} \quad \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \\ a_{m1} & a_{m2} \cdots a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

where the number of control variables n is equal to or greater than the number of equations m, $n \geq m$. We distinguish between vectors and matrices by reserving lower-case letters, $\boldsymbol{x}$, $\boldsymbol{c}$, $\boldsymbol{b}$, and $\boldsymbol{a}_i$, for vectors and upper-case letters, A, for matrices. All vectors in this book are column vectors. A row vector is denoted as a transpose of a column vector. For example,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The row vector of $\boldsymbol{x}$ is

$$\boldsymbol{x}' = [x_1\ x_2 \cdots x_n]$$

EXAMPLE 1.1 A Statically Determinate Minimum-Weight Truss

Consider the truss shown in Figure 1.3a. Formulate a mathematical model to design a simple truss of minimum weight. The critical buckling and maximum allowable tensile stresses of compression and tension members are 10 ksi and 20 ksi, respectively. The truss is to be constructed of steel. All compression and tension members are assumed to have the same cross-sectional area.

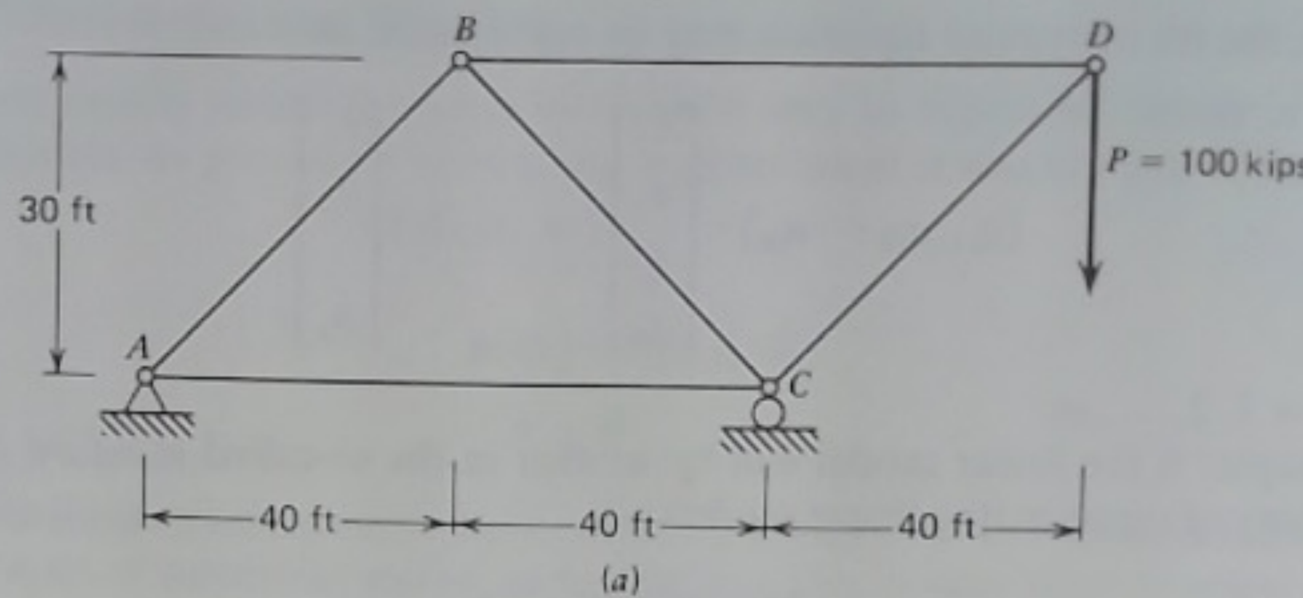


Figure 1.3*a*

Solution

Control Variables The structural members will be sized according to the type of member force, either compression or tension. Thus, the control variables are defined as

$$x_1 = A_1 = \text{Cross-sectional area of a compression member (in.}^2\text{)}$$

$$x_2 = A_2 = \text{Cross-sectional area of a tension member (in.}^2\text{)}$$

The control vector x is $x' = [A_1 \; A_2]$.

Reactions

The support reactions may be determined from Newton's law of static equilibrium. The free-body diagram for the reactions is shown in Figure 1.3*b*.

$$\sum F_x = 0: \quad H_a = 0$$

$$\sum F_y = 0: \quad -V_a + V_c - 100 = 0$$

$$\sum M_A = 0: \quad 80V_c - 120 \cdot 100 = 0$$

Solving this set of reactions results in $H_a = 0$, $V_a = 50$ kips, and $V_c = 150$ kips. (1 kip = 1000 pounds.)

