CIVE 6361 Groundwater Hydrology

Applications of Darcy's Law

- Differential equation behavior
  - Cartesian coordinates
  - Radial coordinates
- Transmissivity
  - Selected "classic" problems in confined aquifers
- Dupuit Assumptions
  - Selected "classic" problems in unconfined aquifers

Darcy's Law - Applications

Darcy's law is a differential equation - it contains a derivative.

It relates the rate of change of head with distance, under given discharge conditions

In ground water engineering one is usually interested in expressions that relate values of head rather than the rate of change of head, under given discharge conditions

To proceed from a differential equation to an algebraic equation is called obtaining a solution to the differential equation.

A variety of techniques are used to find solutions to differential equations - including guessing.

For the purposes of this lesson, it is sufficient to be able to recognize a solution when it is given. This "testing" is simply a matter of differentiation.

To test if an equation is a solution to a differential equation, one differentiates the equation. If the result is equivalent to the differential equation, then the original equation is a solution.

Example: Which of the following algebraic expressions are a solution to

$$\frac{dy}{dx} = K \qquad ?$$

a) $y = Kx^2$ b) $x = 2y + K$ c) $y = Kx + 5$

The correct answer is "c", but for exercise lets test all the possible answers.

a) $y = Kx^2$ $\frac{dy}{dx} = 2Kx$; Not the same.

b) $x = 2y + K$ or $\frac{x-K}{2} = y$

so $\frac{dy}{dx} = \frac{1}{2}$; Also not the same.

c) $y = Kx + 5$ $\frac{dy}{dx} = K$; same - therefore "c" is a solution.

Although "c" is a solution, it is not the only solution.

For example:

$$y = Kx + 7$$
$$y = Kx - 3$$
$$y = Kx + 0$$

are all solutions to $\frac{dy}{dx} = K$.

The constant term on the right does not affect the result of differentiation; regardless of the value of the constant, the derivative of y with respect to x always turns out to be K.

Because there is an infinite number of constants to choose, there are an infinite number of solutions to the differential equation.

This situation is typical for differential equations.

Example: Which of the following three expressions relating head, h, to distance x, are solutions to

$$\frac{Q}{A} = -K\frac{dh}{dx} \quad ?$$

a) $h = -\frac{Q}{KA}x$ b) $h = h_0 - \frac{Q}{KA}x$ c) $h = h_0 - \frac{Q}{KA}x^2 + 7$

The answer is "a" and "b". Again we will show the details

a) $h = \frac{-Q}{KA}x \qquad \frac{dh}{dx} = \frac{-Q}{KA} \quad \Rightarrow \quad -K\frac{dh}{dx} = \frac{Q}{A}$ ✓

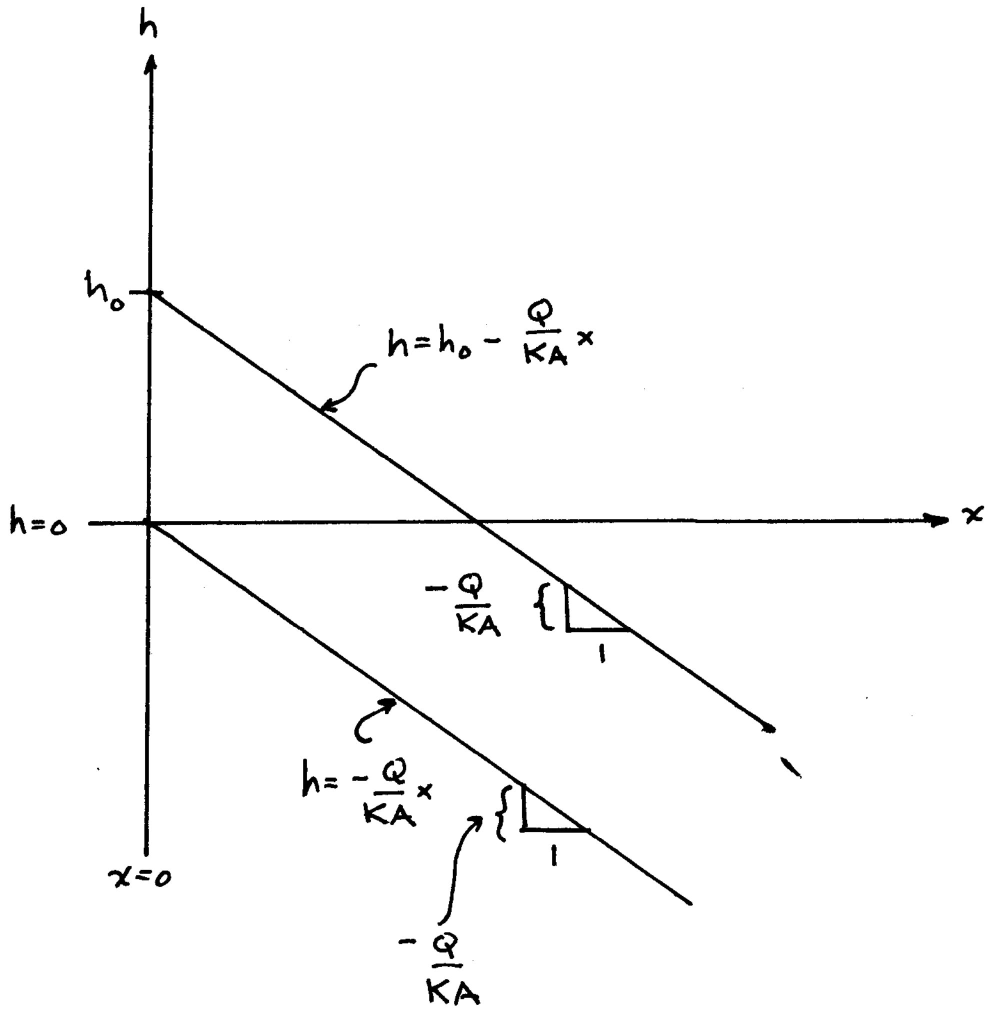
b) $h = h_0 - \frac{Q}{KA}x \qquad \frac{dh}{dx} = -\frac{Q}{KA} \quad \Rightarrow \quad -K\frac{dh}{dx} = \frac{Q}{A}$ ✓

c) $h = h_0 - \frac{Q}{KA}x^2 + 7 \qquad \frac{dh}{dx} = \frac{-2Qx}{KA} \quad \Rightarrow \quad -K\frac{dh}{dx} = -\frac{2Qx}{A}$

Not the same.

In this example the expressions relate <u>values</u> of the head, h, with distance x, from the reference point $x=0$. The differential equation relates the rate of change of head with distance.

The differential equation is Darcy's law. It states that a plot of head versus distance will have constant slope.



The graph shows the two "solutions". Each is a line with slope $-\frac{Q}{KA}$.

The intercept for equation "a" is $h=0$ at $x=0$ while for equation "b" the intercept is $h=h_0$ at $x=0$.

These intercept values give the values of h at $x=0$, the reference location.

These values provide the reference heads from which changes in h are measured.

Now suppose one was to graph <u>all</u> possible solutions to

$$\frac{dh}{dx} = -\frac{Q}{KA}$$

what would be the result?

a) A family of curves, infinite in number with an intercept on the x axis of $x = -\frac{Q}{KA}$

b) An infinite number of parallel lines with slope slope $= -Q/KA$, and different intercepts at $x=0$.

c) A finite number of parallel lines with slope slope $= -Q/KA$, with positive intercepts at $x=0$.

The answer is "b".

DLA-006

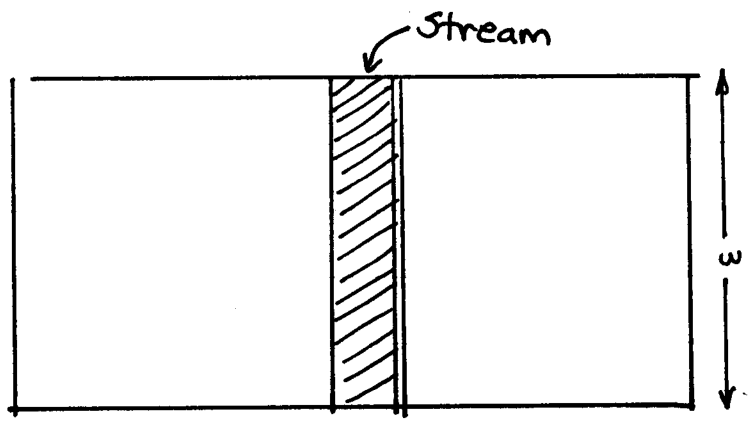
Any line with slope $-\frac{Q}{KA}$ will satisfy the differential equation.

e.g. $h = \underbrace{-\frac{Q}{KA}}_{\text{slope}} x + \underbrace{b}_{\text{intercept}}$ where $b \in \{-\infty, \infty\}$

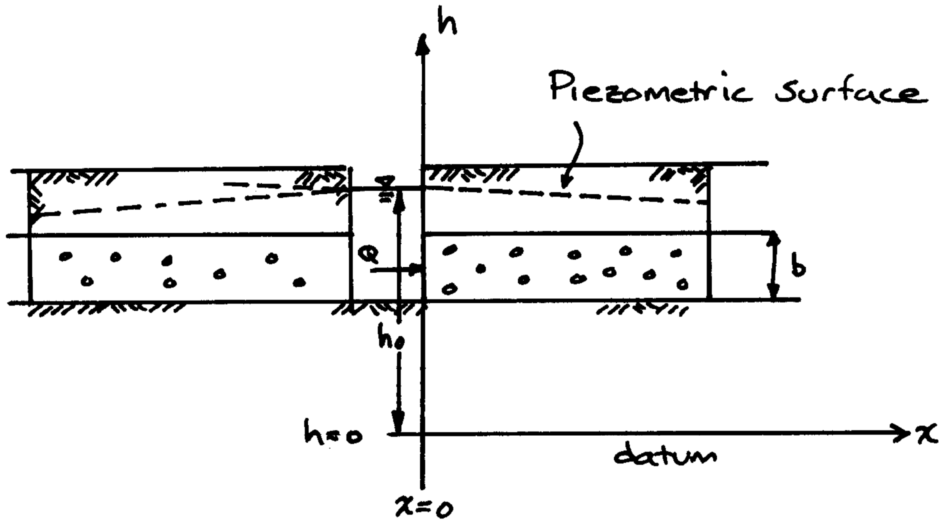
is a solution.

b can take any value. For practical problems, the problem geometry and reference conditions are used to establish a value for b that is unique to the particular problem.

