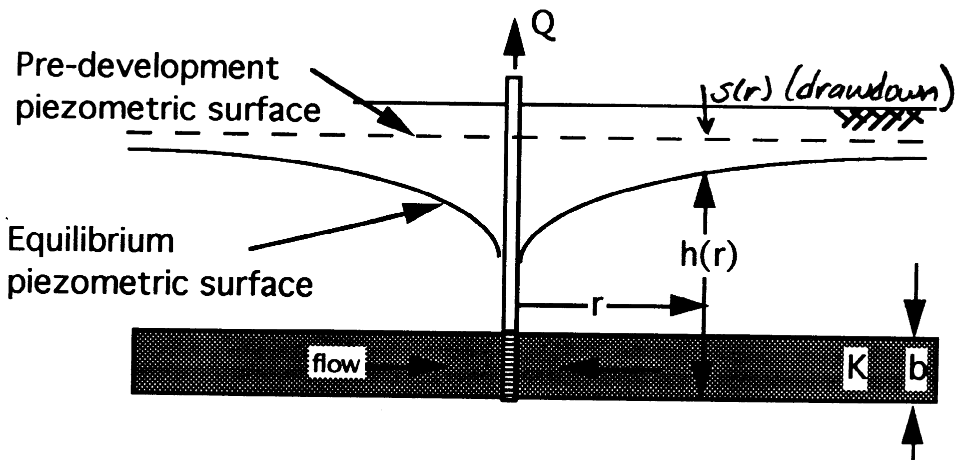
Groundwater & Well Hydraulics

- Steady flow to a well
  confined; unconfined.
- Superposition to represent
  multiple wells; aquifer boundaries
- Transient flow
  Theis solution - confined
  Hantush solution - unconfined
- Superposition to represent
  multiple wells; aquifer boundaries
- Convolution to represent
  time varying pumping rates

Confined aquifer (hydraulic approach)



Homogeneous, isotropic, confined aquifer. Flow in radial direction only. Steady state. No internal sources/sinks.

$$h(r) = h_0 + \frac{Q}{2\pi T} \ln\left(\frac{r}{R}\right)$$

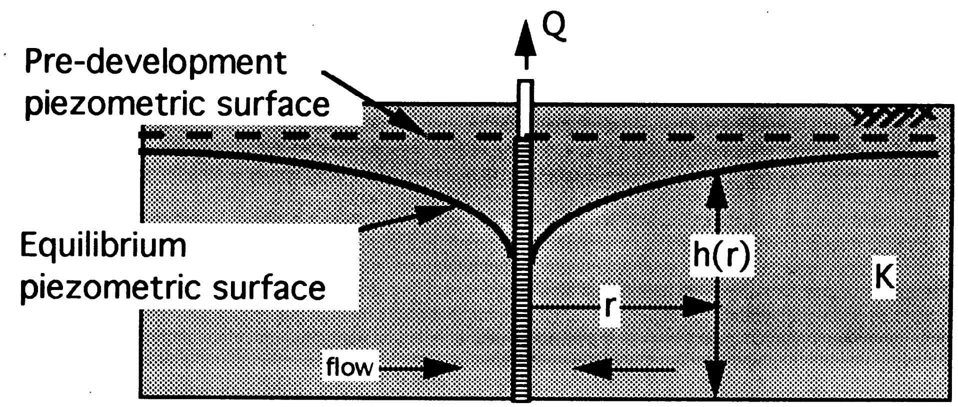
$$s(r) = h_0 - h(r) = -\frac{Q}{2\pi T} \ln\left(\frac{r}{R}\right) = \frac{Q}{2\pi T} \ln\left(\frac{R}{r}\right)$$

$s(r)$ is a solution to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{\partial s}{\partial r}\right)=0.$$

Theim equation $s_1 - s_2 = \frac{Q}{2\pi T}\ln\left(\frac{r_2}{r_1}\right)$ used to infer hydraulic properties.

Unconfined Aquifer



Homogeneous, isotropic, confined aquifer. Flow in radial direction only. Steady state. No internal sources/sinks.

$$h^2(r) = \frac{Q}{\pi K}\ln\left(\frac{r}{R}\right) + h_0^2$$

Corrected drawdown term treats entire aquifer as if sat. thickness is $b = h_0$

$$s'(r) = \frac{Q}{2\pi K h_0}\ln\left(\frac{R}{r}\right)$$

and

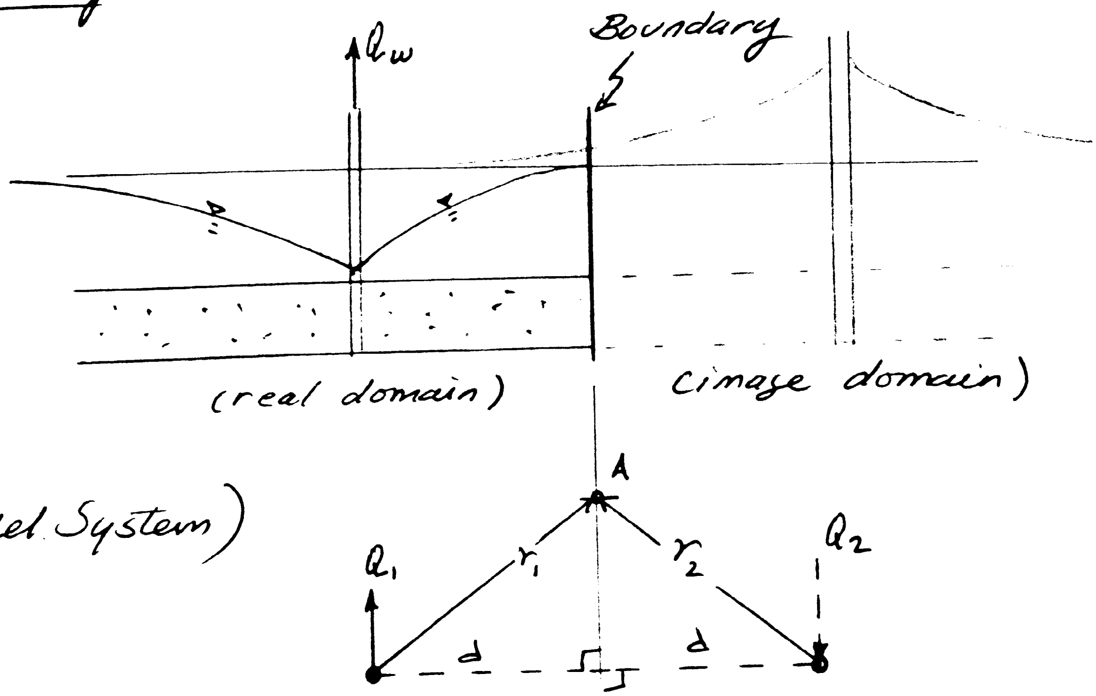
$$s(r) = h_0 - h(r) = h_0 - \sqrt{h_0^2 + \frac{Q}{\pi K}\ln\left(\frac{r}{R}\right)} =$$

$$= h_0\left(1 - \sqrt{1 + \frac{2s'}{h_0}}\right)$$

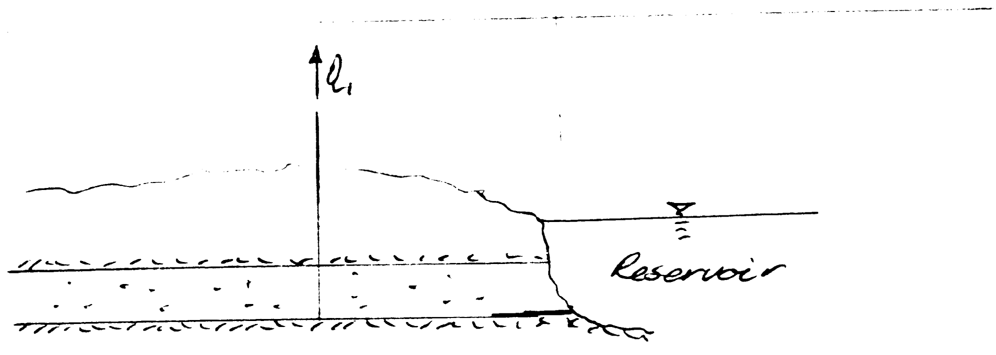
Superposition: Any linear combination of solutions is a solution.

Can be used to construct solutions with multiple wells, or a well in a uniform flow field and so on.

Single Well near a constant head boundary



(Model System)



(Physical System)

$s_A = s_A$ (from well #1) + s_A (from well #2)

But A is a constant head boundary $\therefore s_A = 0$

$$s_A = \frac{Q_1}{2\pi K b} \ln\left(\frac{R}{r_1}\right) - \frac{Q_2}{2\pi K b} \ln\left(\frac{R}{r_2}\right)$$

"−" because well #2 is "modeled" as injection to produce zero drawdown at A (anywhere along boundary)

$$|Q_1| = |Q_2| = |Q_w|$$

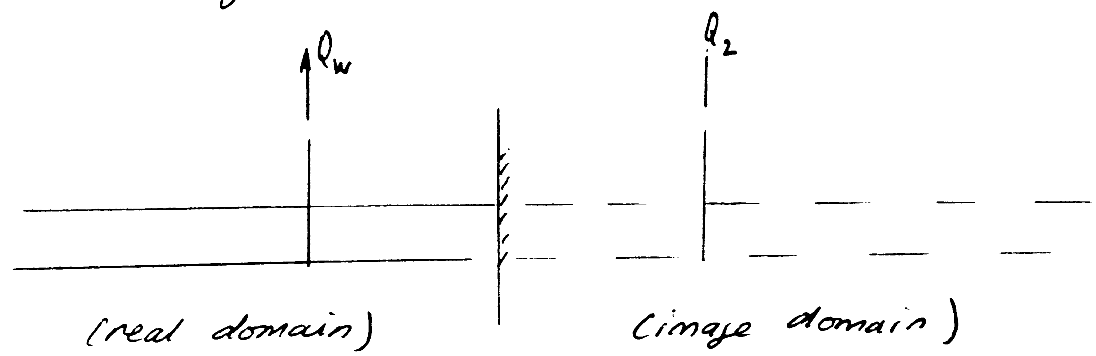
$$\therefore s_A = \frac{Q_w}{2\pi K b} \ln\left|\frac{R}{r_1}\right| - \frac{Q_w}{2\pi K b} \ln\left|\frac{R}{r_2}\right|$$

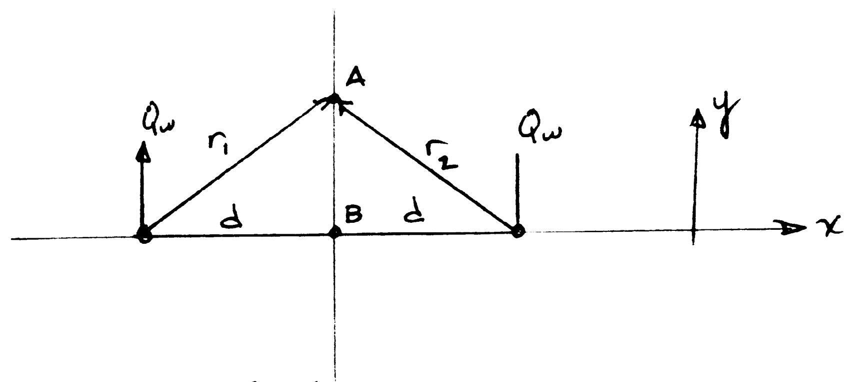
$$= \frac{Q_w}{2\pi K b} \ln\left|\frac{r_2}{r_1}\right|$$

Now if Q_2 is located the same distance from the boundary as Q_1, $r_2 = r_1$

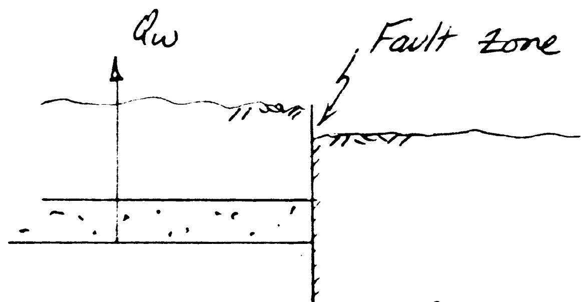
$\Rightarrow s_a = 0$ (as expected)

No-Flow Boundary





Model System



$$s_A = \frac{Q_w}{2\pi K b}\ln\left(\frac{R}{r_1}\right) + \frac{Q_w}{2\pi K b}\ln\left(\frac{R}{r_2}\right) = \frac{Q_w}{2\pi K b}\ln\left(\frac{R^2}{r_1 r_2}\right)$$

$$s_B = \frac{Q_w}{2\pi K b}\ln\left(\frac{R^2}{d^2}\right)$$

$$\left.\frac{\partial s_B}{\partial x}\right|_{\text{Well \#1}} = \frac{Q_w}{2\pi K b d} \quad ; \quad \left.\frac{\partial s_B}{\partial x}\right|_{\text{Well \#2}} = \frac{-Q_w}{2\pi K b d}$$

$$\left.\frac{\partial s_B}{\partial x}\right|_{\text{both wells}} = \frac{Q_w}{2\pi K b d} - \frac{Q_w}{2\pi K b d} = 0$$

(This will be the result for all points along the boundary)