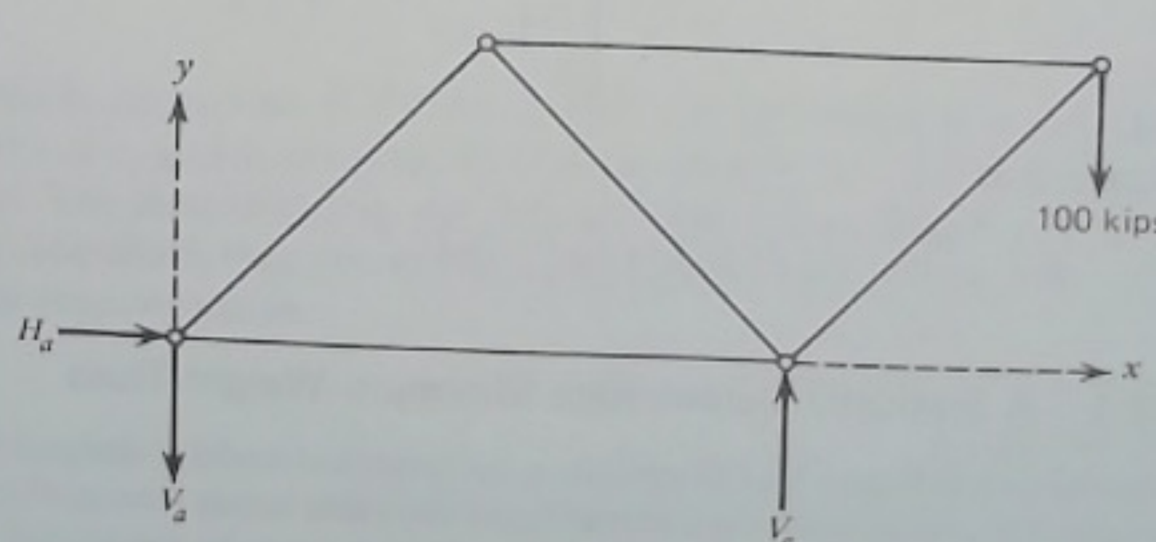


Figure 1.3*b*

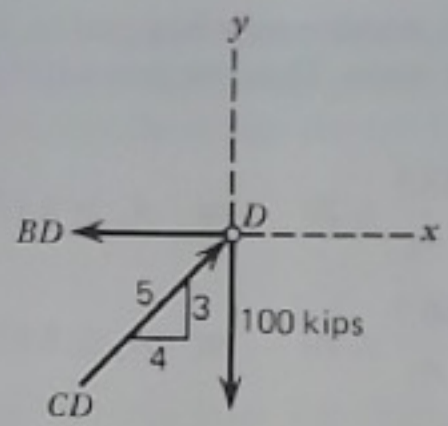


Figure 1.3c

Member Forces and Stresses We next use the method of joints to determine forces in each member. The free-body diagram for joint D is shown in Figure 1.3c.

$$\sum F_x = 0: \quad -BD + \tfrac{4}{5}CD = 0$$

$$\sum F_y = 0: \quad \tfrac{3}{5}CD - 100 = 0$$

The member forces are $CD = 167$ kips (compression) and $BD = 125$ kips (tension). The stress in the members will be equal to the member force divided by the cross-sectional area of the member. In this case, the tension member BD will have a stress σ_{BD} equal to

$$\sigma_{BD} = \frac{125}{A_2}$$

The compression member CD will have a stress equal to

$$\sigma_{CD} = \frac{167}{A_1}$$

The method of joints was used to determine the forces in members BC, AB, and AC. The member forces and reactions are summarized in Figure 1.3*d*. The stress in remaining members AB, AC, and BC will be equal to

$$\sigma_{AB} = \frac{83.3}{A_2}; \qquad \sigma_{AC} = \frac{66.7}{A_1}; \qquad \sigma_{BC} = \frac{83.3}{A_1}$$

respectively.

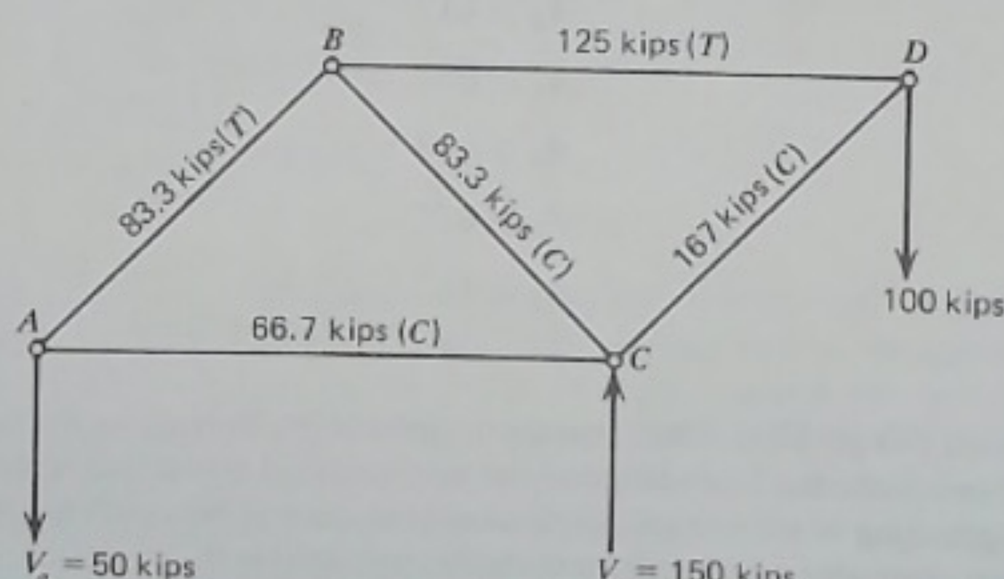


Figure 1.3*d*

Constraint Equations The stresses in each member may be equal to, but must never exceed, the critical buckling stress or allowable tensile stress. Thus, we may express these restrictions by the set of equations:

$$\text{Member } AB: \quad \frac{83.3}{A_2} \leq 20 \qquad \text{or} \quad A_2 \geq 4.17$$

$$\text{Member } AC: \quad \frac{66.7}{A_1} \leq 10 \qquad \text{or} \quad A_1 \geq 6.67$$

$$\text{Member } BC: \quad \frac{83.3}{A_1} \leq 10 \qquad \text{or} \quad A_1 \geq 8.33$$

$$\text{Member } BD: \quad \frac{125}{A_2} \leq 20 \qquad \text{or} \quad A_2 \geq 6.25$$

$$\text{Member } CD: \quad \frac{167}{A_1} \leq 10 \qquad \text{or} \quad A_1 \geq 16.7$$