Such a solution is called a particular solution.



Plan View



Elevation View

The figure is a sketch of a confined aquifer of thickness b.

The aquifer is fully penetrated by a stream, and flow occurs from the stream into the aquifer

The water level in the stream is at elevation h_o above the horizontal datum.

Let $x=0$ be at the stream-aquifer interface. Assume the system is in equilibrium, so that no changes occur with time.

Suppose along the reach of the stream of length w, the total rate of loss from the stream is $2Q$ (volume/time), and half this flow (Q) enters the part of the aquifer in the sketch.

This flow then moves away from the stream in a steady flow along the x-direction.

The resulting distribution of head in the aquifer is indicated by the piezometric surface.

This surface actually traces the static water levels in wells or pipes tapping the aquifer at various points.

Darcy's law for this situation is

$$Q = -Kbw\frac{dh}{dx}$$

Rearrangement into a "typical" differential equation gives

$$\frac{dh}{dx} = -\frac{Q}{Kbw}$$

where K is the hydraulic conductivity.

The head distribution – the piezometric surface – is described by <u>one</u> of the solutions to the differential equation.

This particular solution must both satisfy the differential equation and the boundary condition $h = h_o$ at $x = 0$.

A (the) solution that satisfies the conditions of this problem is

$$h = h_0 - \frac{Q}{Kwb} x .$$

To check this solution, first verify boundary conditions

$x = 0 \Rightarrow h = h_0$ ✓

Then verify it satisfies the differential equation.

$$\frac{dh}{dx} = -\frac{Q}{Kwb}$$ ✓ (This is the original) diff. eq.

The condition that $h = h_0$ at $x = 0$ is called a "boundary condition"; it is a requirement that states that h must have a certain value along one or another boundary of the problem.

The differential equation $\frac{dh}{dx} = -\frac{Q}{Kbw}$

is insufficient to define h as a function of x. All it supplies is information that the graph of h versus x will be a line with slope $= -Q/Kbw$.

But we already know that there are an infinite number of such lines.

The additional information given by the boundary condition $h=h_0$ at $x=0$, selects the one line required for the problem by supplying the intercept.

This information provides the reference value from which changes in head indicated by a differential equation may be measured.

The process of (1) differentation to establish that a given solution is indeed a solution and (2) application of a boundary condition to determine the particular solution is the essence of mathematical analysis of ground water flow problems.

Suppose that, in measuring observation wells tapping a confined aquifer, we observe a linear increase in head with distance away from a stream or channel that completely penetrates the aquifer. Suppose that this pattern remains unchanged through a considerable period of time.

Which of the following conclusions logically follows from the evidence presented?

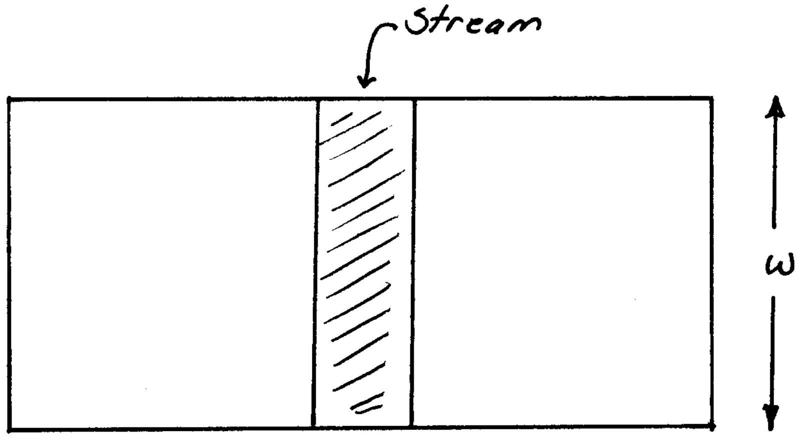
a) There is no flow in the aquifer

b) There is steady flow through the aquifer into the stream

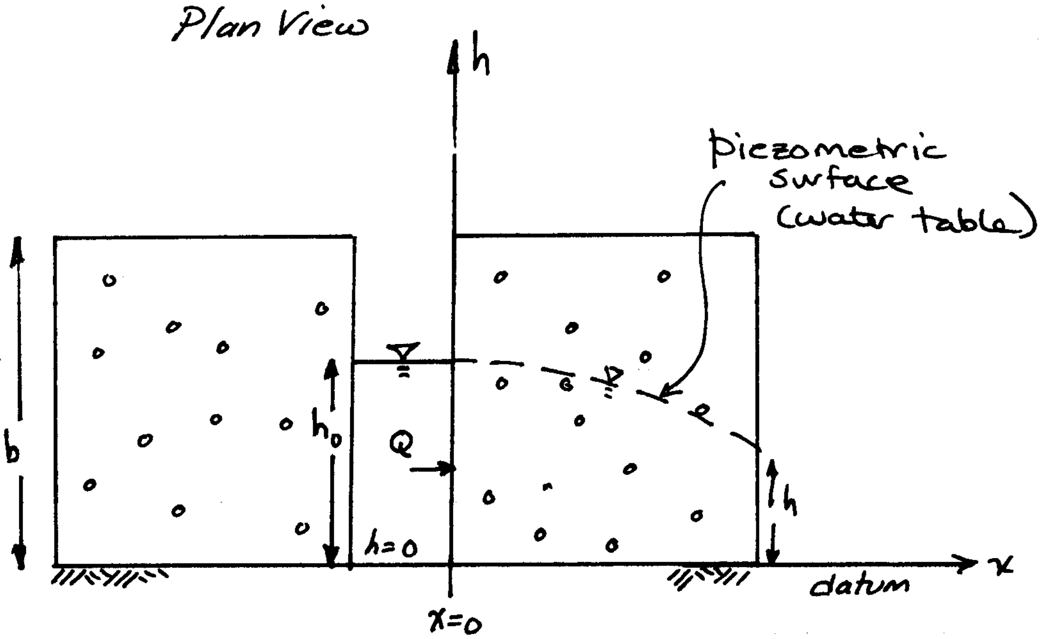
c) A flow which increases in specific discharge as one approaches the stream occurs in the aquifer.

Darcy's Law - Applications

Consider an unconfined aquifer as shown on the sketch



Plan View



Elevation View

The upper limit of flow at any point is the water table itself – the working definition of an unconfined aquifer.

Consider uniform flow away from the stream as shown in the diagram.

The datum is taken as the bottom of the aquifer.

Assume vertical components of flow are negligible.

This assumption is never completely satisfied because movement cannot be lateral near the free surface because of the slope of the free surface itself. Typically the vertical velocity component is very small compared to the lateral component and can be neglected.

This assumption is called the Dupuit Assumption.

An important difference in this situation as compared to the confined-flow problem is that the cross-sectional area of flow decreases along the flow path, where in the confined flow case it remains constant.

As before assume that seepage along a reach of stream of length w is $2Q$ and half this discharge goes into the part of aquifer detailed in the sketch.

Application of Darcy's law to this situation gives

$$Q = -Kwh\frac{dh}{dx}$$

After rearrangement into a "typical" differential equation we have

$$\frac{dh}{dx} = -\frac{Q}{Kwh}$$

To find a general solution to this differential equation one can "guess" a solution or rearrange the differential equation into a more recognizable form.

A slight rearrangement produces

$$h \frac{dh}{dx} = \frac{-Q}{KW}$$

Recall from calculus that,

$$\frac{d}{dx}(h^2) = 2h\frac{dh}{dx}$$

Therefore multiplication of both sides by the constant 2 produces

$$2h\frac{dh}{dx} = -\frac{2Q}{KW} = \frac{d(h^2)}{dx}$$

which is identical to the original differential equation.

The last form of the equation supplies a useful "guess" for a solution.

$$\frac{d(h^2)}{dx} = -\frac{2Q}{Kw}$$

This expression states that the derivative of h^2 with respect to x is the constant $-2Q/KW$

Thus a reasonable "guess" for a solution is

$$h^2 = h_0^2 - \frac{2Q}{Kw} x$$

To test this guess differentiate h^2 with respect to x

$$\frac{d(h^2)}{dx} = \frac{d}{dx}\left[h_0^2 - \frac{2Q}{Kw} x\right] = \frac{-2Q}{Kw}$$

Which indeed is ~~a solution~~ the original differential equation.

The solution indicates that a graph of h versus x will be a parabola.