<u>Summary</u>

① drawdown in confined aquifer due to well (steady flow)

$$s(r) = \frac{Q_w}{2\pi K b} \ln\left|\frac{R}{r}\right|$$

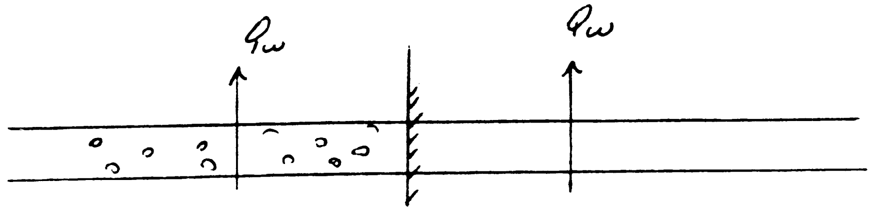
② Constant head boundary:

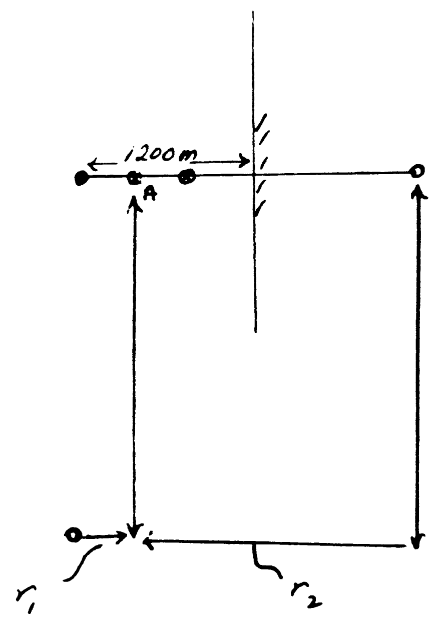
a) locate image well same <u>distance</u> from boundary as real well, opposite sense on Q_w

③ No flow boundary

b) locate image well same distance from boundary as real well, same sense on Q_w

Example





$Q_w = 100 \, m^3/d$

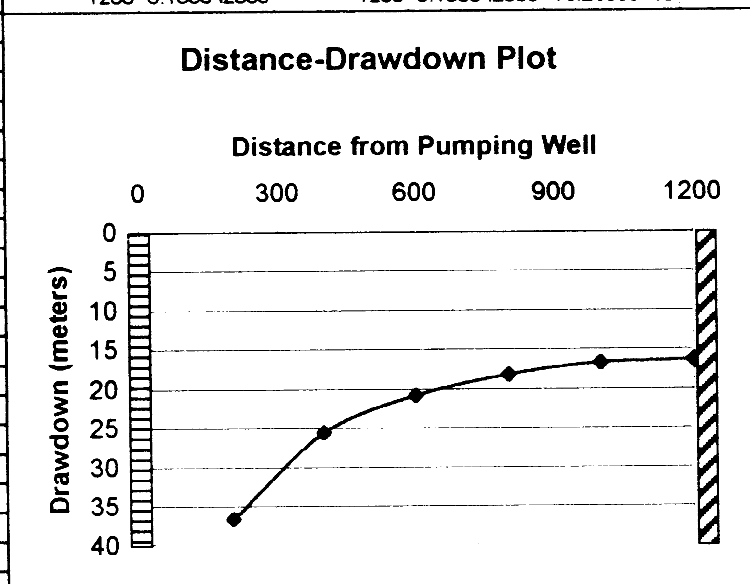
$Kb = 1 \, m^2/d$

$R = 2000 \, m$

Plot distance-drawdown profile between well and boundary

$$s_A = \frac{Q_w}{2\pi Kb} \ln\left(\frac{R}{r_1}\right) + \frac{Q_w}{2\pi Kb} \ln\left(\frac{R}{r_2}\right)$$

	A	B	C	D	E	F
1	Qw	100 m3/d				
2	T	1 m2/d				
3	R	2000 m				
4	real well		image well			
5	distance from well to field point (m)	drawdown due to well	distance from well to field point (m)	drawdown due to well	drawdown at field point	
6	200	36.64677994	2200	0	36.64677994	
7	400	25.61499994	2000	0	25.61499994	
8	600	19.16182232	1800	1.676864687	20.838687	
9	800	14.58321993	1600	3.551439921	18.13465985	
10	1000	11.03178001	1400	5.67665804	16.70843805	
11	1200	8.130042308	1200	8.130042308	16.26008462	



AQUIFER BOUNDARIES

One of the assumptions inherent in the Theis equation (and in most other fundamental ground-water flow equations) is that the aquifer to which it is being applied is infinite in extent. Obviously, no such aquifer exists on Earth. However, many aquifers are areally extensive, and, because pumping will not affect recharge or discharge significantly for many years, most water pumped is from ground-water storage; as a consequence, water levels must decline for many years. An excellent example of such an aquifer is that underlying the High Plains from Texas to South Dakota.

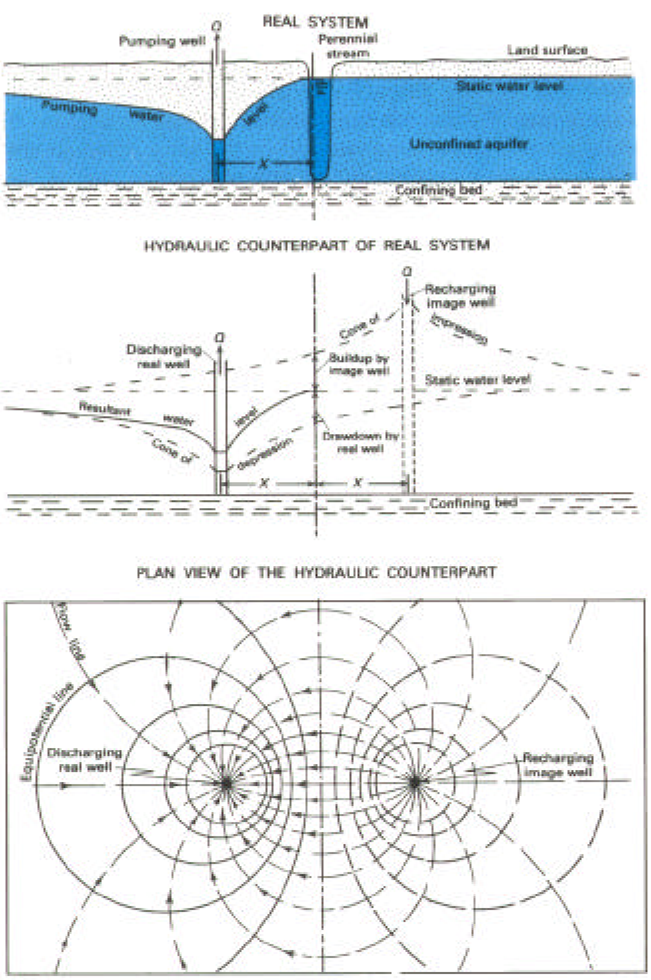
All aquifers are bounded in both the vertical direction and the horizontal direction. For example, vertical boundaries may include the water table, the plane of contact between each aquifer and each confining bed, and the plane marking the lower limit of the zone of interconnected openings—in other words, the base of the ground-water system.

Hydraulically, aquifer boundaries are of two types: recharge boundaries and impermeable boundaries. A *recharge boundary* is a boundary along which flow lines originate. In other words, such a boundary will, under certain hydraulic conditions, serve as a source of recharge to the aquifer. Examples of recharge boundaries include the zones of contact between an aquifer and a perennial stream that completely penetrates the aquifer or the ocean.

An *impermeable boundary* is a boundary that flow lines do not cross. Such boundaries exist where aquifers terminate against "impermeable" material. Examples include the contact between an aquifer composed of sand and a laterally adjacent bed composed of clay.

The position and nature of aquifer boundaries are of critical importance in many ground-water problems, including the movement and fate of pollutants and the response of aquifers to withdrawals. Depending on the direction of the hydraulic gradient, a stream, for example, may be either the source or the destination of a pollutant.

Lateral boundaries within the cone of depression have a profound effect on the response of an aquifer to withdrawals. To analyze, or to predict, the effect of a lateral boundary, it is necessary to "make" the aquifer appear to be of infinite extent. This feat is accomplished through the use of imaginary



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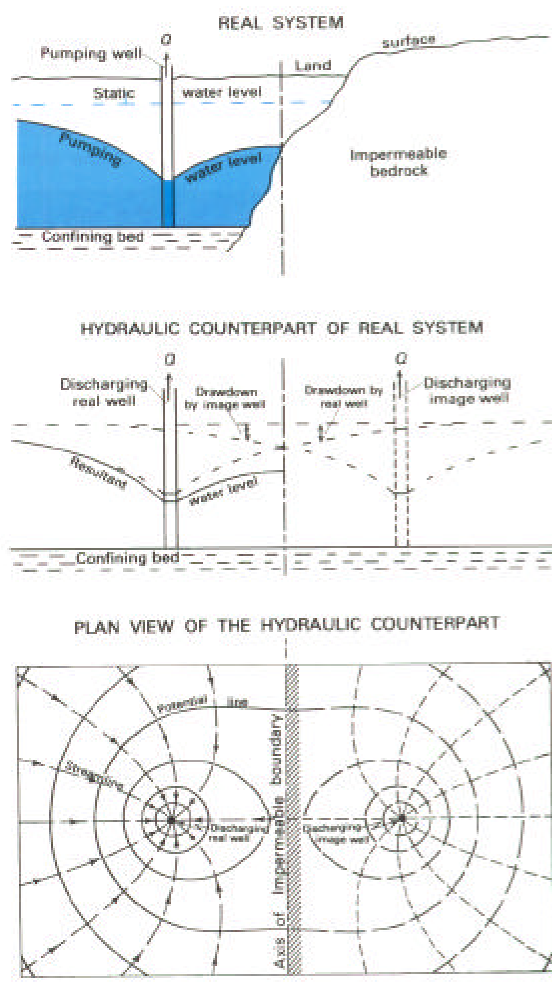


(2)

wells and the *theory of images*. Sketches 1 and 2 show, in both plan view and profile, how image wells are used to compensate, hydraulically, for the effects of both recharging and impermeable boundaries. (See "Well Interference.")

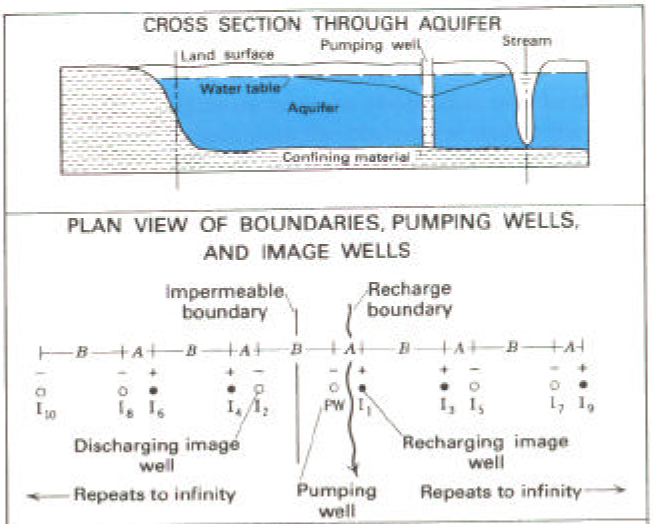
The key feature of a recharge boundary is that withdrawals from the aquifer do not produce drawdowns across the boundary. A perennial stream in intimate contact with an aquifer represents a recharge boundary because pumping from the aquifer will induce recharge from the stream. The hydraulic effect of a recharge boundary can be duplicated by assuming that a recharging image well is present on the side of the boundary opposite the real discharging well. Water is injected into the image well at the same rate and on the same schedule that water is withdrawn from the real well. In the plan view in sketch 1, flow lines originate at the boundary, and equipotential lines parallel the boundary at the closest point to the pumping (real) well.

The key feature of an impermeable boundary is that no water can cross it. Such a boundary, sometimes termed a "no-flow boundary," resembles a divide in the water table or the potentiometric surface of a confined aquifer. The effect of an impermeable boundary can be duplicated by assuming that a discharging image well is present on the side of the boundary opposite the real discharging well. The image well withdraws water at the same rate and on the same schedule as the real well. Flow lines tend to be parallel to an impermeable boundary, and equipotential lines intersect it at a right angle.

The image-well theory is an essential tool in the design of well fields near aquifer boundaries. Thus, on the basis of minimizing the lowering of water levels, the following conditions apply:

1. Pumping wells should be located parallel to and as close as possible to recharging boundaries.
2. Pumping wells should be located perpendicular to and as far as possible from impermeable boundaries.

Sketches 1 and 2 illustrate the effect of single boundaries and show how their hydraulic effect is compensated for through the use of single image wells. It is assumed in these sketches that other boundaries are so remote that they have a negligible effect on the areas depicted. At many places, however, pumping wells are affected by two or more boundaries. One example is an alluvial aquifer composed of sand and gravel bordered on one side by a perennial stream (a recharge boundary) and on the other by impermeable bedrock (an impermeable boundary).