Objective Function The weight of each member is equal to the density of steel times the volume of each member. The density of steel is approximately 490 lb/ft^3 or 3.40 lb/ft-in.^2. The equation for weight of the truss is the sum of the weight of each individual member; thus,

$$z = 3.40[V_{AB} + V_{AC} + V_{BC} + V_{BD} + V_{CD}]$$

where V = volume of each member or the member length multiplied by the member cross-sectional area. The objective function is

$$z = 3.40[50A_2 + 80A_1 + 50A_1 + 80A_2 + 50A_1]$$

or

$$z = 612A_1 + 442A_2$$

Mathematical Model The problem formulation is complete. Summarizing the equations results in the following mathematical model:

$$\text{Minimize } z = 612A_1 + 442A_2$$

subject to the constraints

$$A_2 \geq 4.17$$

$$A_1 \geq 6.67$$

$$A_1 \geq 8.33$$

$$A_2 \geq 6.25$$

$$A_1 \geq 16.7$$

Remarks

The formulation of this problem utilizes basic principles of engineering mechanics and structural design. In most instances, the formulation of the mathematical model will be derived from basic principles of engineering or economics. All physical laws, such as Newton's law, must be satisfied, and all rules of good engineering practice must be incorporated in the model.

Systems analysis gives us a different perspective in solving engineering problems. In this problem, for instance, we have established the limitations on our design with structural engineering principles and combined these limitations in a set of constraint equations with a goal of minimizing the weight of the truss. The optimum solution satisfies the goal and the constraints at the same time. Systems analysis adds a new dimension to the design process that should lead us to a better design.

PROBLEMS

Problem 1

A contractor may purchase material from two different sand and gravel pits. The unit cost of material including delivery from pits 1 and 2 is \$5 and \$7 per cubic yard, respectively. The contractor requires 100 yd^3 of mix. The mix must contain a minimum of 30 percent sand. Pit 1 contains 25 percent sand, and pit 2 contains 50 percent sand.

The objective is to minimize the cost of material.

(a) Define the control variables.
(b) Formulate a mathematical model.

Problem 2

An aggregate mix of sand and gravel must contain no less than 20 percent nor more than 30 percent gravel. The in situ soil contains 40 percent gravel and 60 percent sand. Pure sand may be purchased and shipped to site at \$5.00/yd^3. A total mix of 1000 yd^3 is needed. There is no charge for using in situ material.

The goal is to minimize cost subject to the mix constraints.

(a) Define the control variables.
(b) Formulate the mathematical model.

Problem 3

There are two suppliers of pipe.

SOURCE	UNIT COST (\$/LINEAR FOOT)	SUPPLY (LINEAR FOOT)
1	\$100	100 ft maximum
2	\$125	Unlimited

Nine hundred feet of pipe is required.

The goal is to minimize the total cost of pipe.

(a) Define the control variables.
(b) Formulate a mathematical model.

Problem 4

A company requires at least 4.0 Mgal/day more water than it is currently using. A water-supply facility can supply up to 10 Mgal/day of extra supply. A local stream can supply an additional 2 Mgal/day. The company requires that the water pollution concentration be less than 100 mg/l. BOD, the biological oxygen demand. The water from the water-supply facility and from the stream has a BOD concentration of 50 mg/l and 200 mg/l, respectively. The cost of water from the water supply is \$100/Mgal and from the local stream is \$50/Mgal.

The goal is to minimize the cost of supplying extra water that meets water quality standards.

(a) Define all control variables.
(b) Formulate a mathematical model.

Problem 5

The unit selling price p of an item is \$150/unit, thus the total revenue is $R = 150q$. The production cost C is a function of output level q, $C = 100q^{0.8}$. The maximum output of the firm is 50 units/yr.

The objective is to maximize yearly profit subject to the production constraint.

(a) Define the control variable.
(b) Formulate a mathematical model.

Problem 6

Determine the maximum volume of a sphere subject to an internal pressure of 4000 psi (lb/in.^{2}). The volume of a sphere is $V = (\pi/6)d^3$, where d equals the mean diameter of the tank. The allowable stress of the material is equal to 20,000 psi. The hoop stress is a function of the tank diameter d and tank thickness t.

$$\sigma = \frac{pd}{4t}$$

The objective is to maximize tank volume subject to the limitation on stress.

(a) Define the control variable.
(b) Formulate a mathematical model.

1.2 THE SYSTEMS-ANALYSIS APPROACH

The systems-analysis approach consists of the following steps.

1. Establish an objective and an appropriate measure of effectiveness.
2. Formulate a mathematical model.
3. Determine the optimum solution.

In this chapter, we implicitly utilized steps 1 and 2 in formulating the mathematical model for a minimum-weight truss. In some real-world engineering problems, however, the establishment of the objective and the formulation of the mathematical models will not be so easy. We shall see in our development of systems analysis that most of our discussion in this textbook is devoted to steps 2 and 3. It might appear that step 1 is not as important as the others. But in this section we see that the establishment of a single goal or objective and the establishment of an appropriate measure of effectiveness will have a most important bearing upon the solution to a problem.

Multiobjective Problems

The types of systems-analysis problems we consider here are limited to a single objective. One difficulty in satisfying this restriction is establishing a mathematical model with unlike measures of effectiveness. Consider the purchase of an automobile. Most people decide which automobile to buy through comparison shopping. We weigh the attributes or features of one automobile against the others. All automobiles provide transportation, but it is the selling price of the vehicle, its performance, and styling features that are ultimately used to determine the final selection of the vehicle. We can measure these attributes by vehicle price, speed, fuel economy, reliability, comfort, and prestige of ownership. If we rank each automobile by each one of these attributes, it will be very unlikely that one vehicle would be our first choice in every category. In other words, the lowest-priced vehicle will rank number one in the price category, but it is unlikely that it would rank number one in speed, comfort, and prestige of ownership as well. In order to make a decision, we have to weigh these attributes. If price is important and prestige of ownership is unimportant, a greater weight will be placed on price than on owner prestige. By this *subjective reasoning process*, we can make a choice among the vehicles. It is possible that the vehicle chosen does not receive a top rating in any one category.