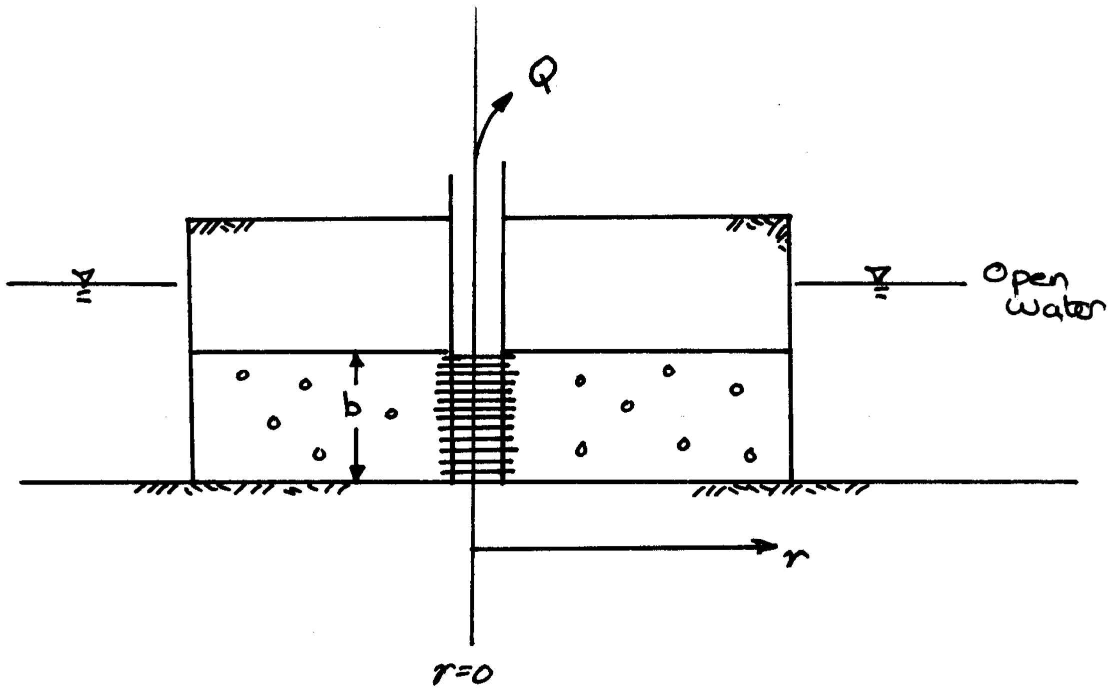
The parabolic shape compensates for the progressive decrease in flow area in such a way that Darcy's law is always satisfied.

This approximate theory of unconfined flow was introduced by Dupuit (1863) and the assumptions involved are usually called the Dupuit assumptions.

If the approximation is applied in cases where the assumptions are not valid, serious errors can be introduced.

In these two flow situations the flow geometry is rectilinear. The next situation is a case of radial flow in a cylindrical geometry.

In this case the cross sectional area of flow diminishes along the flow path creating a progressive steepening of the hydraulic gradient. However the decrease in area will be generated by the problem geometry rather than by the slope of the free surface.



The sketch shows a well centered in a circular island. The well taps a confined aquifer that is recharged by the open water on the perimeter of the island

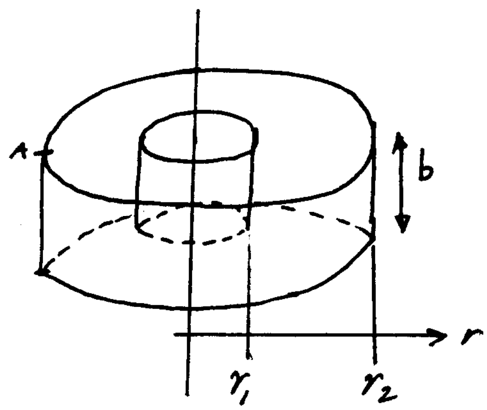
During pumping the water flows inward toward the well (radial flow). Assume that the open water elevation is constant and that recharge along the periphery equals the well's discharge

Because the well is at the center of a circular island, cylindrical symmetry is present and polar coordinates are appropriate for this situation

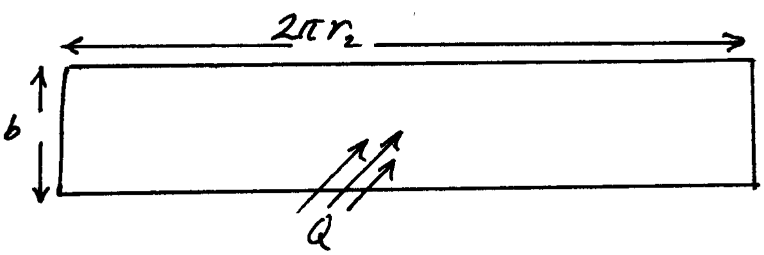
DLA-020

If the aquifer has thickness b, then the cross sectional area of flow at any radial distance, r, from the well is

$$A = 2\pi r b$$



Unwrap at A



Flow in this situation is radially inward in the $-r$ direction, parallel to the r axis. The cross sectional area of flow is a surface which is everywhere perpindicular to this direction of flow, in this case a cylinder

As one procedes inward along the path of flow in this problem, the cylindrical area becomes smaller and smaller.

Because the cross sectional area of flow decreases along the path of flow, the hydraulic gradient must increase along the path of flow to maintain a constant discharge.

When we apply Darcy's law to this problem we observe that Q is oriented towards the well (in the $-r$ direction), thus the algebraic sign of Q is negative

Darcy's law applied to this problem is

$$-Q = -K\,2\pi r b \frac{dh}{dr}$$

Now we need to "guess" a general solution to this problem.

Recall from calculus that $\frac{d}{dr}(\ln r) = \frac{1}{r}$

We will rearrange the differential equation and make some substitutions to arrive at a form where we can make a reasonable guess at the answer (solution)

Also recall from calculus that

$$\frac{dh}{dr} = \frac{dh}{d(\ln r)} \cdot \frac{d(\ln r)}{dr}$$

Rearrange the differential equation of Darcy's law

$$r\frac{dh}{dr} = \frac{Q}{2\pi K b}$$

Now substitute

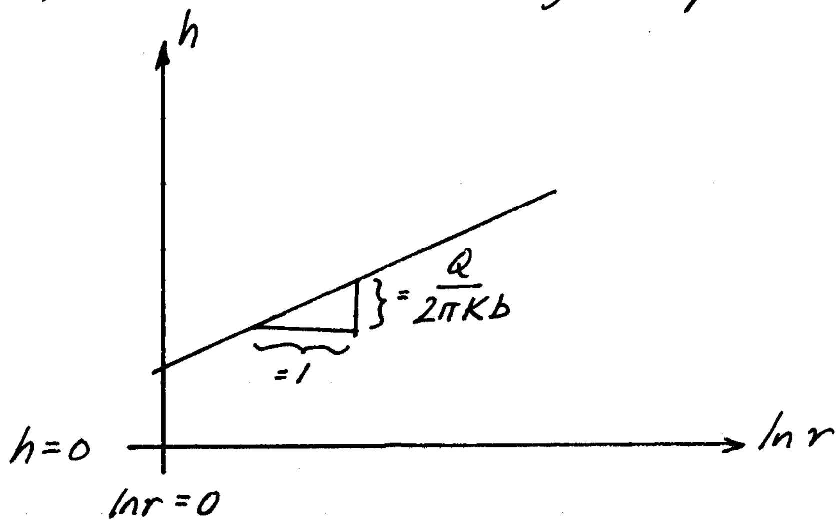
$$r\frac{dh}{d(\ln r)} \cdot \frac{d(\ln r)}{dr} = \cancel{r}\frac{dh}{d(\ln r)} \cdot \frac{1}{\cancel{r}} = \frac{Q}{2\pi K b}$$

And the differential equation can be reexpressed as

$$\frac{dh}{d(\ln r)} = \frac{Q}{2\pi K b}$$

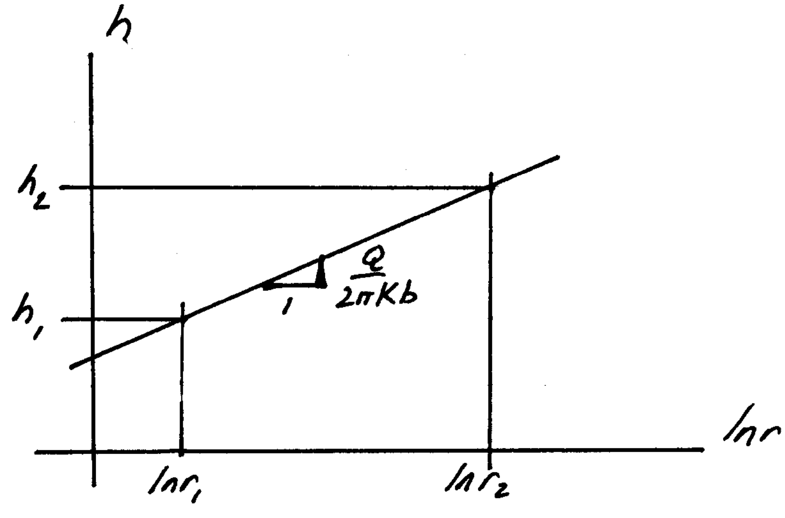
This equation states that the change in head with respect to the natural logarithm of r is a constant.

Now if one were to make a graph of h versus ln r one would expect the slope to be constant, and positive



The graph indicates that h increases as we move away from the well.