Impermeable boundary		Recharge boundary	
I_2	PW	PW	I_1
I_4	I_1	I_2	I_3
I_6	I_3	I_4	I_5
I_8	I_5	I_6	I_7
I_{10}	I_7	I_8	I_9

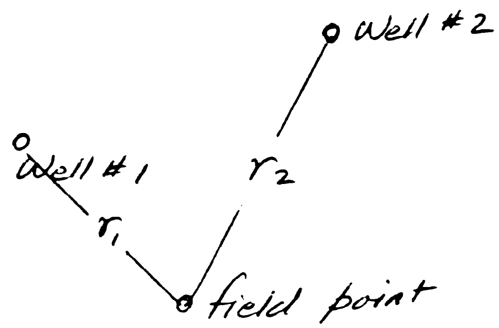
(3)

Contrary to first impression, these boundary conditions cannot be satisfied with only a recharging image well and a discharging image well. Additional image wells are required, as sketch 3 shows, to compensate for the effect of the image wells on the opposite boundaries. Because each new image well added to the array affects the opposite boundary, it is necessary to continue adding image wells until their distances from the boundaries are so great that their effect becomes negligible.

Well Interference

Two wells operating near each other produce a combination drawdown that might affect operation of wells.

This effect is called well interference - it is easily modeled using superposition.



$$s_{\text{field point}} = \frac{Q_1}{2\pi Kb} \ln\left|\frac{R}{r_1}\right| + \frac{Q_2}{2\pi Kb} \ln\left|\frac{R}{r_2}\right|$$

Example

$T = 1\,m^2/d$

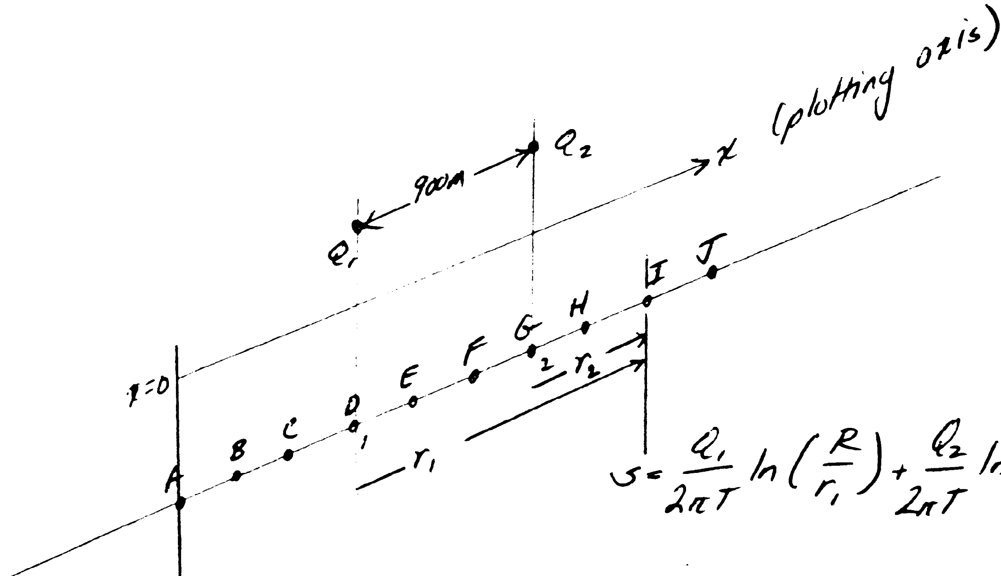
$Q_1 = 100\,m^3/d$

$Q_2 = 200\,m^3/d$

$R = 2000\,m$

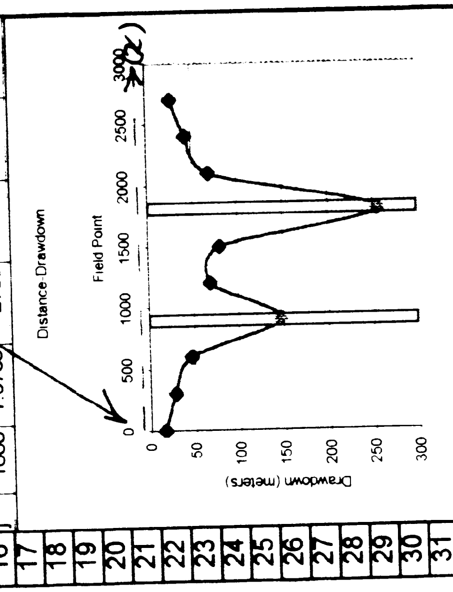
Wells located 900m apart. Show distance-drawdown profile.

Determine "interference" at well #1 due to operation of well #2



$$s = \frac{Q_1}{2\pi T}\ln\left(\frac{R}{r_1}\right) + \frac{Q_2}{2\pi T}\ln\left(\frac{R}{r_2}\right)$$

	A	B	C	D	E	F	G
1			Well#1			Well#2	
2		Qw	100	m3/d	Qw	200	m3/d
3		T	1	m2/d	T	1	m2/d
4		R	2000	m	R	2000	m
5							
6	field point	distance from well to field point (m)	drawdown due to well	distance from A	distance from well to field point (m)	drawdown due to well	drawdown at field point
7	a	900	12.709	0	1800	3.3537	16.062
8	b	600	19.162	300	1500	9.1572	28.319
9	c	300	30.194	600	1200	16.26	46.454
10	d	1	120.97	900	900	$\boxed{25.417}$	146.39
11	e	300	30.194	1200	600	38.324	68.517
12	f	600	19.162	1500	300	60.387	79.549
13	g	900	12.709	1800	1	241.94	254.65
14	h	1200	8.13	2100	300	60.387	68.517
15	i	1500	4.5786	2400	600	38.324	42.902
16	j	1800	1.6769	2700	900	25.417	27.094



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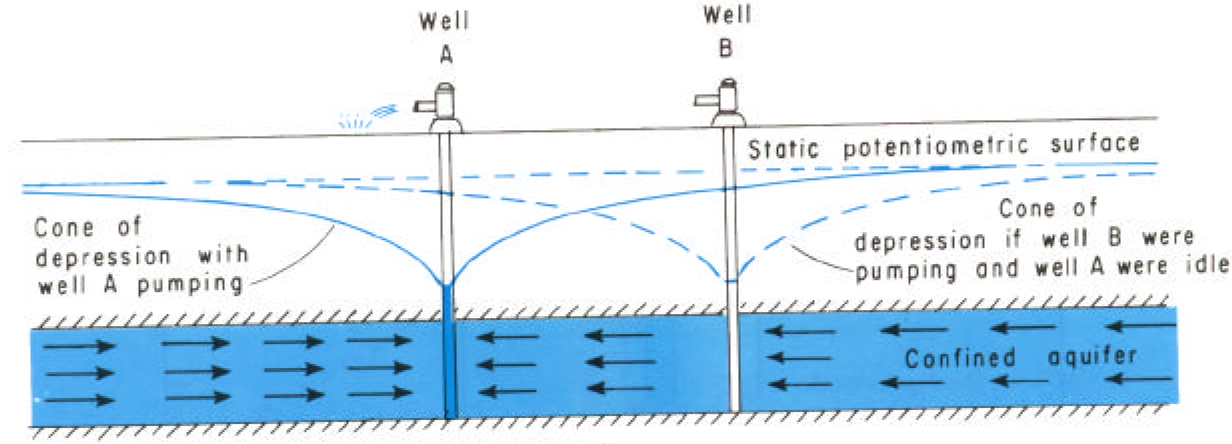
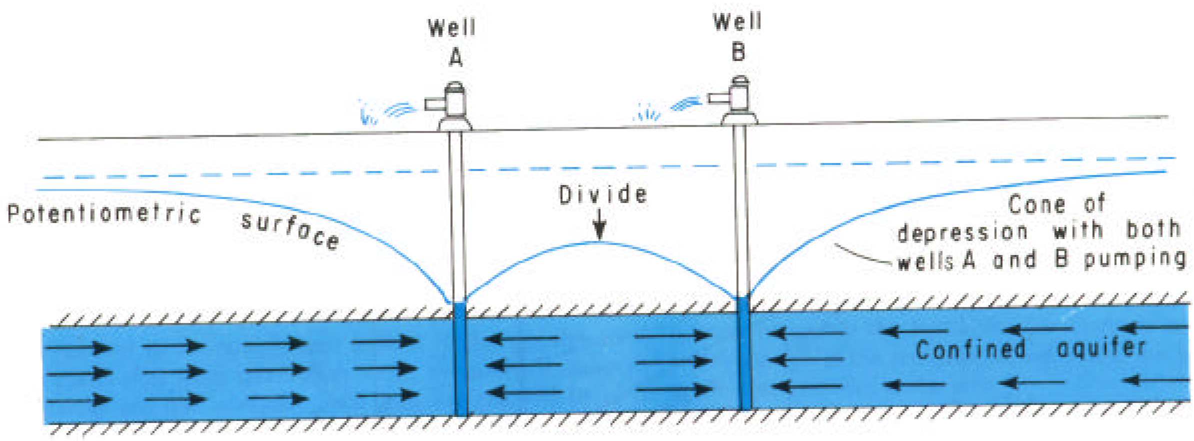
WELL INTERFERENCE



(1)



(2)

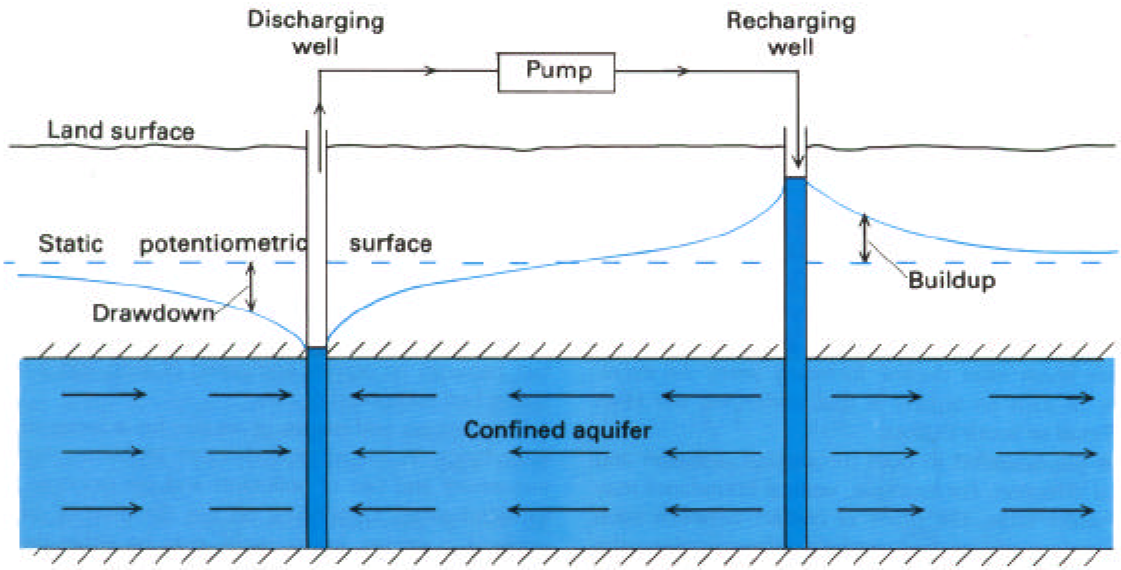
Pumping a well causes a drawdown in the ground-water level in the surrounding area. The drawdown in water level forms a conical-shaped depression in the water table or potentiometric surface, which is referred to as a *cone of depression*. (See "Cone of Depression.") Similarly, a well through which water is injected into an aquifer (that is, a recharge or injection well) causes a buildup in ground-water level in the form of a conical-shaped mound.

The drawdown (s) in an aquifer caused by pumping at any point in the aquifer is directly proportional to the pumping rate (Q) and the length of time (t) that pumping has been in progress and is inversely proportional to the transmissivity (T), the storage coefficient (S), and the square of the distance (r^2) between the pumping well and the point. In other words,

$$s \approx \frac{Q, t}{T, S, r^2} \qquad (1)$$

Where pumping wells are spaced relatively close together, pumping of one will cause a drawdown in the others. Drawdowns are additive, so that the total drawdown in a pumping well is equal to its own drawdown plus the drawdowns caused at its location by other pumping wells (1) (2). The drawdowns in pumping wells caused by withdrawals from other pumping wells are referred to as *well interference*. As sketch 2 shows, a divide forms in the potentiometric surface (or the water table, in the case of an unconfined aquifer) between pumping wells.

At any point in an aquifer affected by both a discharging well and a recharging well, the change in water level is equal to the difference between the drawdown and the buildup. If the rates of discharge and recharge are the same and if the wells are operated on the same schedule, the drawdown and the buildup will cancel midway between the wells, and the water level at that point will remain unchanged from the static level (3). (See "Aquifer Boundaries.")