If we use a mathematical model, we shall have to quantify those attributes and place them in an objective function. Let the control variables $x_1, x_2, \ldots$ represent automobile types 1, 2, ..., and so on.

Here, the control variables are discrete control variables taking on the values of 0 or 1, the automobile type is not selected or is selected for purchase, respectively. Thus, $x_1 \leq 0$ or 1, $x_2 = 0$ or 1, ... Unlike the subjective reasoning process utilized in the comparative shopping discussion, here we must explicitly assign numerical *weights* or a *measure of effectiveness* to each attribute. For instance, the purchase price is measured in dollars, the speed in miles per hour, and the fuel economy in miles per gallon. These are objective measures of effectiveness because they have economic value and performance rating that are distinguishable. The attributes of reliability, comfort, and prestige of ownership are personal preference items. They are considered subjective measures because they have psychological values that are difficult to quantify.

The objective measures of effectiveness are more desirable to use for engineering problems than are subjective measures. Even when all measures of effectiveness are objective measures, they may be *unlike measures of effectiveness* that are difficult to combine into a single objective function. For example, if selling price in dollars per car and fuel economy in miles per gallon are the only two objective measures of effectiveness, they cannot be added together nor meaningfully introduced into the same single objective function, because they do not have the same units. Therefore, they are considered unlike measures of effectiveness. This problem formulation requires two objective functions: one to minimize the selling price and the other to maximize fuel economy.

Obviously, multiobjective problems are important. We suggest that the optimum solution for each single-value objective be determined independently. Each optimum solution can then be compared in the context of the original problem. Subjective reasoning, trade-off analysis, and other methods can be used to make the final decision. The textbook by Giococchea, et al (see bibliography) is recommended for further reading.

Establishing an Appropriate Objective Function

Once a single measure of effectiveness has been chosen, extreme care must be used in establishing an appropriate objective function. For profit-maximization and cost-minimization problems, money measured in dollars, for example, will be the obvious choice of the unit of measure. However, consider the total cost of an engineered structure. There are design, material, shipping, construction, and labor costs to consider. In lieu of considering all costs in a minimum-cost model, let us assume that minimum-cost models are established and solutions for each individual cost category are found. The optimum solution to a minimum-construction-cost problem may not be the same optimum solution to a minimum-material-cost problem, even though the constraints imposed upon the two problems are the same. The solution to the minimum-material-cost model may result in a very light and delicate structure. The cost to construct this structure may be prohibitively expensive. The solution to the minimum-construction-cost model may result in easy construction with materials being wasted. To eliminate these problems, a model considering all costs, designs, materials, shipping, construction, and labor will probably be most effective. Since the same measure of effectiveness, dollars, is used to measure each attribute, the objective function incorporating all attributes can be established by adding them together.

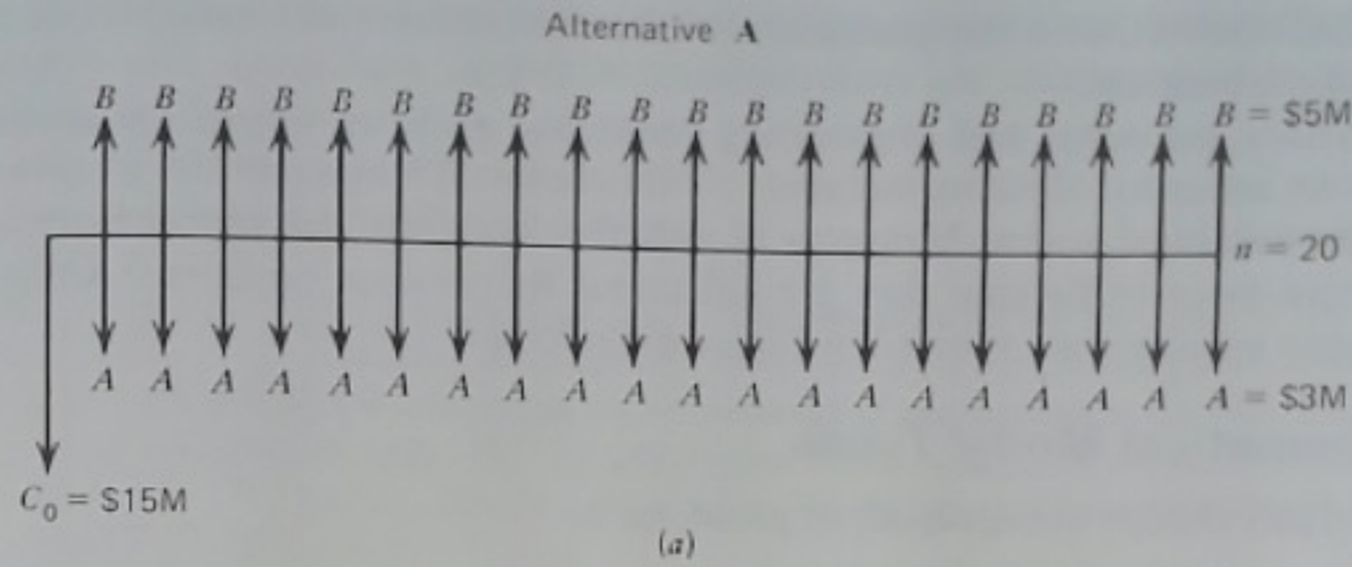
Cash-Flow Problems

Costs and revenues may be affected by time. The cost to construct a system is an initial cost and may be considered as a one-time capital cost. On the other hand, the cost to maintain and operate a system must be paid over the life of the project. When payments are made at different time periods, we cannot simply add them together as we did in establishing a minimum-total-cost model of an engineered structure. For instance, the value of a payment made today versus 10 years from now is different. Clearly, the payment that we receive today has more value than one we expect to receive in the future. Problems of this type are called *cash-flow problems* and will be evaluated with *time value of money relationships* to be discussed in detail in the chapter on engineering economics.