Now if one selects two points on the graph



They are related by the equation of the line. That is

$$h_2 = h_1 + \frac{Q}{2\pi Kb}(\ln r_2 - \ln r_1)$$

Recall that $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

Thus

$$h_2 = h_1 + \frac{Q}{2\pi Kb}\left(\ln\left(\frac{r_2}{r_1}\right)\right)$$

Now the important question is whether or not this equation is a solution to the differential equation.

$$h = h_0 + \frac{Q}{2\pi Kb}(\ln r - \ln r_0)$$

Test the solution:

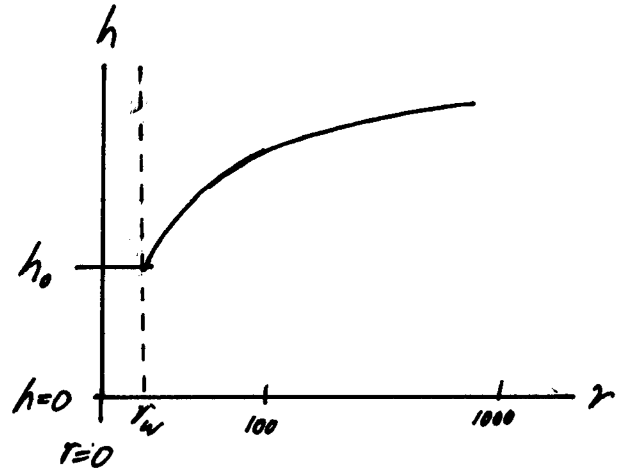
$$\frac{dh}{dr} = \cancelto{0}{\frac{dh_0}{dr}} + \frac{Q}{2\pi Kb}\left(\underbrace{\cancel{\frac{d\ln r}{dr}}}_{=\frac{1}{r}} - \cancelto{0}{\frac{d\ln r_0}{dr}}\right)$$

$$\frac{dh}{dr} = \frac{Q}{2\pi Kb}\cdot\frac{1}{r} \Rightarrow \underbrace{r\frac{dh}{dr} = \frac{Q}{2\pi Kb}}_{\text{original differential eqn.}}$$

Therefore

$$h = h_0 + \frac{Q}{2\pi K b} \ln\left(\frac{r}{r_0}\right)$$ is a solution.

If we were to plot a graph of h vs r (not $\ln(r)$) we would obtain a logarithmic shaped curve for the head distribution



Now if we apply the solution to the island problem, where r_w is the radius of the well, and r_e is the radius of the island, and h_w and h_e are the heads at the well and edge of the island, respectively then the particular solution is

$$h_e - h_w = \frac{Q}{2\pi K b} \ln\left(\frac{r_e}{r_w}\right)$$

If the head in the well and entire aquifer is h_e before pumping began, the term $h_e - h_w$ is the drawdown in the well.

The equation can be used to estimate drawdown for ~~and~~ various pumping rates.

Alternatively the drawdown and pumping rate can be used to estimate K the formation hydraulic conductivity.

The theory of steady flow to a well developed here is called the Theim theory who contributed to its development in 1906.

Readers familiar with differential equations will recognize that all the solutions presented in this lesson can be conviently found by separation of variables and integration.

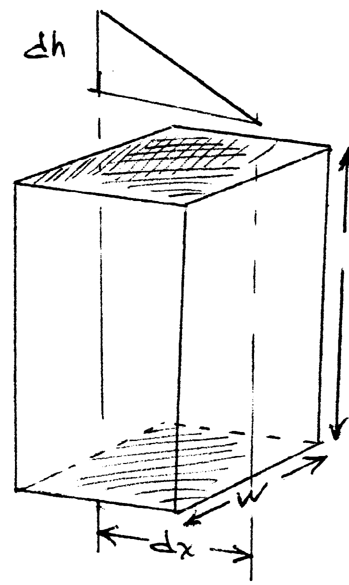
$$\frac{dh}{dx} = -\frac{Q}{KA} \Rightarrow \int_{h_1}^{h_2} dh = \int_{x_1}^{x_2} -\frac{Q}{KA}\, dx \Rightarrow h_2 - h_1 = \frac{-Q}{KA}(x_2 - x_1)$$

$$h\frac{dh}{dx} = \frac{-Q}{Kw} \Rightarrow \int_{h_1}^{h_2} h\, dh = \int_{x_1}^{x_2} -\frac{Q}{Kw}\, dx \Rightarrow h_2^2 - h_1^2 = \frac{-2Q}{Kw}(x_2 - x_1)$$

$$r\frac{dh}{dr} = \frac{Q}{2\pi Kb} \Rightarrow \int_{h_1}^{h_2} dh = \int_{r_1}^{r_2} \frac{Q}{2\pi Kb}\frac{dr}{r} \Rightarrow h_2 - h_1 = \frac{Q}{2\pi Kb}\, ln(r_2) - ln(r_1)$$

Transmissivity

Transmissivity is the term that describes the amount of water that will flow through a unit prism of aquifer

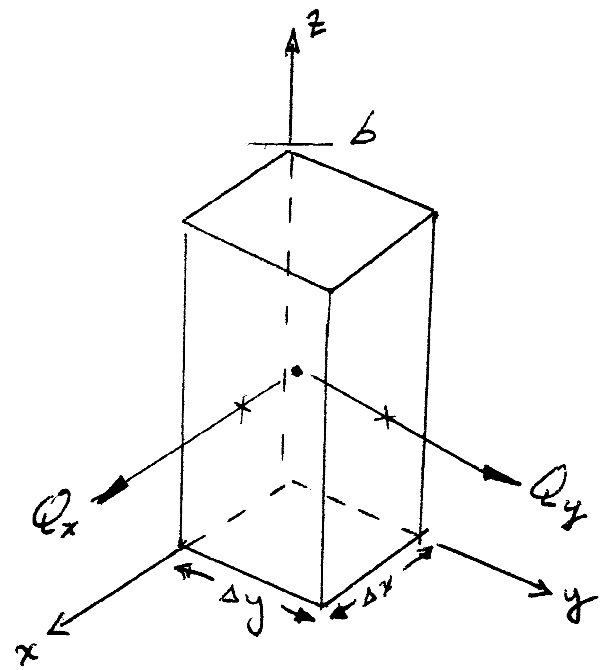


$$Q = K b w \frac{dh}{dx} \quad \text{(Darcy's Law)}$$

$$\frac{Q}{w} = \frac{\text{Discharge}}{\text{Unit width}} = \underbrace{K b}\frac{dh}{dx}$$

Kb is called the aquifer transmissivity

Typically the symbol used is T



Transmissivity is rigorously defined in the following fashion

$$Q_x = \int_0^z v \Delta y \, dz = -\int_0^z K_x \frac{\partial h}{\partial x} \Delta y \, dz$$

$$Q_y = \int_0^z V \Delta x \, dz = -\int_0^z K_y \frac{\partial h}{\partial y} \Delta x \, dz$$

If $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are uniform across the thickness (flow is essentially parallel to x-y plane)

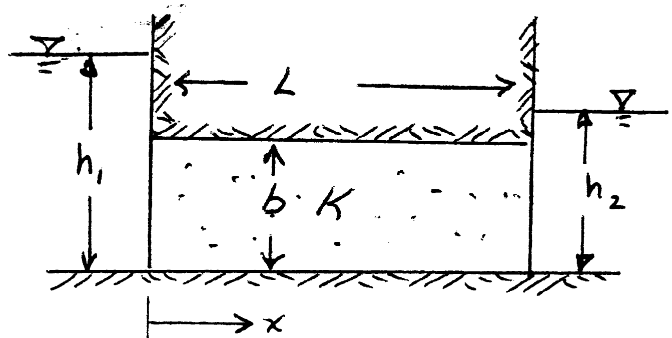
$$Q_x = -\frac{\partial h}{\partial x} \Delta y \int_0^z K_x \, dz$$

$$Q_y = -\frac{\partial h}{\partial y} \Delta x \int_0^z K_y \, dz$$

The integrals are called the Transmissivity

$$\therefore \quad T_x = \int_0^z K_x \, dz \; , \quad T_y = \int_0^z K_y \, dz$$

Direct Application of Darcy's Law



Steady flow, confined aquifer connecting two reservoirs

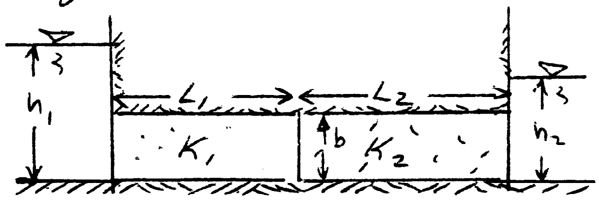
$$Q = KA \frac{\Delta h}{\Delta l} = -KA \frac{dh}{dx}$$

$$A = b \cdot w$$

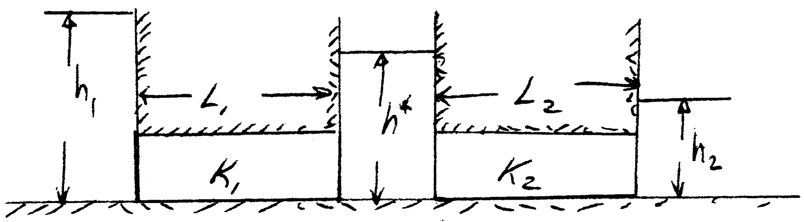
$$\frac{dh}{dx} = \frac{h_2 - h_1}{L}$$

$$Q = -K b w \frac{h_2 - h_1}{L}$$

Steady flow, confined aquifer comprised of two different geologic media.