(3)

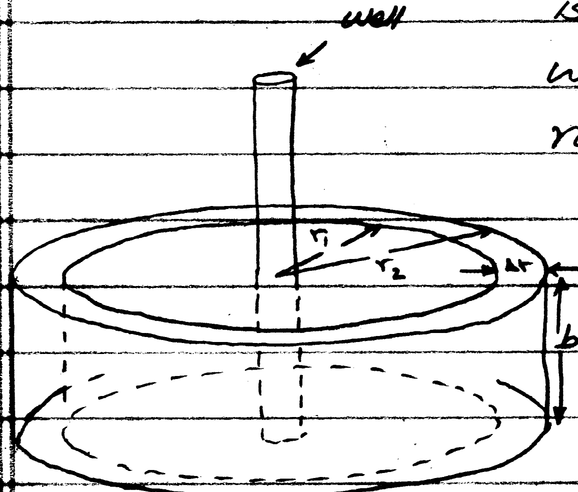
We see from the above functional equation that, in the absence of well interference, drawdown in an aquifer at the effective radius of a pumping well is directly proportional to the pumping rate. Conversely, the maximum pumping rate is directly proportional to the *available drawdown*. For confined aquifers, available drawdown is normally considered to be the distance between the prepumping water level and the top of the aquifer. For unconfined aquifers, available drawdown is normally considered to be about 60 percent of the saturated aquifer thickness.

Where the pumping rate of a well is such that only a part of the available drawdown is utilized, the only effect of well interference is to lower the pumping level and, thereby, increase pumping costs. In the design of a well field, the increase in pumping cost must be evaluated along with the cost of the additional waterlines and powerlines that must be installed if the spacing of wells is increased to reduce well interference. (See "Well-Field Design.")

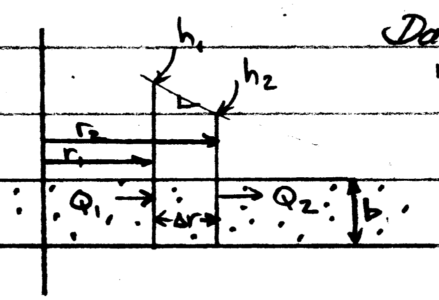
Because well interference reduces the available drawdown, it also reduces the maximum yield of a well. Well interference is, therefore, an important matter in the design of well fields where it is desirable for each well to be pumped at the largest possible rate. We can see from equation 1 that, for a group of wells pumped at the same rate and on the same schedule, the well interference caused by any well on another well in the group is inversely proportional to the square of the distance between the two wells (r^2). Therefore, excessive well interference is avoided by increasing the spacing between wells and by locating the wells along a line rather than in a circle or in a grid pattern.

Transient flow to a well

consider a well in a confined aquifer. A cylindrical shell or prism is depicted, co-axial with the well extending the entire thickness, b, of the aquifer. The thickness of the shell is Δr. The inner surface is located at radius, r_1, from the well, the outer surface is located at radius, r_2, from the well. Flow is assumed to be radial, thus there is no variation in vertical or angular directions (1D radial).



We will study the effect of an impulse (slug) of water injected into the well, so flow is outward, away from the well in the $+r$ direction. Hydraulic conductivity is K, transmissivity, T, and storage coefficient S.



Darcy's law at r_1 is $Q_1 = -2\pi r_1 Kb\left(\frac{\partial h}{\partial r}\right)_1$

" " r_2 is $Q_2 = -2\pi r_2 Kb\left(\frac{\partial h}{\partial r}\right)_2$

The constant Kb is the aquifer transmissivity T. Furthermore $r_1\left(\frac{\partial h}{\partial r}\right)_1$ can be expressed as $\left(r\frac{\partial h}{\partial r}\right)_1$

$\therefore$ At r_1, $Q_1 = -2\pi T\left(r\frac{\partial h}{\partial r}\right)_1$

and

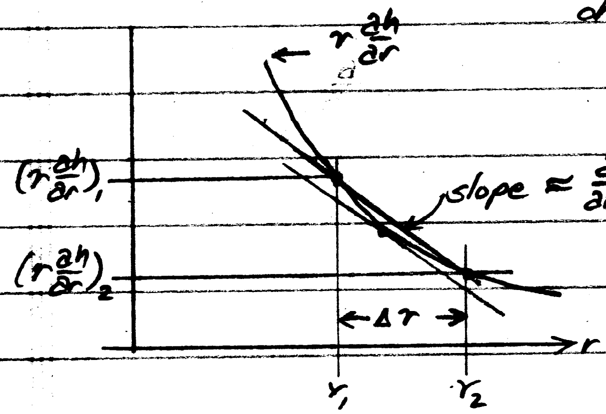
At r_2, $Q_2 = -2\pi T\left(r\frac{\partial h}{\partial r}\right)_2$

thus $\dot{I} - \dot{O} = Q_1 - Q_2$

$= 2\pi T\left(\left[r\frac{\partial h}{\partial r}\right]_2 - \left[r\frac{\partial h}{\partial r}\right]_1\right)$

The term $\left(r\frac{\partial h}{\partial r}\right)_2 - \left(r\frac{\partial h}{\partial r}\right)_1$ represents the change in the value of $r\frac{\partial h}{\partial r}$ between r_1 and r_2. We can approximate this change as

$$\left(r\frac{\partial h}{\partial r}\right)_2 - \left(r\frac{\partial h}{\partial r}\right)_1 \approx \Delta r \frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right)$$



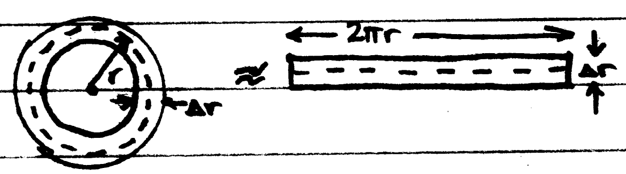
(obviously as $\Delta r \to 0$ the approximation becomes more accurate)

Now if we apply the product rule from calculus we obtain

$$\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) = r\frac{\partial^2 h}{\partial r^2} + \frac{\partial h}{\partial r}\frac{\partial r}{\partial r} = \frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}$$

$$\therefore\ Q_1 - Q_2 \approx 2\pi T\left(\Delta r\left\{\frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right\}\right)$$

Now we want to relate this expression for $\dot{I}-\dot{O}$ to the change in storage in the prism. The surface area of the top of the prism is approximately $A = 2\pi r\,\Delta r$. Again as $\Delta r \to 0$ the approximation becomes more accurate.



Recall that $\frac{d\forall}{dt} = SA\frac{dh}{dt}$, thus $\frac{d\forall}{dt}$ for the prism is

$$\frac{d\forall}{dt} = S\,2\pi r\,\Delta r\,\frac{\partial h}{\partial t}$$

Now relate $\dot{I}-\dot{O}$ to $\frac{d\forall}{dt}$ $\rightarrow$ $\dot{I}-\dot{O} = \frac{d\forall}{dt}$

$$\dot{I} - \dot{O} = 2\pi T \Delta r \left\{ \frac{\partial h}{\partial r} + r \frac{\partial^2 h}{\partial r^2} \right\} ; \quad \frac{d\forall}{dt} = 2\pi r \Delta r S \frac{\partial h}{\partial t}$$

$$\therefore \quad \cancel{2\pi} T \cancel{\Delta r} \left\{ \frac{\partial h}{\partial r} + r \frac{\partial^2 h}{\partial r^2} \right\} = \cancel{2\pi} r \cancel{\Delta r} S \frac{\partial h}{\partial t}$$

Rearrange as: $\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = \frac{S}{T}\frac{\partial h}{\partial t}$ } This is the "classic" PDE for radial flow to/from a well.

The next challenge is to find one particular (fundamental) solution to the PDE. Once we have a fundamental solution we can construct many different solutions as linear combinations of the fundamental solutions.

For example consider: $h = \frac{\forall}{4\pi T t} \exp\left(-\frac{r^2 S}{4Tt}\right)$. How to determine if it is a solution? Simply substitute into PDE and see if equality is preserved:

$$\frac{\partial h}{\partial r} = \frac{\forall}{4\pi T t} \exp\left(\frac{-r^2 S}{4Tt}\right) \cdot \left(\frac{-2rS}{4Tt}\right)$$

$$\frac{\partial^2 h}{\partial r^2} = \frac{\forall}{4\pi T t} \left\{ \exp\left(\frac{-r^2 S}{4Tt}\right) \cdot \left(\frac{-2S}{4Tt}\right) + \left(\frac{-2rS}{4Tt}\right) \exp\left(\frac{-r^2 S}{4Tt}\right) \cdot \left(\frac{-2rS}{4Tt}\right) \right\}$$

$$\frac{\partial h}{\partial t} = \frac{\forall}{4\pi T t} \exp\left(\frac{-r^2 S}{4Tt}\right)\left(\frac{r^2 S}{4Tt^2}\right) - \frac{\forall}{4\pi T t^2} \exp\left(\frac{-r^2 S}{4Tt}\right)$$

Now substitute

$$\frac{1}{r}\frac{\forall}{4\pi T t} \exp\left(\frac{-r^2 S}{4Tt}\right) \cdot \left(\frac{-2rS}{4Tt}\right) + \frac{\forall}{4\pi T t} \left\{ \exp\left(\frac{-r^2 S}{4Tt}\right) \cdot \left(\frac{-2S}{4Tt}\right) + \left(\frac{-2rS}{4Tt}\right)^2 \exp\left(\frac{-r^2 S}{4Tt}\right) \right\}$$

$$\stackrel{?}{=} \frac{S}{T}\left[\frac{\forall}{4\pi T t} \exp\left(\frac{-r^2 S}{4Tt}\right)\left(\frac{r^2 S}{4Tt^2}\right) - \frac{\forall}{4\pi T t^2} \exp\left(\frac{-r^2 S}{4Tt}\right) \right]$$

let $\frac{\forall}{4\pi T t} \exp\left(\frac{-r^2 S}{4Tt}\right) = A$

Then

$$\frac{1}{r}A\left(\frac{-2rS}{4Tt}\right) + A\left(\frac{-2S}{4Tt}\right) + A\left(\frac{-2rS}{4Tt}\right)^2 \overset{?}{=} \frac{S}{T}\left[A\left(\frac{r^2S}{4Tt^2}\right) - \frac{1}{t}A\right]$$

$$A\left(-\frac{4S}{4Tt}\right) + A\left(\frac{-2rS}{4Tt}\right)^2 \overset{?}{=} A\left[\frac{S}{T}\left(\frac{r^2S}{4Tt^2}\right) - \frac{1}{t}\frac{S}{T}\right]$$

$$A\left[\frac{-S}{Tt} + \cancel{4}\left(\frac{r^2S}{\cancel{4}Tt}\right)\frac{S}{Tt}\right] \overset{?}{=} A\left[\frac{S}{T}\left(\frac{r^2S}{4Tt^2}\right) - \frac{1}{t}\frac{S}{T}\right]$$

$$A\left[\frac{S}{T}\left(\frac{r^2S}{4Tt^2}\right) - \frac{1}{t}\frac{S}{T}\right] \overset{?}{=} A\left[\frac{S}{T}\left(\frac{r^2S}{4Tt^2}\right) - \frac{1}{t}\frac{S}{T}\right] \checkmark \text{ yes}$$

$\therefore$ $h = \frac{\forall}{4\pi Tt}\exp\left(\frac{-r^2S}{4Tt}\right)$ is a solution to $\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = \frac{S}{T}\frac{\partial h}{\partial t}$

This "particular" solution describes the head in an infinite (area), horizontal, homogeneous, isotropic confined aquifer when a fixed volume of water $\forall$ is instantly inserted into a fully penetrating well of negligible radius ($r_w \to 0$) at $r=0$. It assumes h everywhere prior to injection was $h=0$ at $t \leq 0$.

Next we need to investigate boundary conditions so this solution can be applied to various problems

The charge of water is added to well at $t=0$, so $h=0$ everywhere except at $r=0$. At $r=0$, $\Delta h = \frac{\forall}{\pi r_w^2}$. But $r_w \to 0$, thus Δh @ $r=0$ is huge ($\Delta h \to \infty$ at $r=0$). Now we need to test the solution to see if such conditions are approximated.

Does $h = \dfrac{V}{4\pi T t}\exp\left(\dfrac{-r^2 S}{4Tt}\right)$ have required properties at $t \to 0, r \to$

At $r=0$ $h = \dfrac{V}{4\pi T t}$; as $t \to 0$, $h \to \infty$ as required

Furthermore at $t=0, r \neq 0$ we have

$$h = \frac{V}{4\pi T t}\left[\frac{1}{\underbrace{1 + \frac{r^2 S}{4Tt} + \frac{\left(\frac{r^2 S}{4Tt}\right)^2}{2!} + \frac{\left(\frac{r^2 S}{4Tt}\right)^3}{3!} + \cdots}_{\text{Series expansion of } \exp\left(-\frac{r^2 S}{4Tt}\right)}}\right]$$

$$h = \frac{V}{4\pi T t + \frac{\cancel{4\pi T t}\, r^2 S}{\cancel{4Tt}} + \frac{\cancel{4\pi T t}\,(r^2 S)^2}{2!\,(\cancel{4Tt})(4Tt)} + \cdots}$$

$$h = \frac{V}{4\pi T t + \pi r^2 S + \frac{\pi (r^2 S)^2}{2!\,4Tt} + \cdots}$$

at $t \to 0$

$$\left.\begin{array}{l} 4\pi T t \to 0 \\ \pi r^2 S \to \text{constant} \\ \frac{\pi (r^2 S)^2}{2!\,4Tt} \text{ \& all H.O.T} \to \infty \end{array}\right\} \quad h \to \frac{V}{\infty} \to 0$$

as required.