For illustration, consider two different types of construction plans. Alternative **A** calls for the construction of the entire system at the present time. Alternative **B**, a staged-construction plan, calls for construction of part of the system now and the remaining part in year 10. The cash flow of benefits and costs are shown in Figure 1.4.

The design life of the project is 20 years. For plan **A**, the capital cost of construction is \$15M, $C_0 = \$15\text{M}$, and the monetary annual benefit and annual operating costs are shown to be uniform over the entire life of the project. The benefits are $B_1 = B_2 = \cdots = B_{20} = B = \$5\text{M/yr}$, and the annual operating and maintenance costs are $A_1 = A_2 = \cdots = A_{20} = A = \$3\text{M/yr}$. The subscript refers to the year the payment is made. The annual net benefit $B - A$ is \$2M/yr. For plan **B**, construction takes place in years 0 and 10, $C_0 = \$9\text{M}$ and $C_{10} = \$8\text{M}$. The annual benefits and costs are not uniform over the 20-year life of the project as shown. For the first 10 years the annual net benefit $B - A$ is \$1M/yr, and for the last 10 years $B' - A'$ is \$3M/yr.

If we study the two cash-flow diagrams, we can see different advantages for the two plans. Plan **B** has the advantage of lower initial construction cost; however, plan **A** offers a greater net annual benefit for the first 10 years. In later years, plan **B** offers a greater net annual benefit. These advantages are derived by making the *same-year comparisons*. Moreover, it is meaningless to say that the total cost of construction of plan **B** is \$9M + 8M = \$17M. This addition should not be performed, because the cash flow, or payments, takes place in different time periods, years 0 and 10. Thus, we need a method of combining present and future cash flows.



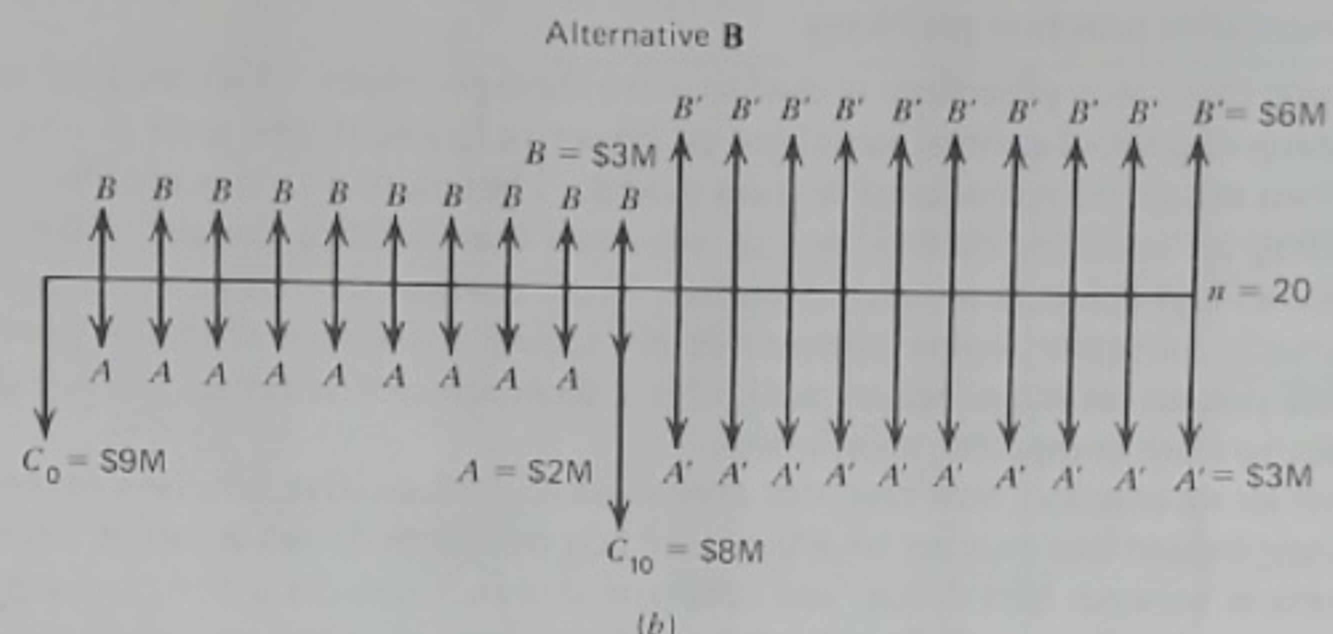


Figure 1.4 Cash-flow diagrams. (*a*) Alternative **A**, construction in year 0. (*b*) Alternative **B**, construction in year 0 and year 10.

Time of value of money relationships, incorporating factors of *time of payment* and *value of money*, which is measured in terms of an interest rate, will permit the transferral of a future payment to an equivalent present-worth payment. Now, simple arithmetic calculations can be performed. Moreover, all present and future benefits and costs can be combined into new value called the *net present-worth* value.

$$NPW = B_0 - A_0 - C_0$$

where A_0, B_0, C_0 represent the present worth of all annual costs, all annual benefits, and all capital costs, respectively. The net present worth of the competing alternatives $NPW^{\mathbf{A}}$ and $NPW^{\mathbf{B}}$ can be compared and the one offering the greater net present worth will be selected. If $NPW^{\mathbf{A}} > NPW^{\mathbf{B}}$, then construction plan **A** should be selected. If not, plan **B** is the better one. This is the essence of the *net present-worth selection method*. to be introduced in Chapter 3.