Typically we want to know K_{mean}, the apparent mean hydraulic conductivity



$$Q_1 = K_1 A \frac{h_1 - h^*}{L_1} \qquad Q_2 = K_2 A \frac{h^* - h_2}{L_2} \qquad Q_T = \bar{K} A \frac{h_1 - h_2}{L_1 + L_2}$$

$$Q_1 = Q_2 = Q_T$$

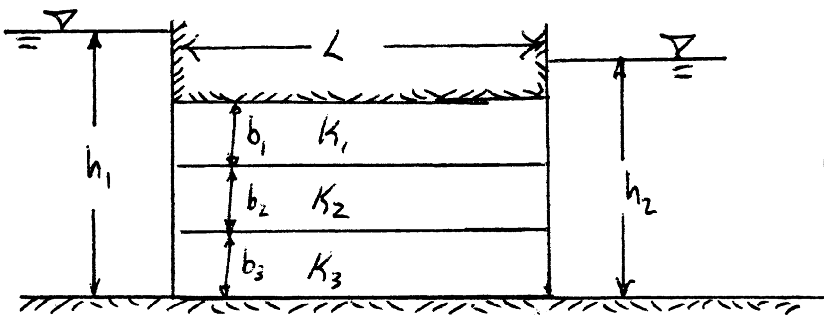
$$\therefore \quad h_1 - h^* = \frac{Q_T L_1}{K_1 A} \qquad h^* - h_2 = \frac{Q_T L_2}{K_2 A} \qquad h_1 - h_2 = \frac{Q_T (L_1 + L_2)}{\bar{K} A}$$

but $h_1 - h_2 = h_1 - h^* + h^* - h_2$

$$\therefore \frac{\cancel{Q_T}(L_1 + L_2)}{\bar{K} \cancel{A}} = \frac{\cancel{Q_T} L_1}{K_1 \cancel{A}} + \frac{\cancel{Q_T} L_2}{K_2 \cancel{A}}$$

$$\frac{\bar{K}}{(L_1 + L_2)} = \frac{1}{\frac{L_1}{K_1} + \frac{L_2}{K_2}} \Rightarrow \bar{K} = \frac{L_1 + L_2}{\frac{L_1}{K_1} + \frac{L_2}{K_2}}$$

Steady flow, confined aquifer, comprised of several layers of geologic media



Again we wish to know K_{mean}, the apparent hydraulic conductivity

Each layer is exposed to same gradient.

$$\therefore \quad Q_1 = K_1 b_1 w \frac{h_1 - h_2}{L}$$

$$Q_2 = K_2 b_2 w \frac{h_1 - h_2}{L}$$

$$Q_3 = K_3 b_3 w \frac{h_1 - h_2}{L}$$

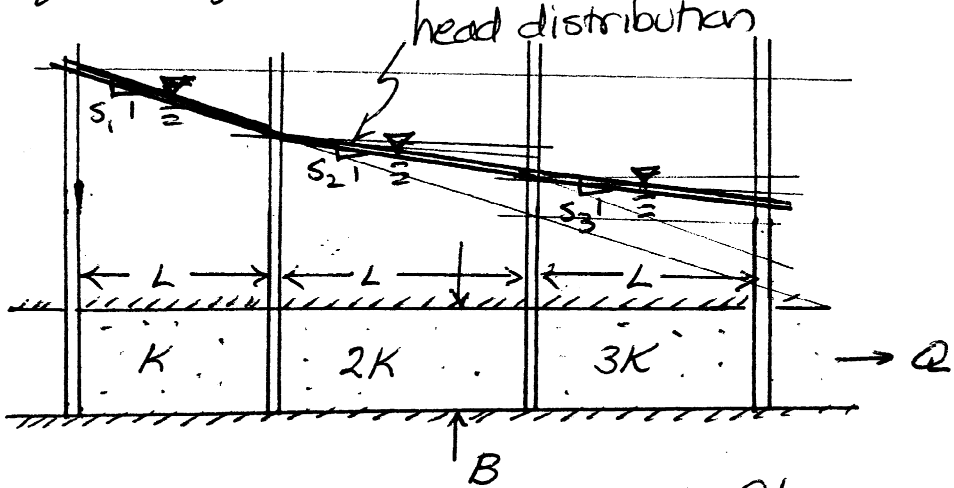
$$Q_T = \bar{K}(b_1 + b_2 + b_3) w \frac{h_1 - h_2}{L} = Q_1 + Q_2 + Q_3$$

$$\therefore \bar{K}(b_1 + b_2 + b_3)\cancel{w}\,\cancel{\frac{h_1 - h_2}{L}} = (K_1 b_1 \cancel{w} + K_2 b_2 \cancel{w} + K_3 b_3 \cancel{w}) * \cancel{\frac{h_1 - h_2}{L}}$$

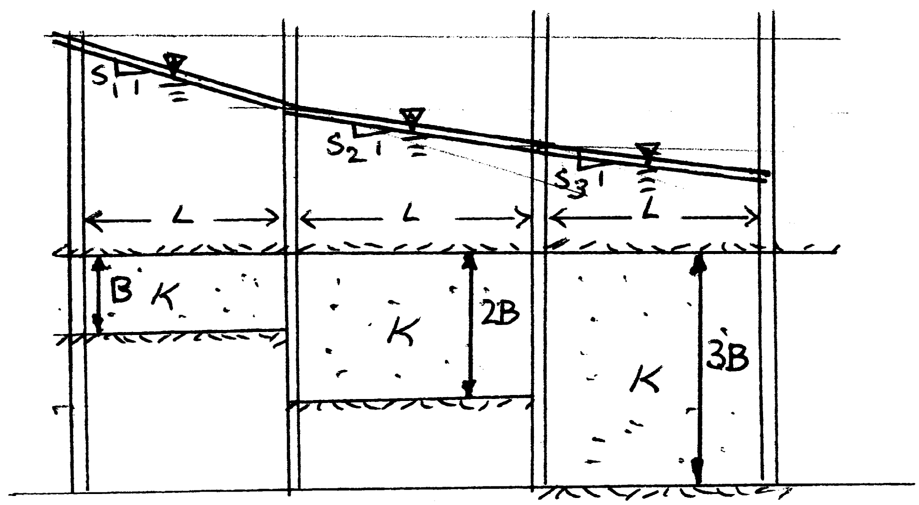
So $$\bar{K} = \frac{K_1 b_1 + K_2 b_2 + K_3 b_3}{b_1 + b_2 + b_3}$$

Transmissivity can be Identical in different systems

Two aquifer systems



$$s_1 = \frac{QL}{KBW} \qquad s_2 = \frac{QL}{2KBW} \qquad s_3 = \frac{QL}{3KBW}$$



$$s_1 = \frac{QL}{KBW} \qquad s_2 = \frac{QL}{K2BW} \qquad s_3 = \frac{QL}{K3BW}$$