$\therefore$

$h = \dfrac{V}{4\pi T t}\exp\left(\dfrac{-r^2 S}{4Tt}\right)$ behaves properly at $r=0, t=0$

$r \neq 0, t=0$

Next consider behavior at large values of r. A finite volume $\forall$ injected into the aquifer would be expected to be undetectable in terms of h a large distance from the injection, regardless of time. (Think of the ripples from a stone dropped into a huge pond).

Mathematically we require $h \to 0$ as $r \to \infty$.

Does $h = \frac{\forall}{4\pi T t} \exp\left(\frac{-r^2 S}{4Tt}\right)$ have required behavior?

as $r \to \infty$ $\frac{-r^2 S}{4Tt} \to -\infty$ and $r^2 \to \infty$ faster than $t \to \infty$

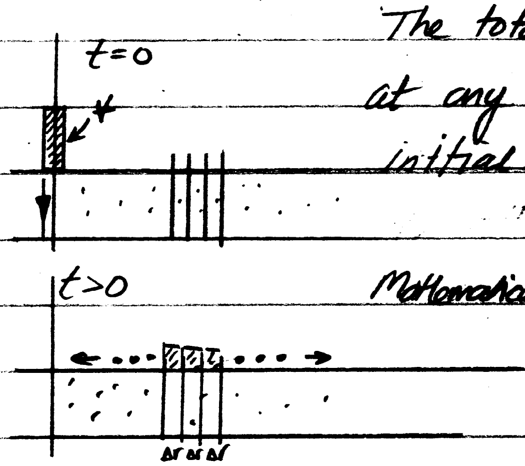
$\therefore$ $r \to \infty$ $\frac{-r^2 S}{4Tt} \to \infty$ regardless of t.

Thus at $r \to \infty$ $h \to \frac{\forall}{4\pi T t} \exp(-\infty)$

$$\exp(-\infty) = \frac{1}{\exp(\infty)} = \frac{1}{\infty} = 0$$

$\therefore$ $h = \frac{\forall}{4\pi T t} \exp\left(\frac{-r^2 S}{4Tt}\right)$ has required properties at large value of r.

The last condition to check is conservation of mass (volume).



The total added water volume in the aquifer at any time should equal the initial charge of water added.

Mathematically $\forall = \int_0^\infty h S 2\pi r \, dr$

Now we have the lovely task of testing this requirement

$$\int_0^\infty h S 2\pi r\,dr = S2\pi \int_0^\infty h r\,dr = S2\pi \int_0^\infty \frac{V}{4\pi T t} \exp\left(\frac{-r^2 S}{4Tt}\right) r\,dr$$

$$= S2\pi \frac{\forall}{4\pi T t} \int_0^\infty \exp\left(\frac{-r^2 S}{4Tt}\right) r\,dr$$

Now let $u = r^2 \qquad du = 2r\,dr$

$$S\pi \frac{\forall}{4\pi T t} \int_0^\infty \exp\left(\frac{-r^2 S}{4Tt}\right) 2r\,dr = S\pi \frac{\forall}{4\pi T t} \int_0^\infty \exp\left(\frac{-uS}{4Tt}\right) du$$

$$= \cancel{S\pi} \frac{\forall}{\cancel{4\pi T t}} \cdot \frac{-\cancel{4Tt}}{\cancel{S}} \exp\left(\frac{-uS}{4Tt}\right)\Big|_0^\infty = -\forall \exp\left(\frac{-uS}{4Tt}\right)\Big|_0^\infty$$

$$= \forall \exp\left(\frac{-uS}{4Tt}\right)\Big|_\infty^0 = \forall [1 - 0] = \forall$$

$\therefore$ The solution satisfies the mass conservation condition.

To review

(i) Derive governing PDE

(ii) Postulate & test solution

(iii) Test boundary, initial & mass conservation conditions

Once we have a fundamental solution we can construct additional solutions. In this case the governing PDE is linear in h — Thus any linear combination of solutions is also a solution (to the PDE).

consider two injections one at $t=0$ a second at $t=t_1$.

$$h = \frac{\mathcal{V}_1}{4\pi T(t-0)} \exp\left(\frac{-r^2 S}{4T(t-0)}\right) + \frac{\mathcal{V}_2}{4\pi T(t-t_1)} \exp\left(\frac{-r^2 S}{4T(t-t_1)}\right)$$

Now the time term contains a "lag" term $(0, t_1)$. If the injections have identical volume we can write

$$h = \frac{\mathcal{V}}{4\pi T}\left(\frac{1}{t-0} \exp\left(\frac{-r^2 S}{4T(t-0)}\right) + \frac{1}{t-t_1} \exp\left(\frac{-r^2 S}{4T(t-t_1)}\right)\right)$$

Now consider constant injection at rate $\frac{d\mathcal{V}}{d\tau} = Q$

Then

$$dh = \frac{Q\,d\tau}{4\pi T(t-\tau)} \exp\left(\frac{-r^2 S}{4T(t-\tau)}\right)$$

To obtain the expression for any time t we integrate with respect to τ

$$\left.\left(\begin{array}{c} \int \frac{d\mathcal{V}}{d\tau}\, d\tau = \int Q\, d\tau \\ \mathcal{V} = Q\tau \end{array}\right)\right\} \Rightarrow \int_0^t \frac{Q}{4\pi T(t-\tau)} \exp\left(\frac{-r^2 S}{4T(t-\tau)}\right) d\tau$$

$$\rightarrow h = \frac{Q}{4\pi T} \int_0^t \frac{1}{(t-\tau)} \exp\left(\frac{-r^2 S}{4T(t-\tau)}\right) d\tau$$

If we let $u = \dfrac{r^2 S}{4T(t-\tau)}$ then

$$h = \frac{Q}{4\pi T} \int_{\frac{r^2 S}{4Tt}}^{\infty} \frac{1}{u} \exp(-u)\, du$$

This last integral is called the exponential integral

$$\int_z^\infty \frac{1}{u} e^{-u} du$$

It is "integrated" by a series expansion to produce

$$\int_z^\infty \frac{1}{u} e^{-u} du = -0.5772 - \ln(z) + z - \frac{z^2}{2 \cdot 2!} + \frac{z^3}{3 \cdot 3!} - \frac{z^4}{4 \cdot 4!} \pm \cdots$$

The integral is also called the well function in groundwater hydraulics and is denoted by $W(\ \ )$

$$\therefore \quad h(r,t) = \frac{Q}{4\pi T} W\left(\frac{r^2 S}{4Tt}\right) = \frac{Q}{4\pi T}\left[-0.5772 - \ln\left(\frac{r^2 S}{4Tt}\right) + \frac{r^2 S}{4Tt} - + \cdots\right]$$

This last equation is also called the Theis equation and is fundamental in well hydraulics.

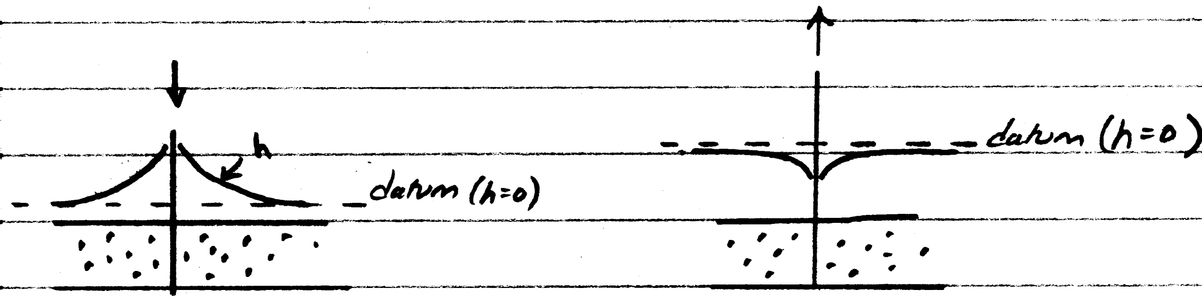
Summary:

(i) developed: $\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = \frac{S}{T}\frac{\partial h}{\partial t}$

assumes: confined aquifer; no vertical flow; all flow radial; T, S, b constant (homogeneous, isotropic); no recharge

(ii) added: infinite extent; no discharge except at well; $h(r,0)=0$; all water added was taken into storage; $r_w \to 0$.

Injection increases head; extraction decreases head



Usually deal with drawdown $(s = h_0 - h)$

Injection $\forall$ or Q is $+$
Extraction $\forall$ or Q is $-$

$\therefore$

Injection: $h = \frac{Q}{4\pi T} W\left(\frac{r^2 S}{4Tt}\right)$; $h_0 - h = s = -\frac{Q}{4\pi T} W\left(\frac{r^2 S}{4Tt}\right)$

Extraction: $h = \frac{-Q}{4\pi T} W\left(\frac{r^2 S}{4Tt}\right)$; $h_0 - h = s = \frac{Q}{4\pi T} W\left(\frac{r^2 S}{4Tt}\right)$

This particular solution is well studied and is programmed into a spreadsheet module MODEL010.XLS available on the course web page.

Transient Groundwater Flow

Now will deal with solutions to

$$\operatorname{div}(T \operatorname{grad}(h)) = S \frac{\partial h}{\partial t} \quad (2\text{-}D)$$

or

$$\operatorname{div}(K \operatorname{grad}(h)) = S_s \frac{\partial h}{\partial t} \quad (3\text{-}D)$$

One Dimensional - Rectangular Coordinates

$$T \frac{\partial^2 h}{\partial x^2} = S \frac{\partial h}{\partial t}$$

let $\alpha = \frac{T}{S}$

$$\therefore \quad \alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t}$$

Observe ① h is a function of x and t

② h differentiated twice with respect to x differs only by a constant with h differentiated once with respect to t

Case 1 Impulse of Water at Origin

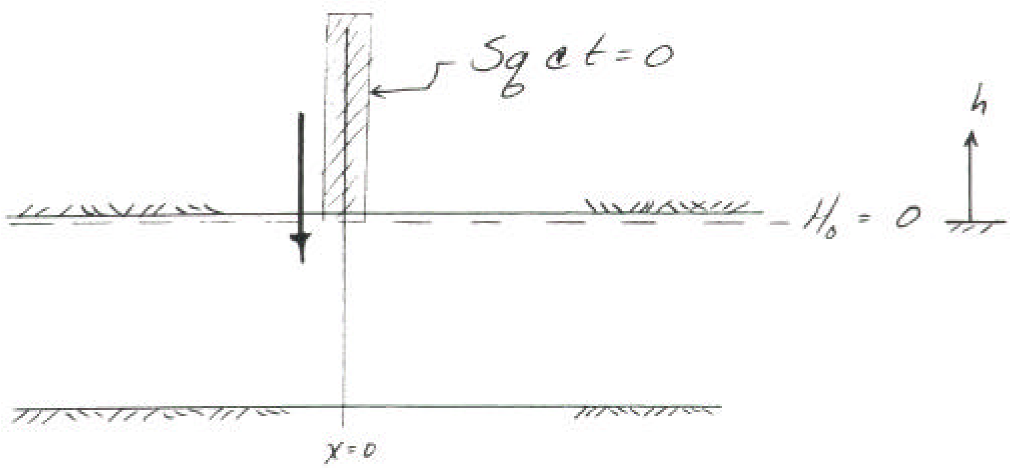
(Fundamental Solution)

$$\alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t}$$

$$h(x,0) = S_q\, \delta(x)$$ ← Dirac delta function

$$\int_{-\infty}^{\infty} h(x,t)\,dx = S_q$$

$$\lim(h,t) = 0 \quad x \to \pm\infty$$



At $x=0$ we instantly add a volume of liquid S_q at $t=0$ at the origin (per unit width of aquifer)

Solution: $$h(x,t) = \frac{q}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right)$$

Check

① Does solution satisfy governing equation?

$$\frac{\partial^2 h}{\partial x^2} = \frac{4x^2 q}{16\alpha^2 t^2 \sqrt{4\pi\alpha t}} \exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{2q}{4\alpha t\sqrt{4\pi\alpha t}} \exp\left(\frac{-x^2}{4\alpha t}\right)$$

$$\frac{\partial h}{\partial t} = \frac{q \exp\left(\frac{-x^2}{4\alpha t}\right)}{\sqrt{4\pi\alpha t}}\left(\frac{x^2}{4\alpha t^2}\right) - \frac{q}{2t\sqrt{4\pi\alpha t}} \exp\left(\frac{-x^2}{4\alpha t}\right)$$

$$\alpha \frac{\partial^2 h}{\partial x^2} \stackrel{?}{=} \frac{\partial h}{\partial t}$$

$$\alpha\frac{\partial^2 h}{\partial x^2} = \frac{q x^2}{4\alpha t^2\sqrt{4\pi\alpha t}} \exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q \exp}{2t\sqrt{4\pi\alpha t}}\left(\frac{-x^2}{4\alpha t}\right)$$

$$\frac{\partial h}{\partial t} = \frac{q x^2}{4\alpha t^2\sqrt{4\pi\alpha t}} \exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q \exp}{2t\sqrt{4\pi\alpha t}}\left(-\frac{x^2}{4\alpha t}\right)$$