Nonmonetary measures of effectiveness are also important. In some instances, a physical measure, such as weight, is more important than a monetary measure. In the aircraft industry, reducing the overall weight of the frame and other components is of prime importance. Safety is of prime concern in all civil engineering designs.

Esthetics, comfort, and the protection of the environment are examples of other important objectives.

Monetary measures and engineering measures, such as weight and pollution emissions, are called objective measures of effectiveness. These quantities are measured in well-established scales. Measures of esthetics, comfort, and pleasure are called subjective measure because they are influenced by personal preference. Generally, subjective measures are avoided in engineering design.

Mathematical Model Types

We broadly classify system-analysis problems as

1. Resource allocation problems.
2. Alternative selection problems.

The same three-step procedure is used to solve these problems. Mathematical models consisting of a set of control variables, an objective function, and a set of constraint equations are called resource allocation models. The solution to this problem results in finding an *optimum combination of resources* that satisfies the set of constraint conditions and achieves a stated objective. Mathematical models consisting of a set of *mutually exclusive design alternatives* are called alternative selection problems. The net present-worth selection method is a technique for selecting the *best design alternative* from competing alternatives.

Now let us consider how resource allocation and alternative selection procedures can complement one another to achieve the best optimum design solution. There is a distinction between best design and optimum design. Consider a design of a bridge. Different technology and design configurations may be used to satisfy the basic requirement to span a given distance safely. If a steel truss and a concrete beam are considered the only alternatives, the one with the least combined construction and maintenance costs is considered the better choice. However, it may not be the best optimum design! Only if each design alternative is a minimum-cost design will the use of the alternative selection method lead to the best optimum design. It is possible to determine the least-cost designs by formulating separate resource allocation models for the steel truss and the concrete beam designs. These models will consist of an objective function to minimize cost subject to a set of constraint equations to ensure bridge safety and performance. Thus, the methods of solving resource allocation problems and the alternative selection method are complementary procedures that can be used to achieve the best optimum design.

A major difficulty of utilizing mathematical models is that not all of them can be easily solved. As a result, the use of combined methods of resource allocation and alternative selection may not be fully realized in the mathematical sense. Engineering is as much an art as an applied science. The best optimum design may be obtained by experimentation and trial and error. Thus, sometimes it is only practical to use the conceptual aspects of resource allocation and alternative selection to formulate problems and to seek solutions by mathematical means, by experimentation, or by trial and error. This book deals only with the mathematical methods of solution. Chapters 2 and 5 through 10 deal with methods of solving resource allocation problems, and Chapters 3 and 4 deal with methods of selecting the best design alternative. Chapter 5 also shows how cash-flow considerations are incorporated in resource allocation problems.

EXAMPLE 1.2 Bridge Location Study

Establish an objective function and measure of effectiveness to satisfy the goals of the City of Kingsbury. Owing to industrial growth, the City of Kingsbury is experiencing new demands for housing. City officials want to encourage the development of new housing in the West End. Since the economic future looks bright, the city government wants to improve the accessibility of the West End and find the best location for a new bridge. Two bridge sites have been chosen and are shown on the map in Figure 1.5.