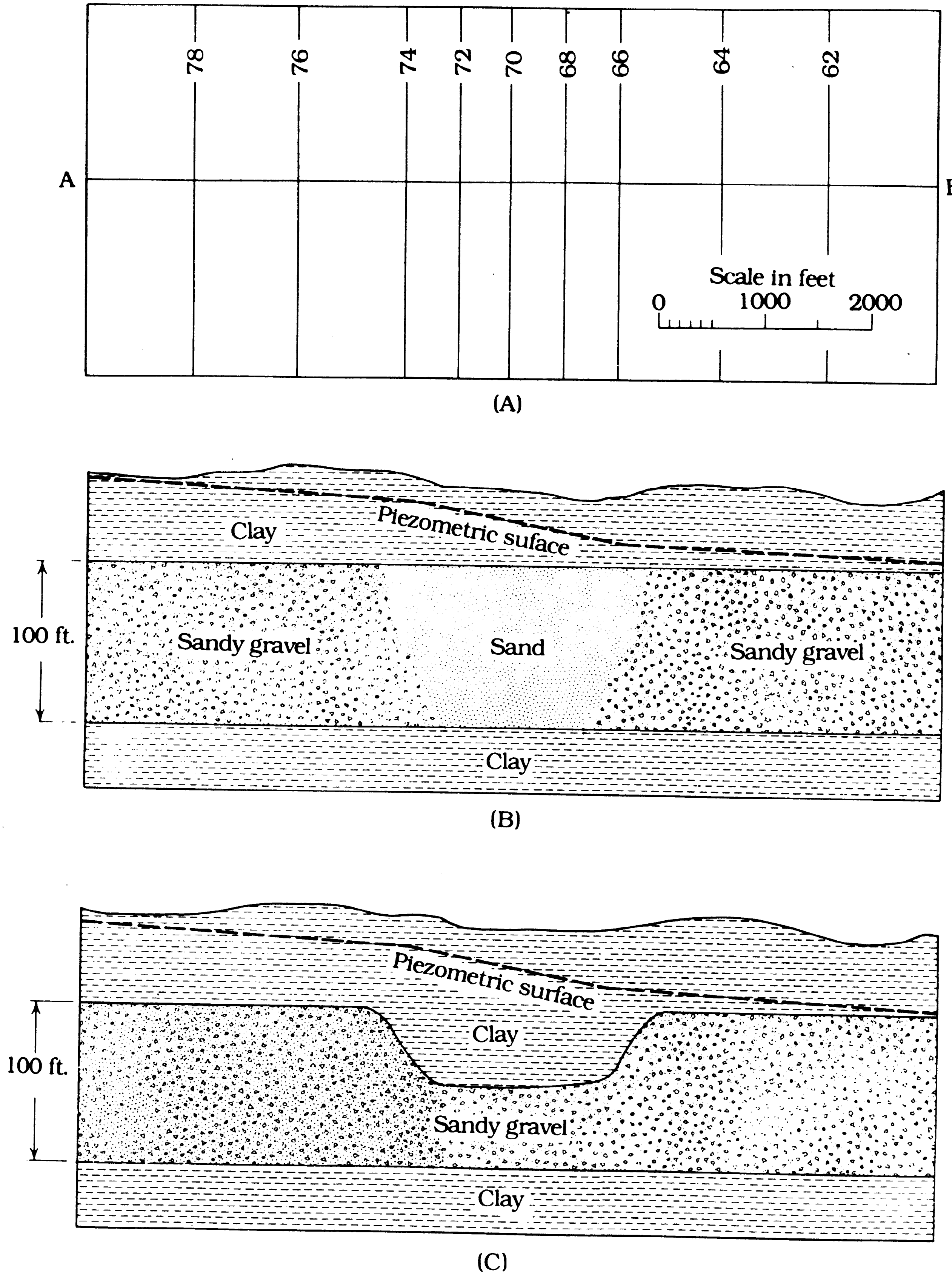


Figure 8.5. The effect of permeability and aquifer thickness on hydraulic gradient. A, Map of piezometric surface (contours in feet). B, Cross section illustrating change in permeability (K). C, Cross section illustrating change in thickness (b).

TRANSMISSIVITY

The capacity of an aquifer to transmit water of the prevailing kinematic viscosity is referred to as its transmissivity. The *transmissivity* (T) of an aquifer is equal to the hydraulic conductivity of the aquifer multiplied by the saturated thickness of the aquifer. Thus,

$$T = Kb \tag{1}$$

where T is transmissivity, K is hydraulic conductivity, and b is aquifer thickness.

As is the case with hydraulic conductivity, transmissivity is also defined in terms of a unit hydraulic gradient.

If equation 1 is combined with Darcy's law (see "Hydraulic Conductivity"), the result is an equation that can be used to calculate the quantity of water (q) moving through a unit width (w) of an aquifer. Darcy's law is

$$q = KA\left(\frac{dh}{dl}\right)$$

Expressing area (A) as bw, we obtain

$$q = Kbw\left(\frac{dh}{dl}\right)$$

Next, expressing transmissivity (T) as Kb, we obtain

$$q = Tw\left(\frac{dh}{dl}\right) \tag{2}$$

Equation 2 modified to determine the quantity of water (Q) moving through a large width (W) of an aquifer is

$$Q = TwW\left(\frac{dh}{dl}\right)$$

or, if it is recognized that T applies to a unit width (w) of an aquifer, this equation can be stated more simply as

$$Q = TW\left(\frac{dh}{dl}\right) \tag{3}$$

If equation 3 is applied to sketch 1, the quantity of water flowing out of the right-hand side of the sketch can be calculated by using the values shown on the sketch, as follows:

$$T = Kb = \frac{50\text{ m}}{\text{d}} \times \frac{100\text{ m}}{1} = 5{,}000\text{ m}^2\text{ d}^{-1}$$

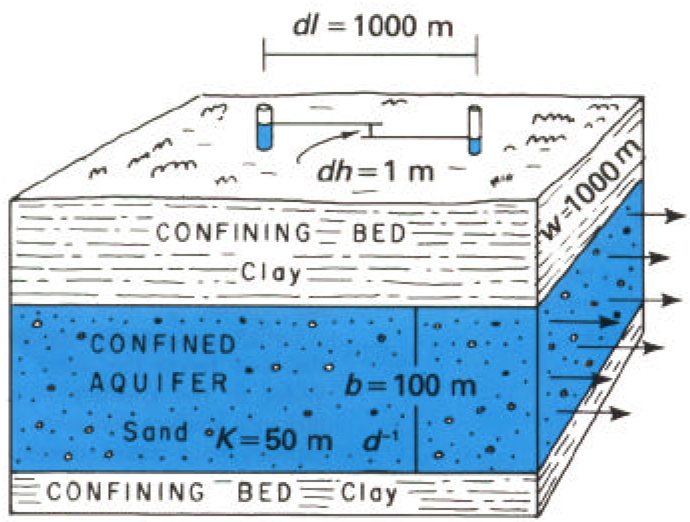
$$Q = TW\left(\frac{dh}{dl}\right) = \frac{5{,}000\text{ m}^2}{\text{d}} \times \frac{1{,}000\text{ m}}{1} \times \frac{1\text{ m}}{1{,}000\text{ m}} = 5{,}000\text{ m}^3\text{ d}^{-1}$$

Equation 3 is also used to calculate transmissivity, where the quantity of water (Q) discharging from a known width of aquifer can be determined as, for example, with streamflow measurements. Rearranging terms, we obtain

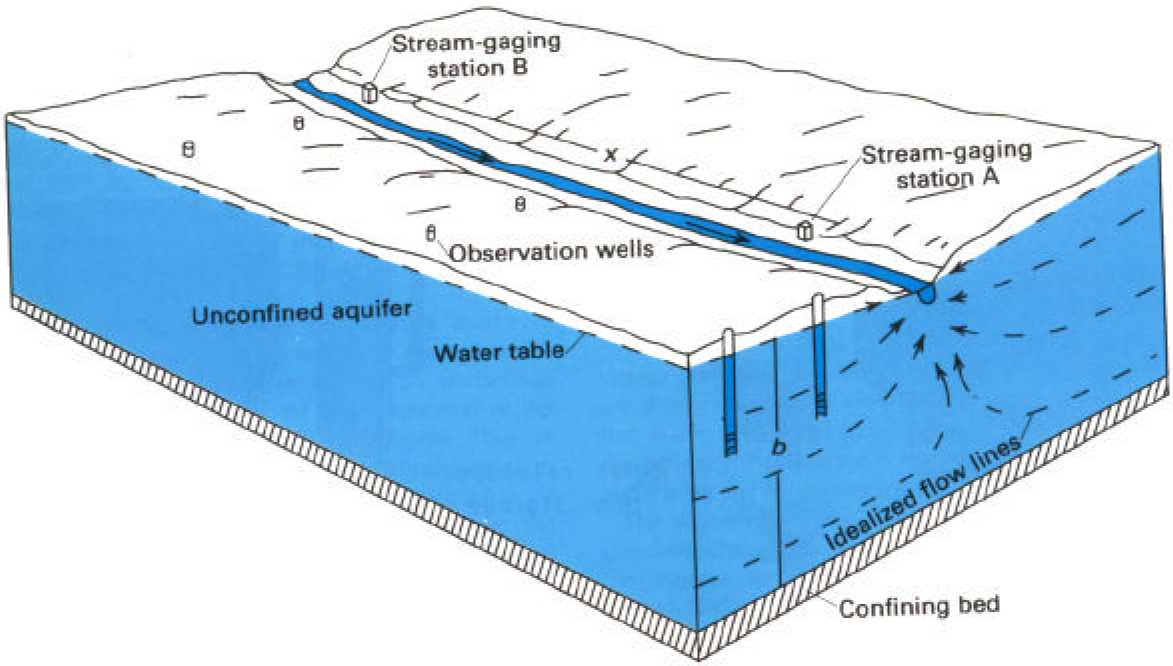
$$T = \frac{Q}{W}\left(\frac{dl}{dh}\right) \tag{4}$$

The units of transmissivity, as the preceding equation demonstrates, are

$$T = \frac{(\text{m}^3\text{ d}^{-1})(\text{m})}{(\text{m})(\text{m})} = \frac{\text{m}^2}{\text{d}}$$



(1)



(2)

Sketch 2 illustrates the hydrologic situation that permits calculation of transmissivity through the use of stream discharge. The calculation can be made only during dry-weather (baseflow) periods, when all water in the stream is derived from ground-water discharge. For the purpose of this example, the following values are assumed:

Average daily flow at stream-gaging station A:	2.485 $m^3\ s^{-1}$
Average daily flow at stream-gaging station B:	2.355 $m^3\ s^{-1}$
Increase in flow due to ground-water discharge:	0.130 $m^3\ s^{-1}$
Total *daily* ground-water discharge to stream:	11,232 $m^3\ d^{-1}$
Discharge from half of aquifer (one sid of the stream):	5,616 $m^3\ d^{-1}$
Distance (x) between stations A and B:	5,000 m
Average thickness of aquifer (*b*):	50 m
Average slope of the water table (*dh/dl*) determined from measurements in the observation wells:	1 m/2,000 m

By equation 4,

$$T = \frac{Q}{W} \times \frac{dl}{dh} = \frac{5{,}616\ \mathrm{m}^3}{\mathrm{d} \times 5{,}000\ \mathrm{m}} \times \frac{2{,}000\ \mathrm{m}}{1\mathrm{m}} = 2{,}246\ \mathrm{m}^2\ \mathrm{d}^{-1}$$

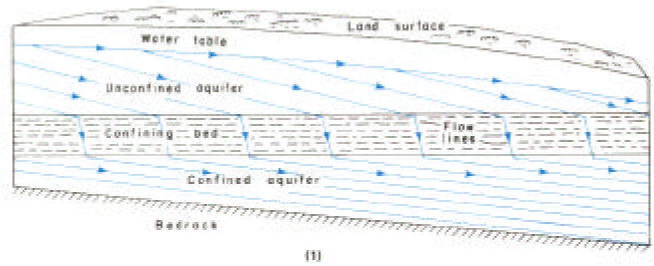
The hydraulic conductivity is determined from equation 1 as follows:

$$K = \frac{T}{b} = \frac{2{,}246\ \mathrm{m}^2}{\mathrm{d} \times 50\ \ \mathrm{m}} = 45\ \mathrm{m}\ \mathrm{d}^{-1}$$

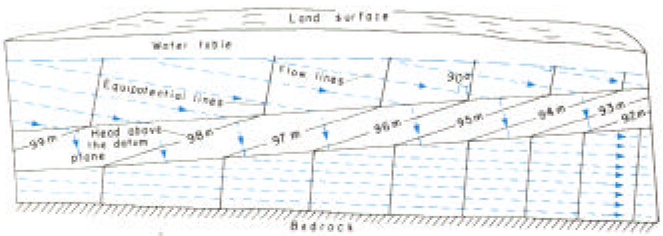
Because transmissivity depends on both *K* and *b*, its value differs in different aquifers and from place to place in the same aquifer. Estimated values of transmissivity for the principal aquifers in different parts of the country range from less than 1 $m^2\ d^{-1}$ for some fractured sedimentary and igneous rocks to 100,000 $m^2\ d^{-1}$ for cavernous limestones and lava flows.