∴ Yes, governing equation is satisfied

② Are B.C.'s satisfied

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\alpha t}} \exp\left(\frac{-x^2}{4\alpha t}\right) = \int_{-\infty}^{\infty} \frac{S^{1/2}}{\sqrt{4\pi T t}} \exp\left(\frac{-x^2 S}{4Tt}\right) = S$$

$$\therefore \int_{-\infty}^{\infty} \frac{q}{\sqrt{4\pi\alpha t}} \exp\left(\frac{-x^2}{4\alpha t}\right) = Sq \quad \therefore \text{ Yes, B.C. satisfied}$$

Impulse Solution (Last Time)

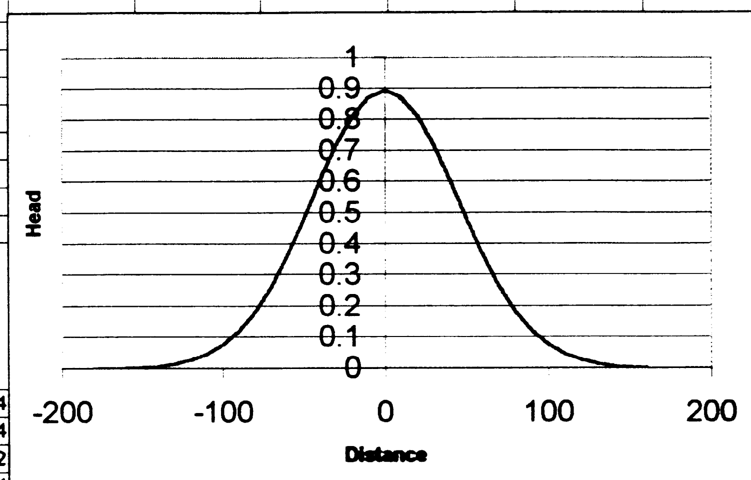
$$h(x,t) = \frac{q}{\sqrt{4\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right)$$

where $\forall = qS$

or $q = \frac{\forall}{S}$

$\forall$ is volume added

	A	B	C	D	E
1	1D Impulse Cartesian				
2					
3	T	1	m/d		
4	S	0.001			
5	dx	10			
6	α	1000			
7					
8	V	0.1	m3/m2		
9					
10	x	t	h(x,t)	q/SQRT(4PI()α)	C2
11	-180	1	3E-04	0.892	3E-04
12	-170	1	6E-04	0.892	7E-04
13	-160	1	0.001	0.892	0.002
14	-150	1	0.003	0.892	0.004



Now suppose we have two impulses, 1 day apart.

By superposition we can write

$$h(x,t) = h_1(x,t) + h_2(x,t)$$

h_1 = head from first impulse

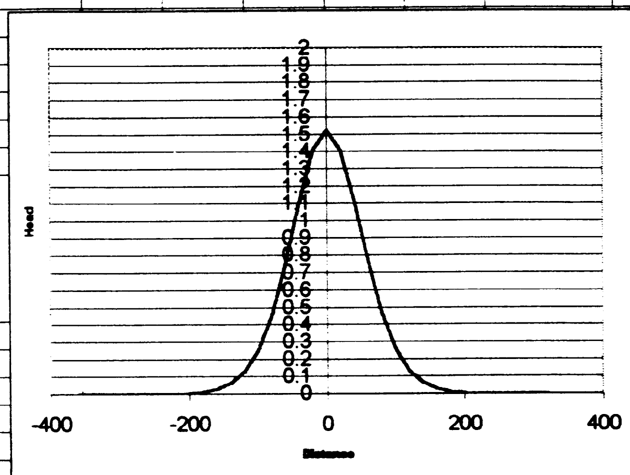
h_2 = head from second impulse

Then

$$h(x,t) = \frac{q_1}{\sqrt{4\pi\alpha(t-\tau)}} \exp\left(\frac{-x^2}{4\alpha(t-\tau)}\right) + \frac{q_2}{\sqrt{4\pi\alpha(t)}} \exp\left(\frac{-x^2}{4\alpha t}\right)$$

Where impulse #2 starts at time $t=0$ and impulse #1 starts at time $t=\tau$

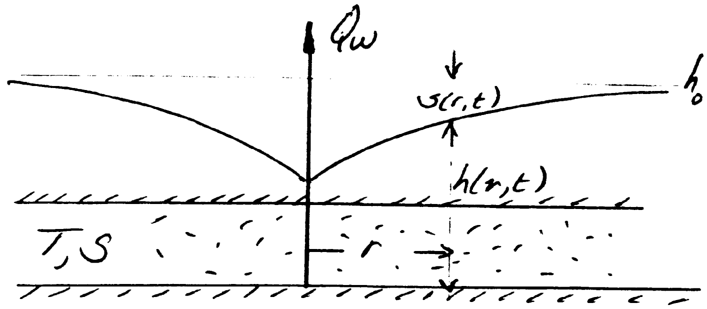
	A	B	C	D	E	F	G	H	I	J	K	L	M
1	1D Impulse Cartesian												
2													
3	T	1	m/d										
4	S	0.001											
5	dx	20											
6	α	1000											
7	τ	1											
8	V	0.1	m3/m2										
9													
10	x	t	h(x,t)	q/SQRT(4PI()α)	C2								h1+h2
11	-360	2	6E-08	0.631	9E-08								E-08
12	-340	2	3E-07	0.631	5E-07								E-07
13	-320	2	2E-06	0.631	3E-06								E-06
14	-300	2	8E-06	0.631	1E-05								E-06
15	-280	2	3E-05	0.631	6E-05								E-05
16	-260	2	1E-04	0.631	2E-04								E-04
17	-240	2	5E-04	0.631	7E-04		-240	1	5E-07	0.8921	6E-07		5E-04



In the limiting case we apply an infinite series of impulses, dt units apart. The solution for this condition is

$$h(x,t) = \int_0^t \frac{q}{\sqrt{4\pi\alpha(t-\tau)}} \exp\left(\frac{-x^2}{4\alpha(t-\tau)}\right) d\tau = \frac{q}{4\pi\alpha x} \operatorname{erfc}\left(\sqrt{\frac{x^2}{4\alpha t}}\right)$$

Radial Flow to a Well



r = radial distance from well

s = drawdown $(h_0 - h(r,t))$

horizontal, radial flow

homogeneous, isotropic, aquifer

$$\operatorname{div}(T \operatorname{grad}(h)) = S\frac{\partial h}{\partial t}$$

$$s = h_0 - h(r,t)$$

$$\frac{\partial s}{\partial t} = -\frac{\partial h}{\partial t} \qquad \frac{\partial^2 s}{\partial r^2} = -\frac{\partial^2 h}{\partial r^2} \qquad \frac{\partial s}{\partial r} = -\frac{\partial h}{\partial r}$$

$$\therefore \quad T\frac{\partial^2 s}{\partial r^2} + \frac{T}{r}\frac{\partial s}{\partial r} = S\frac{\partial s}{\partial t} \qquad \text{or}$$

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t}$$

Boundary Conditions

$$r = \infty, \; s = 0; \quad r \to 0, \; \lim_{r\to 0}\left(2\pi T \frac{\partial s}{\partial r}\right) = -Q_\omega$$

Initial Conditions

$$t = 0, \; s = 0$$

<u>Obtaining a Solution</u>

$$\text{let } u = \frac{r^2 S}{4Tt} \qquad \begin{aligned} &t \to 0, \; s \to 0 \Rightarrow u \to \infty \\ &r \to \infty, \; s \to 0 \Rightarrow u \to \infty \end{aligned}$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = -\frac{Q_\omega}{2\pi T}$$

$$\lim_{u\to 0} r\frac{\partial s}{\partial r} = \lim_{u\to 0} r\frac{\partial s}{\partial u}\frac{\partial u}{\partial r} = \lim_{u\to 0} 2u\frac{\partial s}{\partial u} = -\frac{Q_\omega}{2\pi T}$$

$$\therefore \lim_{u\to 0} u\frac{\partial s}{\partial u} = -\frac{Q_\omega}{4\pi T}$$

Now transform governing equation into an ODE

$$\frac{\partial s}{\partial t} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial t} = -\frac{u}{t}\frac{\partial s}{\partial u}$$

$$\frac{\partial s}{\partial r} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial r} = \frac{2u}{r}\frac{\partial s}{\partial u}$$

$$\frac{\partial^2 s}{\partial r^2} = \frac{2u}{r^2}\frac{\partial s}{\partial u} + \frac{4u^2}{r^2}\frac{\partial^2 s}{\partial u^2}$$

Subsitute into PDE

$$-\frac{u}{t}\frac{\partial s}{\partial u}\frac{S}{T} = \frac{4u^2}{r^2}\frac{\partial^2 s}{\partial u^2} + \frac{2u}{r^2}\frac{\partial s}{\partial u} + \frac{2u}{r}\frac{\partial s}{\partial u}$$

Multiply by $r^2/4$, divide by u^2, rearrange:

$$\frac{\partial^2 s}{\partial u^2} + \left(\frac{u+1}{u}\right)\frac{\partial s}{\partial u} = 0 \qquad \text{let } x = \frac{\partial s}{\partial u}$$

$$\frac{dx}{du} = -\left(\frac{u+1}{u}\right)x \Rightarrow -\int\frac{dx}{x} = \int\frac{u+1}{u}\,du$$

$$-\ln|x| = \ln|u| + u + \ln|c|$$

$$u = -\ln|x| - \ln|u| - \ln|c| = -\ln|xuc|$$

$$e^{-u} = xuc \qquad \text{but } x = \frac{ds}{du}$$

So:

$$\frac{ds}{du} = \frac{1}{c}\frac{e^{-u}}{u}$$

Apply B.C. $\quad u\frac{ds}{du} \to -\frac{Q_w}{4\pi T} \quad u \to 0$

$$\therefore \frac{1}{c} = -\frac{Q_w}{4\pi T}$$

$$\frac{ds}{du} = -\frac{Q_w}{4\pi T}\frac{e^{-u}}{u} \Rightarrow \int_0^s ds = -\int_\infty^u \frac{Q_w}{4\pi T}\frac{e^{-u}}{u}\,du$$

$$s(u) = -\frac{Q_w}{4\pi T}\underbrace{\int_\infty^u \frac{e^{-u}}{u}\,du}_{-Ei(u)\ \text{exponential integral}}$$

EVALUATE $Ei(u)$ BY

- SERIES EXPANSION:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$Ei(u) = \int_u^{\infty} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} dx = \ln|x| - x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \cdots \Big|_u^{\infty}$$

NOTE: $$\left[\ln|x| - x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \cdots\right]\Big|_{x=\infty} = \gamma \approx -0.5772$$

$$\therefore\ Ei(u) = \gamma - \ln|u| + u - \frac{u^2}{2\cdot 2!} + \frac{u^3}{3\cdot 3!} \mp \cdots$$

- POLYNOMIAL APPROXIMATION

$$Ei(u) \approx -\ln|u| + a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5$$

$a_0 = -0.577\,215\,66$ $\quad$ $a_3 = 0.055\,199\,68$

$a_1 = 0.999\,991\,93$ $\quad$ $a_4 = -0.009\,760\,04$

$a_2 = -0.249\,910\,55$ $\quad$ $a_5 = 0.001\,078\,57$

FOR $0 < u \leq 1$

$$Ei(u) \approx \left(\frac{u^4 + a_1 u^3 + a_2 u^2 + a_3 u + a_4}{u^4 + b_1 u^3 + b_2 u^2 + b_3 u + b_4}\right)\left(\frac{1}{u\exp(u)}\right)$$

$a_1 = 8.573\,328\,740\,1$ $\quad$ $b_1 = 9.573\,322\,345\,4$

$a_2 = 18.059\,016\,973\,0$ $\quad$ $b_2 = 25.632\,956\,148\,6$

$a_3 = 8.634\,760\,892\,5$ $\quad$ $b_3 = 21.099\,653\,082\,7$

$a_4 = 0.267\,773\,734\,3$ $\quad$ $b_4 = 3.958\,496\,922\,8$

FOR $1 \leq u \leq \infty$

- DON'T TRUNCATE CONSTANTS (UNLESS U WANT WRONG RESULT)

MODEL5.XLS

ADDITIONAL SOLUTIONS BY SUPERPOSITION

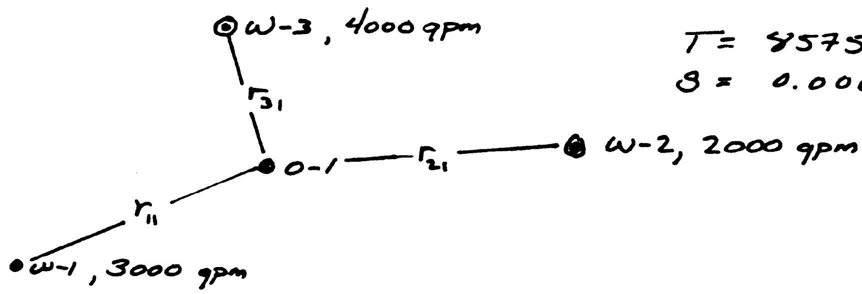
$$\frac{S}{T}\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} \quad \text{IS LINEAR IN } s \,\&\, t$$

$\therefore$ CAN DEVELOP ADDITIONAL SOLUTIONS BY SUPERPOSITION

EXAMPLE

SUPPOSE WELLFIELD BELOW IS PLANNED TO OPERATE AS SHOWN.

WHAT IS THE DRAWDOWN AT O-1 AFTER 365 DAYS OF PUMPING?