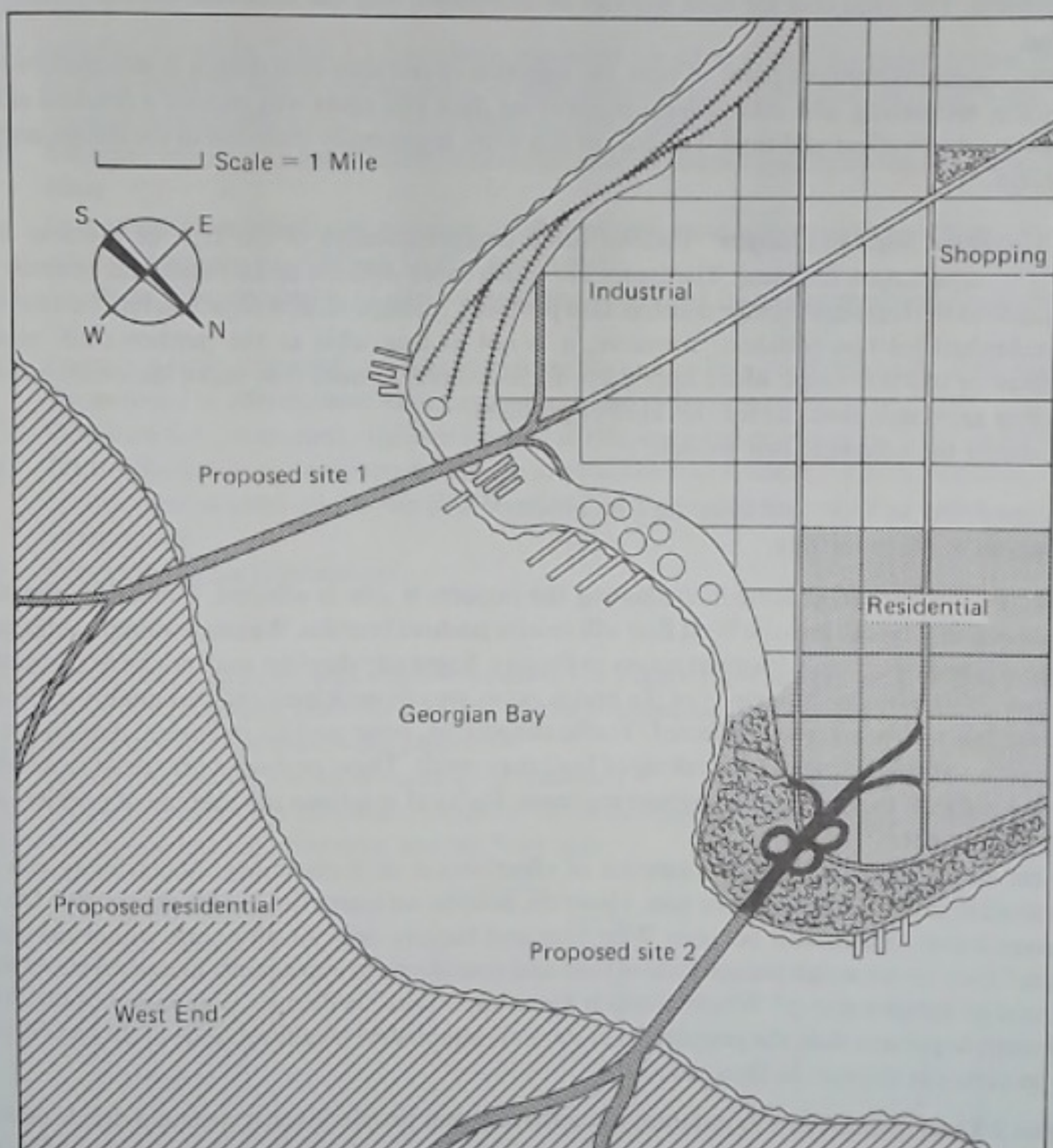


Figure 1.5

Discuss the use of systems analysis in relation to the following goals:

(a) Select a bridge site that will be the least costly to construct.
(b) Select a bridge site that will provide maximum societal benefit to the city.

Solution

(a) Minimize Construction Cost Minimum-cost problems are representative of the type of problem that a structural engineer faces when he designs a bridge. He must find the most economical means to construct the bridge. In this case, the measure of effectiveness will be specified in units of money or dollars. Costs associated with the design, the fabrication, and the construction of the bridge structure and foundation will be considered. Environmental conditions such as soil strength and fluid flow about the bridge piers will be investigated as part of the design process. The total cost of design, fabrication, construction, and material is considered an appropriate measure of effectiveness. The total cost for each site can be compared, and the least-cost alternative can be selected.

From a systems-analysis point of view, the selection of the least-cost design is straightforward. Acquiring, estimating, and establishing engineering data and costs will require a detailed study demanding much effort and time. The cost of this work is generally included in the design cost of the bridge.

(b) Maximize Societal Benefits Facility siting is representative of the type of problem that faces a transportation engineer. The issues are much more difficult to establish and quantify as compared to the least-cost bridge construction problem. A single-valued objective function cannot be established for this problem; therefore, it is not as amenable to the mathematical model presented in this text book. More important, there are other issues that make the mathematical modeling approach ill-suited for this kind of problem.

Consider the following two impacts.

1. Accessibility to West End from the City of Kingsbury.
2. Quality of life in the city.

Impact 1 An important issue in evaluating the impacts is *who* is affected. Obviously, the new population that moves into the West End will receive positive benefits. We can assume that the new bridge will provide them with good access to the city. Some city dwellers may not be as fortunate, however. If they live in the vicinity of the bridge ramp, we can anticipate that the quality of life for these people will be adversely affected. Traffic congestion, noise and air pollution, demolition of neighborhood buildings, and the taking of land may result. These problems are social issues that are very difficult to solve in the engineering sense. Political solutions to this type of problem are generally sought.

Now let us discuss different measures of effectiveness of accessibility. Accessibility can be measured in units of travel time or cost. However, how do we assess the impact on society? Do we measure travel time or cost between West End and the city only, or all points within the study region? How do we weigh reduced travel time and cost to one group against increased travel time and cost to another group? Which group is more important? Is the new population of the West End more important than the population of the well-established neighborhoods of the city? There are no clear-cut answers to these questions.

Impact 2 Quality of life means different things to different people. Even if we could decide upon a set of issues to be considered, how do we collect this information and use it effectively? Possibly we could interview people, asking them which bridge site they prefer, how the new bridge will affect them personally, and how they think it will affect their neighborhood. Obviously, incorporating this information into a model will be subjective at best.

Clearly, we have only scratched the surface of this problem. It should be evident that mathematical modeling is not well suited to the solution of this type of problem. Incidently, an environmental impact statement and public hearings are required for most major projects such as the one being considered in this hypothetical example. Local, state, and federal laws and regulations specify this need.

Systems analysis is an important decision-making tool but not a panacea. When suitable, it provides insight and understanding into optimization problems. Its advantages and shortcomings must be understood for it to be used effectively.

PROBLEMS

Problem 1

The statistics shown in Table 1.1 have been compiled for cities operating major transit systems.

(a) In your opinion which one measure of effectiveness most appropriately reflects the transit systems' overall efficiency? Why? Define overall efficiency.

(b) Are there other measures of effectiveness that reflect a transit system's overall efficiency? List them.

(c) In your opinion which one measure of effectiveness most appropriately reflects the transit systems' productivity? Why? Define productivity.

(d) Are there other measures of effectiveness that reflect the transit system's productivity? List them.

(e) Prepare a table, and rank the transit systems by city as being effective and productive. Use the measure of effectiveness from parts a, b, c, and d for the ranking.

(f) Which transit system ranks the best for overall efficiency and productivity? The worst? Why?

(g) Try to establish a single objective function to reflect overall efficiency utilizing all the measures of effectiveness listed under overall efficiency. List the problems associated with establishing this measure.

(h) Repeat step g for productivity.