Finally, transmissivity replaces the term "coefficient of transmissibility" because, by convention, an aquifer is transmissive, and the water in it is transmissible.

GROUND-WATER MOVEMENT AND STRATIFICATION



(1)



(2)

Nearly all ground-water systems include both aquifers and confining beds. Thus, ground-water movement through these systems involves flow not only *through* the aquifers but also *across* the confining beds (1).

The hydraulic conductivities of aquifers are tens to thousands of times those of confining beds. Thus, aquifers offer the least resistance to flow, the result being that, for a given rate of flow, the head loss per unit of distance along a flow line is tens to thousands of times less in aquifers than it is in confining beds. Consequently, lateral flow in confining beds usually is negligible, and flow lines tend to "concentrate" in aquifers and be parallel to aquifer boundaries (2).

Differences in the hydraulic conductivities of aquifers and confining beds cause a refraction or bending of flow lines at their boundaries. As flow lines move from aquifers into confining beds, they are refracted toward the direction perpendicular to the boundary. In other words, they are refracted in the direction that produces the shortest flow path in the confining bed. As the flow lines emerge from the confining bed, they are refracted back toward the direction parallel to the boundary (1).

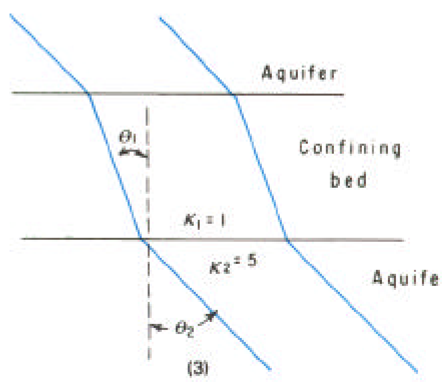
The angles of refraction (and the spacing of flow lines in adjacent aquifers and confining beds) are proportional to the differences in hydraulic conductivities (K) (3) such that

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{K_1}{K_2}$$

In cross section, the water table is a flow line. It represents a bounding surface for the ground-water system; thus, in the development of many ground-water flow equations, it is assumed to be coincident with a flow line. However, during periods when recharge is arriving at the top of the capillary fringe, the water table is also the point of origin of flow lines (1).

The movement of water through ground-water systems is controlled by the vertical and horizontal hydraulic conductivities and thicknesses of the aquifers and confining beds and the hydraulic gradients. The maximum difference in head exists between the central parts of recharge areas and discharge areas. Because of the relatively large head loss that occurs as water moves across confining beds, the most vigorous circulation of ground water normally occurs through the shallowest aquifers. Movement becomes more and more lethargic as depth increases.

The most important exceptions to the general situation described in the preceding paragraph are those systems in which one or more of the deeper aquifers have transmissivities significantly larger than those of the surficial and other shallower aquifers. Thus, in eastern North Carolina, the Castle Hayne Limestone, which occurs at depths ranging from about 10 to about 75 m below land surface, is the dominant aquifer because of its very large transmissivity, although it is overlain in most of the area by one or more less permeable aquifers.

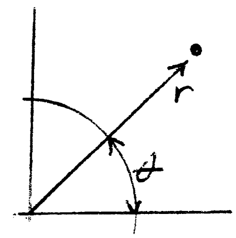


(3)

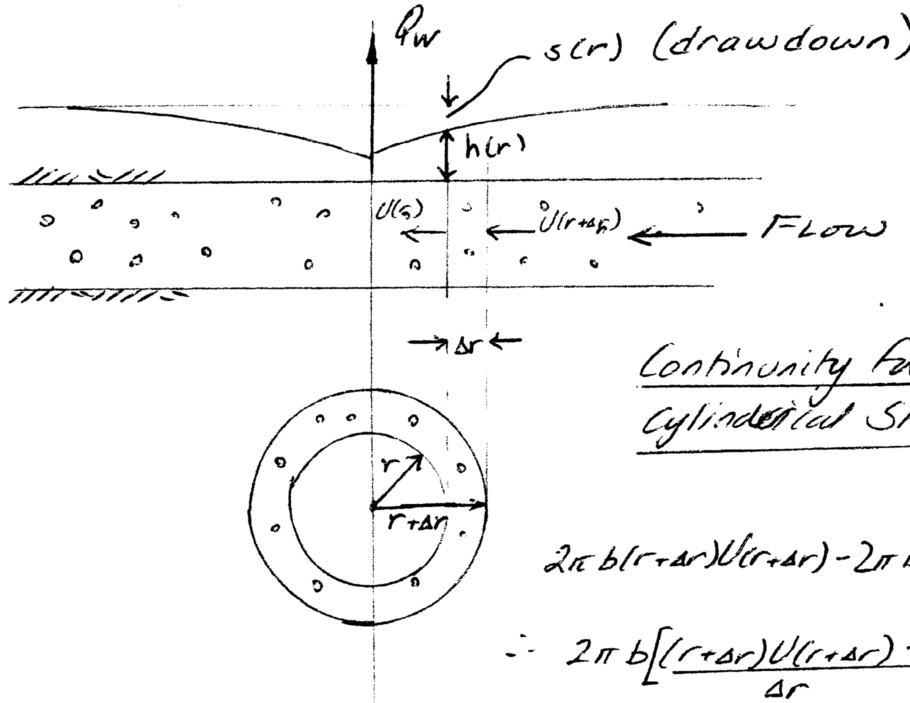
Radial Flow

Polar Coordinate system

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$



Confined Aquifer



Continuity for Cylindrical Shell

$$2\pi b(r+\Delta r)U(r+\Delta r) - 2\pi b r U(r) = 0$$

$$\therefore\ 2\pi b\left[\frac{(r+\Delta r)U(r+\Delta r) - rU(r)}{\Delta r}\right] = 0$$

In lim as $\Delta r \to 0$

$$2\pi b \frac{d}{dr}(rU) = 0$$

Darcys law: $U = K\dfrac{\partial h}{\partial r}$

$$\therefore\ 2\pi b \frac{d}{dr}\left(rK\frac{\partial h}{\partial r}\right) = 0$$

Groundwater Flow Equation

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{\partial\phi}{dr}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}}_{=0 \text{ (radial flow)}} = 0$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = 0$$

but $\phi = Kbh$ so $\frac{d}{\partial r}\left(r\,Kb\frac{\partial h}{\partial r}\right) = 0$ (a)

Boundary Conditions

$$\lim_{r\to r_w}\left(2\pi r\,Kb\frac{\partial h}{dr}\right) = Q_w$$

$\therefore$ Complete problem is

(a) $\frac{d}{\partial r}\left(r\,Kb\frac{\partial h}{dr}\right) = 0$

(b) $\lim_{r\to r_w}\left(2\pi r\,Kb\frac{\partial h}{\partial r}\right) = Q_w$ (c) $h(R) = H$