$T = 8575\ ft^2/d$
$S = 0.0008$

SOLUTION:

- FIND DRAWDOWN AT O-1 FROM EACH PUMPING WELL
- TOTAL DRAWDOWN IS SIMPLY SUM OF INDIVIDUAL DRAWDOWNS

SUPPOSE: $r_{11} = 1500\ ft$ $\quad t = 365\ day$
$r_{21} = 1470\ ft$
$r_{31} = 1000\ ft$

COMPUTE:

$$U_{11} = \frac{r_{11}^2 S}{4Tt} = 0.000144$$

$$U_{21} = \frac{r_{21}^2 S}{4Tt} = 0.000138$$

$$U_{31} = \frac{r_{31}^2 S}{4Tt} = 0.000064$$

EVALUATE $Ei(u)$

		Q (gpm)	Q (ft³/day)
$Ei(u_{11}) = 8.270182$	;	3000	577540
$Ei(u_{21}) = 8.310582$	;	2000	385027
$Ei(u_{31}) = 9.081032$	;	4000	770053

COMPUTE INDIVIDUAL DRAWDOWNS

$$s_{11} = \frac{Q_1}{4\pi T} Ei(u_{11}) = \frac{577540}{4\pi(8575)} 8.270182 = 44.325$$

$$s_{21} = \frac{Q_2}{4\pi T} Ei(u_{21}) = 29.694$$

$$s_{31} = \frac{Q_3}{4\pi T} Ei(u_{31}) = 64.895$$

TOTAL DRAWDOWN

$$s = \sum_{i=1}^{3} s_{i1} \qquad \Sigma = 139'$$

$\therefore$ TOTAL PREDICTED DRAWDOWN AT O-1 FROM THE PUMPING WELL ENSEMBLE IS 139' AFTER 365 DAYS OF PUMPING

GENERAL FORM:

$$s_j = \sum_{i=1}^{NW} \frac{Q_i}{4\pi T} Ei\left(\frac{r_{ij}^2 S}{4Tt}\right)$$

r_{ij} IS RADIUS FROM i-th WELL TO FIELD POINT j

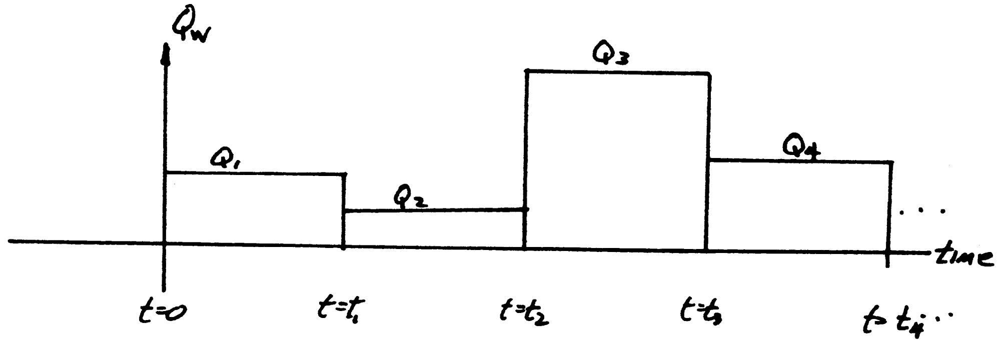
Q_i IS PUMPING RATE OF j-th WELL

VARIABLE PUMPING RATES

USE CONVOLUTION IN TIME:

$$s(r,t) = \frac{Q}{4\pi T} Ei(u); \quad u = \frac{r^2 S}{4Tt}$$

ESTIMATE RESPONSE OVER SEVERAL PLANNING PERIODS (SEQUENTIAL) WITH DIFFERENT PUMP RATES:



FROM $t=0$ to $t=t_1$; $Q_w = Q_1$

$t=t_1$ to $t=t_2$; $Q_w = Q_2$

$\vdots \qquad \vdots$

RESPONSE AT SOME ARBITRARY FIELD POINT:

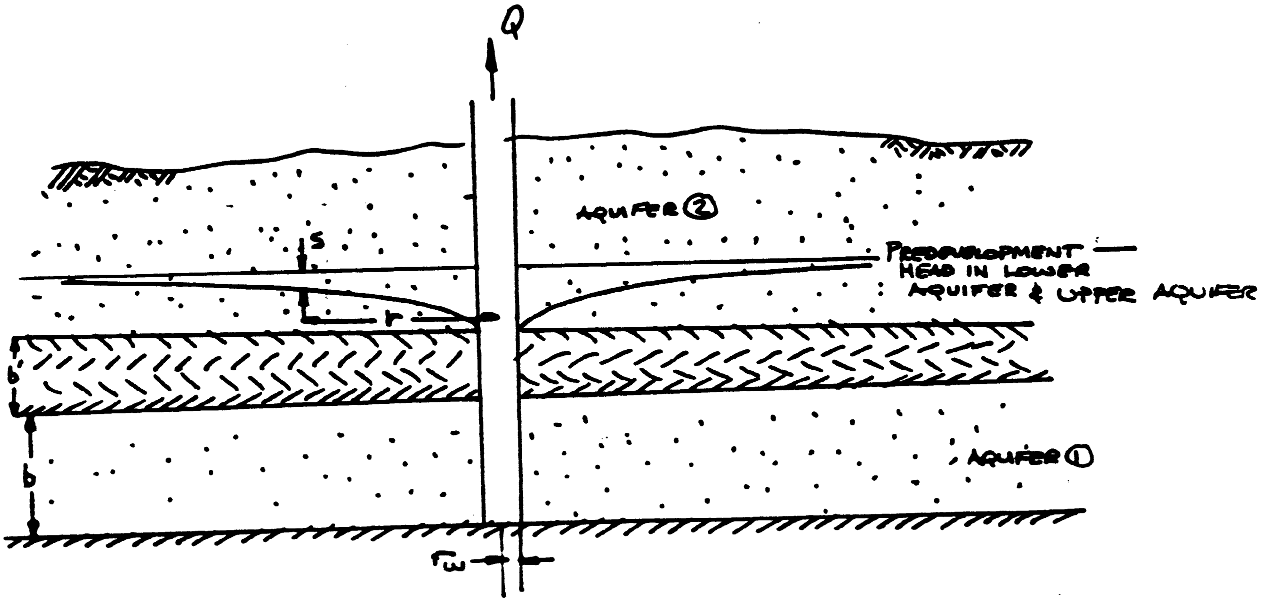
$$0 \le t \le t_1; \quad s = \frac{Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4Tt}\right)$$

$$t_1 \le t \le t_2; \quad s = \frac{Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4Tt}\right) + \frac{Q_2 - Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_1)}\right)$$

$$t_2 \le t \le t_3; \quad s = \frac{Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4Tt}\right) + \frac{Q_2 - Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_1)}\right) + \frac{Q_3 - Q_2}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_2)}\right)$$

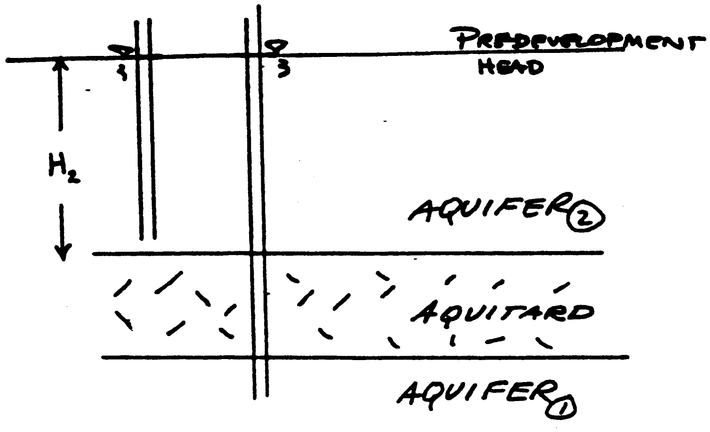
$$t_3 \le t \le t_4; \quad s = \frac{Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4Tt}\right) + \frac{Q_2 - Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_1)}\right) + \frac{Q_3 - Q_2}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_2)}\right)$$

FULLY PENETRATING WELL IN A LEAKY AQUIFER



- WELL DISCHARGES AT CONSTANT RATE Q
- INFINITESMAL WELL DIAMETER
- AQUIFER ① OVERLAIN BY CONFINING BED (AQUITARD) OF THICKNESS b', HYDRAULIC CONDUCTIVITY K'
- AQUIFER ② OVERLIES AQUITARD AND HAS CONSTANT HEAD
- HYDRAULIC GRADIENT ACROSS CONFINING BED CHANGES INSTANTLY — NO STORAGE IN AQUITARD
- AQUIFER FLOW IS 2D HORIZONTAL, AQUITARD FLOW IS VERTICAL

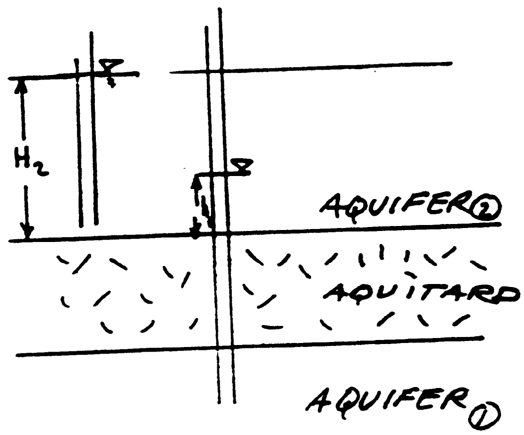
LEAKAGE CONCEPTS



$t < 0$

NO FLOW:

$$q_v = -K' \underbrace{grad(h)}_{0}$$



$t > 0$

FLOW:

$$q_v = -K' \underbrace{grad(h)}_{\neq 0}$$

$$q_v = -K' \cdot \frac{h_1 - H_2}{b'}$$

$$q_v = K' \cdot \frac{H_2 - h_1}{b'}$$

BUT: $H_2 - h_1 = s$ (DRAWDOWN)

$$\therefore \; q_v = \frac{K' s}{b'}$$

- NO STORAGE IN AQUITARD $\Rightarrow$ CHANGE IN HEAD CAUSES INSTANTANEOUS CHANGE IN FLOW
- H_2 = CONSTANT (ASSUMPTION 4)

BASIC EQUATIONS

$$S\frac{\partial h}{\partial t} = \mathrm{div}\left(\mathbf{T}\,\mathrm{grad}(h)\right) \pm \text{sources}$$

OR

$$S\frac{\partial h}{\partial t} = T\frac{\partial^2 h}{\partial r^2} + \frac{T}{r}\frac{\partial h}{\partial r} - \frac{(H_2 - h)K'}{b'}$$

IN TERMS OF DRAWDOWN:

$$\frac{S}{T}\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{sK'}{Tb'}$$

SUBJECT TO:

$$s(r,0) = 0$$

$$s(\infty, t) = 0$$

$$\lim_{r=r_w \to 0} \frac{\partial s}{\partial r} = -\frac{Q}{2\pi T}$$

SOLUTION BY

LÁPLACE TRANSFORM:

$$s = \frac{Q_w}{4\pi T}\int_{u}^{\infty} \frac{e^{-u-\frac{v^2}{u}}}{u}\,du$$

WHERE

$$u = \frac{r^2 S}{4Tt}$$

$$v^2 = \frac{r^2 K'}{4Tb'}$$

THE EXPONENTIAL INTEGRAL:

$$\int_u^{\infty} \frac{e^{-u-\frac{v^2}{u}}}{u} \, du = L(u,v)$$

IS USUALLY EVALUATED BY TABLE LOOK-UP OR NUMERICALLY (REED, PG. 21)

$L(u,v)$ IS SOMETIMES DENOTED BY

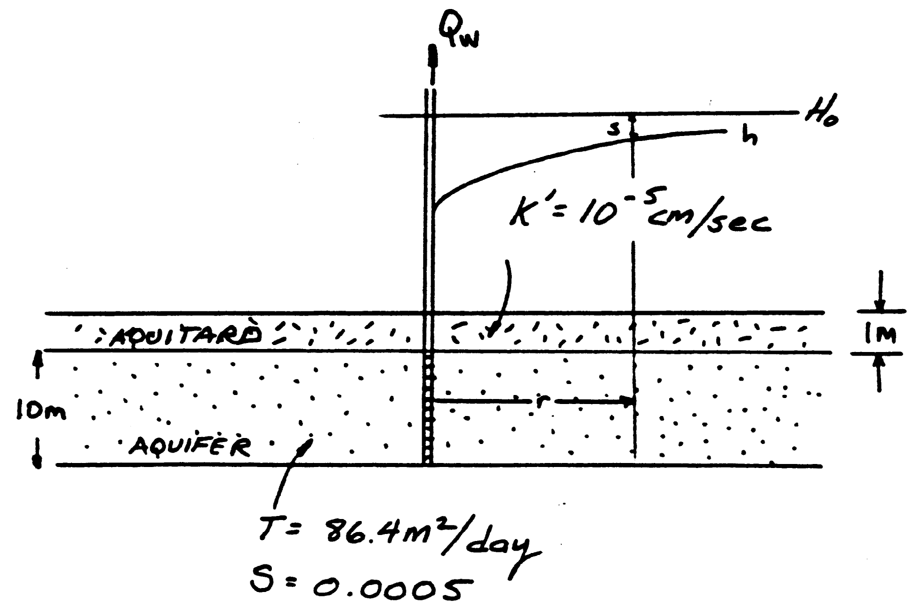
$W(u, r/B)$.