Problem 2

Consider a cube, a cylinder, and a sphere as shown in Figure 1.6. They are to be used for storage. For each of the following, state the measure of effectiveness and determine:

(a) Which one has the maximum capacity.

(b) Which one utilizes the most material to construct.

(c) Which one offers the maximum capacity while utilizing the least building material.

(d) Which one has the maximum useable floor area.

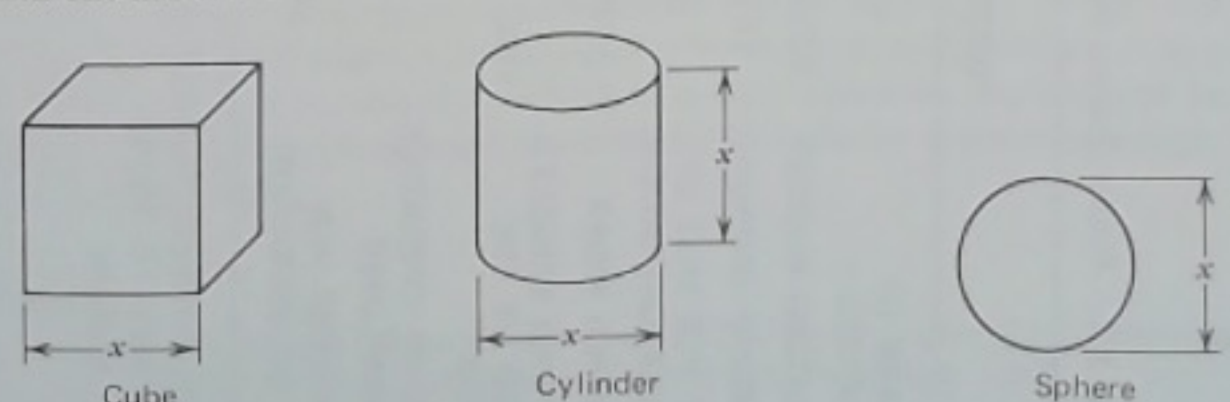


Figure 1.6

Table 1.1 Public Transit Systems[a]

	BOSTON	TORONTO	CHICAGO	NEW YORK	PHILADELPHIA	CLEVELAND	LOS ANGELES	WASHINGTON	SAN FRANCISCO
NUMBER OF EMPLOYEES	5,999	8,405	13,245	36,655	5,904	2,459	6,661	6,145	2,994
OPERATING BUDGET	$235.6M	$213.8M	$394.4M	$972.7M	$163.1M	$80.4M	$206.7M	$168.2M	$85.9M
RIDERSHIP (ANNUAL)	151.4M	337.6M	374.6M	1334.4M	212.9M	115.5M	230.5M	149.9M	120.2M
COST PER MILE OF VEHICLE OPERATION	$6.22	$2.18	$2.96	$3.22	$3.13	$3.05	$1.98	$2.82	$3.39
COST PER RIDER	$1.56	$0.63	$1.05	$0.72	$0.77	$0.69	$0.90	$1.12	$0.71
COST PER VEHICLE OWNED BY THE SYSTEM	$143,982	$92,960	$112,692	$103,984	$73,424	$73,531	$96,403	$76,945	$85,729
MILES OF OPERATION PER EMPLOYEE	6,333	11,672	10,057	8,242	8,825	10,736	15,646	9,690	8,466
WAGE COST PER MILE OF OPERATION	$4.16	$1.54	$2.35	$3.01	$2.20	$1.94	$1.55	$2.13	$2.70

Source: American Public Transit Association, Toronto Transit System, Massachusetts Bay Transportation Authority, Compiled by *Boston Globe*, Nov. 16, 1979.

[a] M = Millions

Problem 3

Identify five factors that may affect land developers in building the following types of structures. In other words briefly explain why:

New high-rise buildings are usually constructed in cities.
New fabrication plants tend to be located in suburbs.

Problem 4

Refer to Example 1.2.

(a) For each land classification—residential, shopping, and industrial—determine three different impacts that may affect the quality of life of the region owing to the construction of the new bridge.

(b) Discuss the problems associated with formulating a single measure of effectiveness to objective function that weighs both positive and negative impacts on the region.

Summary

The systems-analysis approach consists of the following steps.

1. State a goal, establish an appropriate measure of effectiveness, and develop an objective function.
2. Determine the financial, physical, and institutional limitations, and establish a set of constraint equations.
3. Determine a solution, the so-called optimum solution, that achieves the stated goal and satisfies all constraint conditions.

The mathematical model can be written as

$$\text{Minimize or maximize} \quad z = f(x) \quad \text{(objective)}$$

$$\text{subject to} \quad g(x)\{=, \leq, \geq\}b \quad \text{(constraint set)}$$

$$\text{and} \quad x \geq 0$$

where the constraint set consists of m constraint equations and x consists of a set of n control variables, $x_1, x_2, \ldots, x_n$. A control variable is a parameter that the engineer is free to vary during the design, planning, or management process.

The primary attribute of systems analysis is that it provides a systematic step-by-step procedure for formulating mathematical models and obtaining the optimum solution to these models. In this textbook, we discuss graphical and mathematical methods for solving various linear and nonlinear mathematical models. The systems-analysis approach may always be used to formulate problems. However, it is not a panacea; there is no guarantee that an optimum solution will be found. In some cases, the mathematics becomes so complex that a solution cannot be easily determined. For problems consisting of unlike measures of effectiveness or multiple objectives, the formulation of the problem as a mathematical model is difficult. In any event, thinking of an engineering problem in terms of satisfying a goal or objective, subject to a set of financial, physical, and institutional limitations, does provide a convenient framework for formulating engineering design, planning, and management problems.

Bibliography

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