THE TERM $B = \sqrt{\frac{Tb'}{K'}}$ IS CALLED

THE LEAKAGE FACTOR

OBSERVE: $W(u, r/B) = L(u, 2v)$

EXAMPLE



AQUIFER 10m THICK IS OVERLAIN BY A 1m THICK AQUITARD. STORAGE IN AQUITARD IS ASSUMED NEGLIGIBLE. THE WELL PUMPS AT 500 m³/DAY. WHAT IS THE DRAWDOWN AT 1, 5, 10, 50, 100, 500, AND 1000 METERS AFTER ONE DAY OF PUMPING?

① MODEL: $s(r,t) = \frac{Q_w}{4\pi T} L(u,v)$

$$u = \frac{r^2 S}{4Tt}$$

$$v^2 = \frac{r^2 K'}{4Tb'}$$

(2) REDUCE DATA

$t = 1$ day $\qquad Q_w = 500\ m^3/d$

$K' = 8.64 \cdot 10^{-3}\ m/d$

$T = 86.4\ m^2/d$

$S = 0.0005$

$b' = 1m$

$$u = \frac{r^2 (0.0005)}{4(86.4)(1)} = 1.45 \cdot 10^{-6} r^2$$

$$v^2 = \frac{r^2 (8.64 \cdot 10^{-3})}{4(86.4)(1)} = 2.5 \cdot 10^{-5} r^2$$

$$\frac{Q_w}{4\pi T} = \frac{500}{4(\pi)(86.4)} = 0.46$$

(3) MAKE A TABLE

r	u	v^2	v	$2v\left(=\frac{r}{B}\right)$	$W\left(u,\frac{r}{B}\right)$
1 m	$1.45 \cdot 10^{-6}$	$2.5 \cdot 10^{-5}$	$5.0 \cdot 10^{-3}$	0.01	9.44
5 m	$3.63 \cdot 10^{-5}$	$6.25 \cdot 10^{-4}$	$2.5 \cdot 10^{-2}$	0.05	6.23
10 m	$1.45 \cdot 10^{-4}$	$2.5 \cdot 10^{-3}$	$5.0 \cdot 10^{-2}$	0.1	4.85
50 m	$3.63 \cdot 10^{-3}$	$6.25 \cdot 10^{-2}$	$2.5 \cdot 10^{-1}$	0.5	1.85
100 m	$1.45 \cdot 10^{-2}$	$2.5 \cdot 10^{-1}$	$5.0 \cdot 10^{-1}$	1	0.842
500 m	$3.63 \cdot 10^{-1}$	6.25	2.5	5	0.007
1000 m	1.45	25	5.0	10	0.0001

(4) APPLY: $s = \frac{Q_w}{4\pi T} W\left(u, \frac{r}{B}\right)$

(5) Solution

r (meters)	s (meters)
1	4.34
5	2.87
10	2.23
50	0.85
100	0.39
500	0.003
1000	0.000046

TABLE 4.1.—*Selected values of* W(u,r/B)

[From Hantush (1961e)]

u	r/B 0.001	0.003	0.01	0.03	0.1	0.3	1	3
1×10^{-6}	13.0031	11.8153	9.4425	7.2471	4.8541	2.7449	0.8420	0.0695
2	12.4240	11.6716						
3	12.0581	11.5098	9.4425					
5	11.5795	11.2248	9.4413					
7	11.2570	10.9951	9.4361					
1×10^{-5}	10.9109	10.7228	9.4176					
2	10.2301	10.1332	9.2961	7.2471				
3	9.8288	9.7635	9.1499	7.2470				
5	9.3213	9.2818	8.8827	7.2450				
7	8.9863	8.9580	8.6625	7.2371				
1×10^{-4}	8.6308	8.6109	8.3983	7.2122				
2	7.9390	7.9290	7.8192	7.0685				
3	7.5340	7.5274	7.4534	6.9068	4.8541			
5	7.0237	7.0197	6.9750	6.6219	4.8530			
7	6.6876	6.6848	6.6527	6.3923	4.8478			
1×10^{-3}	6.3313	6.3293	6.3069	6.1202	4.8292			
2	5.6393	5.6383	5.6271	5.5314	4.7079	2.7449		
3	5.2348	5.2342	5.2267	5.1627	4.5622	2.7448		
5	4.7260	4.7256	4.7212	4.6829	4.2960	2.7428		
7	4.3916	4.3913	4.3882	4.3609	4.0771	2.7350		
1×10^{-2}	4.0379	4.0377	4.0356	4.0167	3.8150	2.7104		
2	3.3547	3.3546	3.3536	3.3444	3.2442	2.5688		
3	2.9591	2.9590	2.9584	2.9523	2.8873	2.4110	.8420	
5	2.4679	2.4679	2.4675	2.4642	2.4271	2.1371	.8409	
7	2.1508	2.1508	2.1506	2.1483	2.1232	1.9206	.8360	
1×10^{-1}	1.8229	1.8229	1.8227	1.8213	1.8050	1.6704	.8190	
2	1.2226	1.2226	1.2226	1.2220	1.2155	1.1602	.7148	.0695
3	.9057	.9057	.9056	.9053	.9018	.8713	.6010	.0694
5	.5598	.5598	.5598	.5596	.5581	.5453	.4210	.0681
7	.3738	.3738	.3738	.3737	.3729	.3663	.2996	.0639
1×10^{0}	.2194	.2194	.2194	.2193	.2190	.2161	.1855	.0534
2	.0489	.0489	.0489	.0489	.0488	.0485	.0444	.0210
3	.0130	.0130	.0130	.0130	.0130	.0130	.0122	.0071
5	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0008
7	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

(1)

all same (6.23) (2)

(4.35) (3)

all same

all same

all same

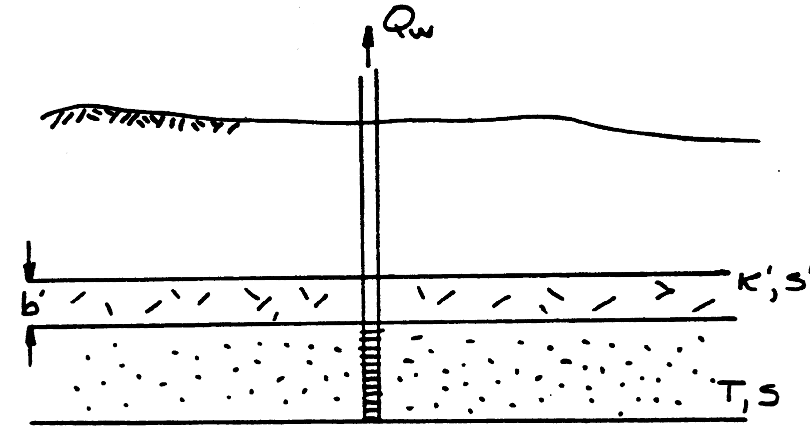
(1.45) (4)

etc.

LOG SCALE

USE LINEAR INTERPOLATION FOR MISSING VALUES

LEAKY AQUIFER - STORAGE IN AQUITARD



ASSUMPTIONS

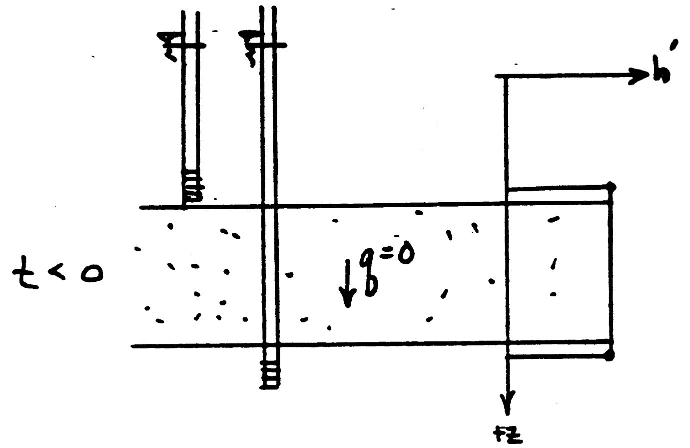
- WELL DISCHARGES AT CONSTANT RATE
- WELL OF INFINITESIMAL DIAMETER
- FULLY PENETRATING
- AQUIFER IS OVERLAIN BY AQUITARD OF K', S'
- FLOW IN AQUIFER IS HORIZONTAL
  FLOW IN AQUITARD IS VERTICAL
- CONSTANT HEAD ABOVE AQUITARD

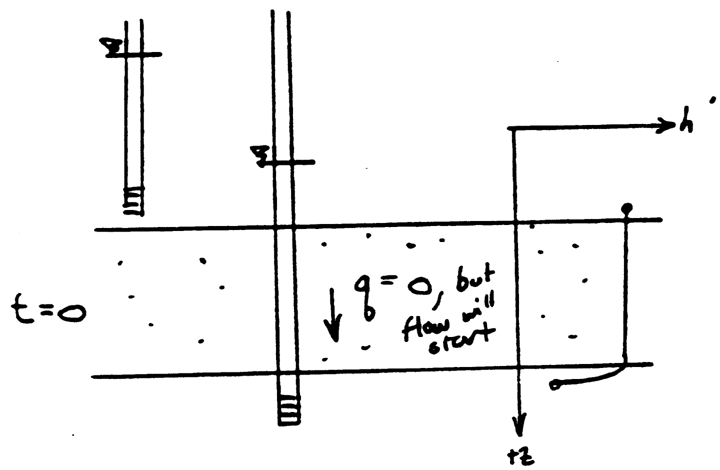
BASIC EQUATIONS

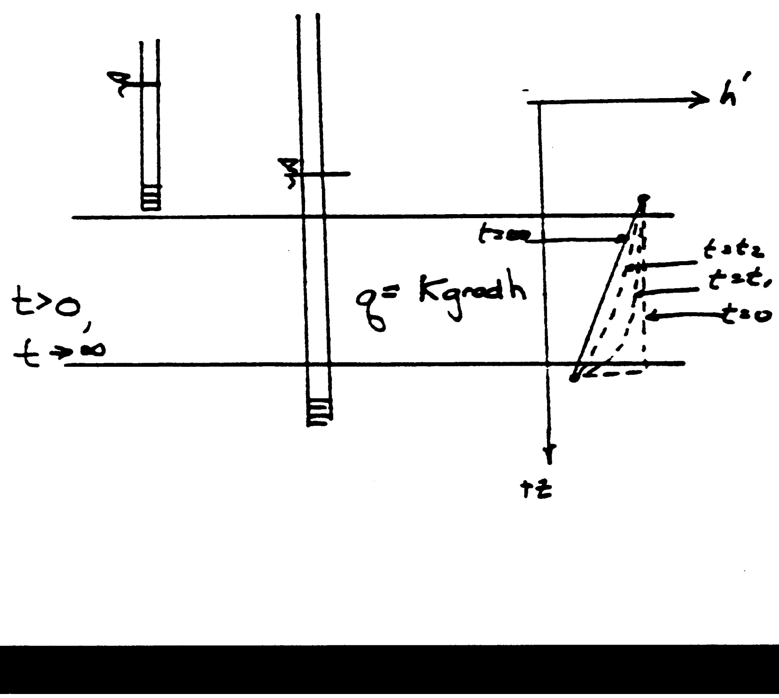
AQUITARD: $$\frac{\partial^2 s'}{\partial z^2} = \frac{S'}{K'b'} \frac{\partial s'}{\partial t}$$

AQUIFER : $$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{K'}{T} s' = \frac{S}{T}\frac{\partial s}{\partial t}$$

LEAKAGE CONCEPTS







GOVERNING EQUATION IN AQUITARD

$$\frac{\partial^2 h'}{\partial z^2} = \frac{S'}{K'b'} \frac{\partial h'}{\partial t}$$

OR

$$\frac{\partial h'}{\partial t} = \left(\frac{K'b'}{S}\right) \frac{\partial^2 h'}{\partial z^2}$$

$$\downarrow$$

$$\frac{T'}{S'}$$

$$\Downarrow$$

$$\frac{\partial h'}{\partial t} = \frac{T'}{S'} \frac{\partial^2 h}{\partial z^2}$$

(UNI-DIRECTIONAL NON-EQUILIBRIUM FLOW)

SOLUTION (BY LAPLACE TRANSFORM)

$$s(r,t) = \frac{Q}{4\pi T} H(u,\beta)$$

$$u = \frac{r^2 S}{4Tt}$$

$$\beta = \frac{r}{4}\left(\sqrt{\frac{K'S'}{b'TS}}\right)$$

$$H(u,\beta) = \int_u^\infty \frac{e^{-y}}{y} \operatorname{erfc}\left[\frac{\beta\sqrt{u}}{\sqrt{y(y-u)}}\right] dy$$

"complimentary error function"

USUALLY EVALUATED NUMERICALLY OR BY TABLE LOOK-UP.

(SEE HANDOUT)

TABLES OF $H(u,\beta)$ ARE INCLUDED IN REED pg. 28. COMPUTER PROGRAM IN SAME BOOK.

AS t INCREASES BEHAVIOR APPROACHES THEIS SOLUTION.