"Solution"

$$\int \frac{d}{\partial r}\left(r\,Kb\frac{\partial h}{\partial r}\right)dr = \int 0\,dr$$

$$r\,Kb\frac{\partial h}{\partial r} = C_1$$

$$\int Kb\,\partial h = \int C_1 \frac{dr}{r}$$

$$Kbh = C_1 \ln(r) + C_2$$

Evaluate C_1 & C_2 from boundary conditions

$$C_1 = rKb\frac{\partial h}{\partial r} = \frac{Q_w}{2\pi} \quad @\ r \to r_w$$

$$\therefore Kbh = \frac{Q_w}{2\pi}\ln(r) + C_2$$

$$h = \frac{Q_w}{2\pi Kb}\ln(r) + C_2$$

$$h(R) = H = \frac{Q_w}{2\pi Kb}\ln(R) + C_2$$

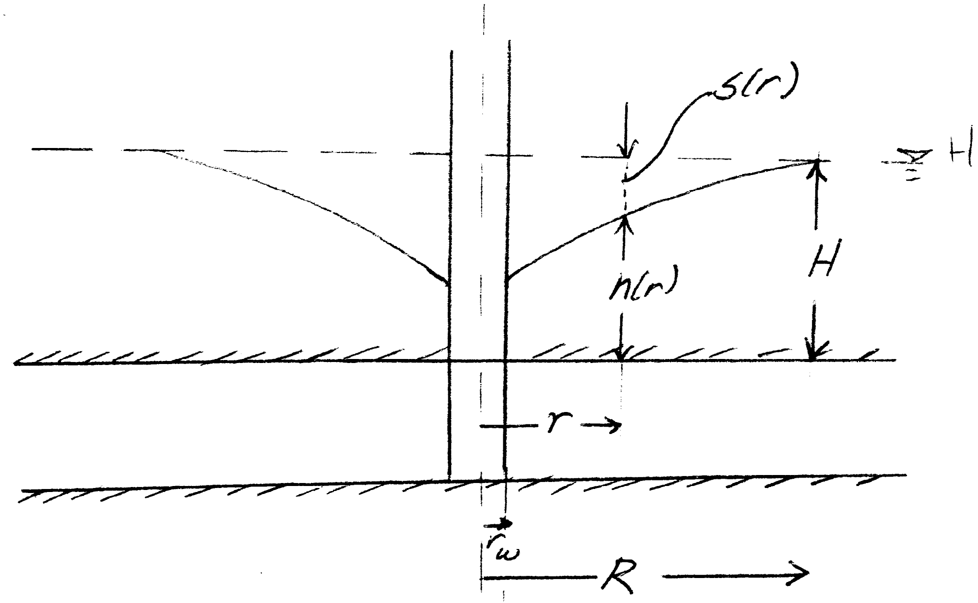
$$\therefore C_2 = H - \frac{Q_w}{2\pi Kb}\ln(R)$$

So

$$h(r) = \frac{Q_w}{2\pi Kb}\ln(r) - \frac{Q_w}{2\pi Kb}\ln(R) + H$$ or

$$h(r) = H + \frac{Q}{2\pi Kb}\ln\left(\frac{r}{R}\right)$$

Sketch of result



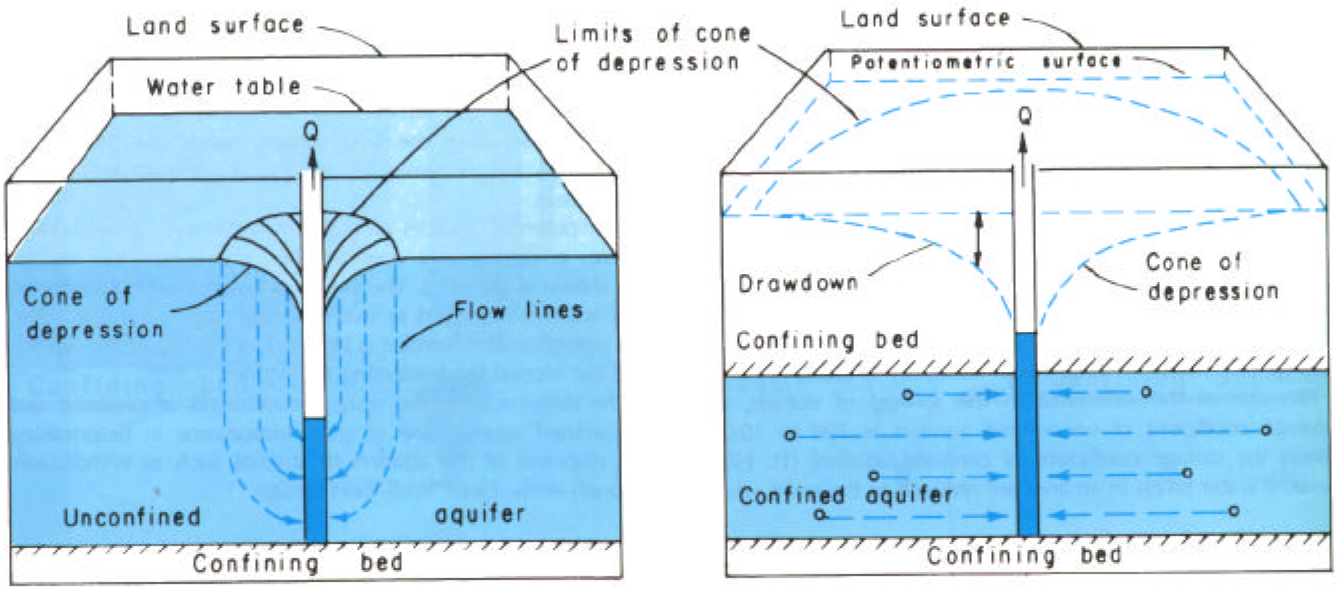
$s(r) = H - h(r)$ (Drawdown)

R is called "radius of action" of the well (Radius of influence)

When $r > R$, $s(r) \triangleq 0$ (definition)

$$s(r) = -\frac{Q}{2\pi K b}\ln\left(\frac{r}{R}\right) = \frac{Q}{2\pi K b}\ln\left(\frac{R}{r}\right)$$

CONE OF DEPRESSION



(1) (2)

Both wells and springs serve as sources of ground-water supply. However, most springs having yields large enough to meet municipal, industrial, and large commercial and agricultural needs occur only in areas underlain by cavernous limestones and lava flows. Therefore, most ground-water needs are met by withdrawals from wells.

The response of aquifers to withdrawals from wells is an important topic in ground-water hydrology. When withdrawals start, the water level in the well begins to decline as water is removed from storage in the well. The head in the well falls below the level in the surrounding aquifer. As a result, water begins to move from the aquifer into the well. As pumping continues, the water level in the well continues to decline, and the rate of flow into the well from the aquifer continues to increase until the rate of inflow equals the rate of withdrawal.

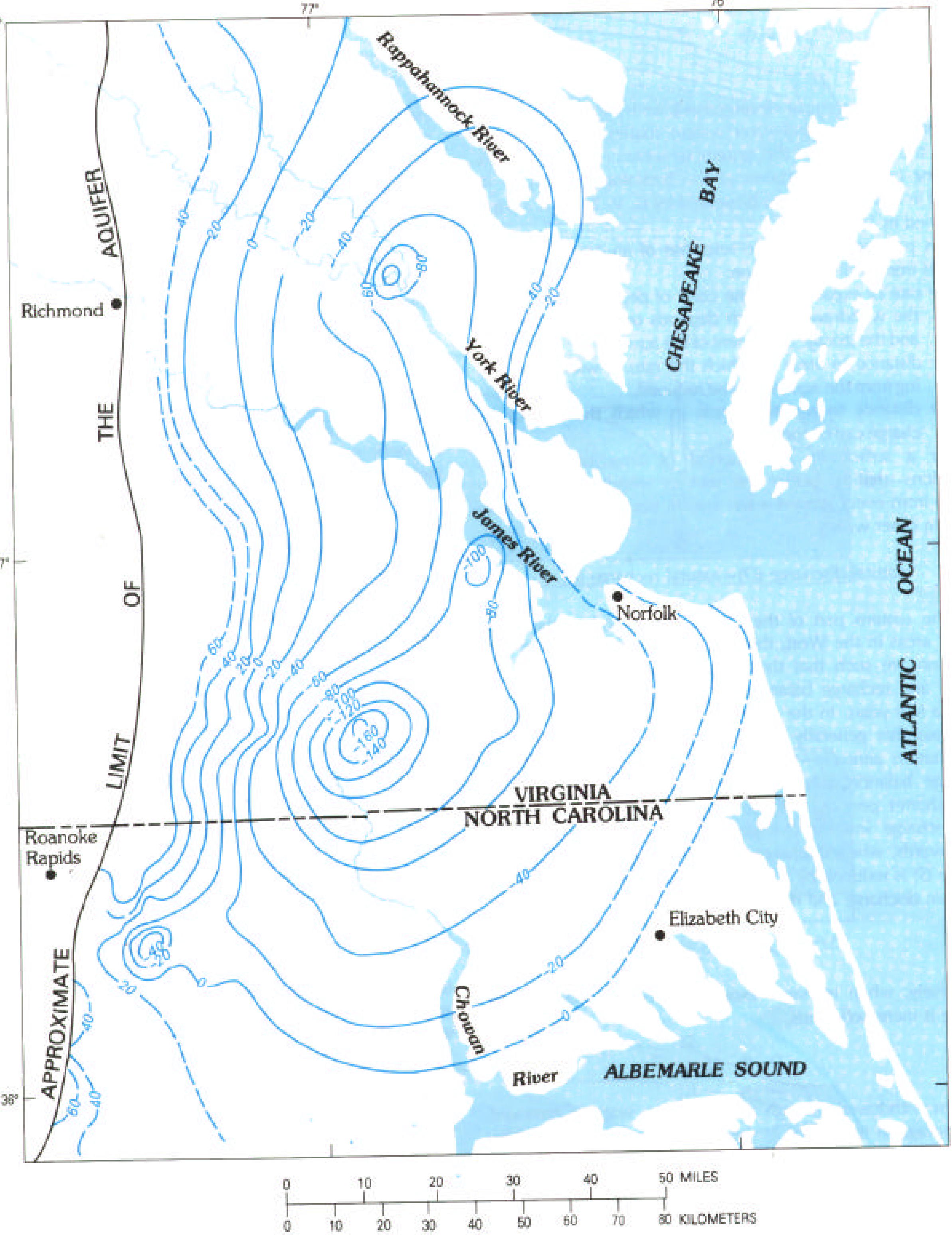
The movement of water from an aquifer into a well results in the formation of a *cone of depression* (1) (2). Because water must converge on the well from all directions and because the area through which the flow occurs decreases toward the well, the hydraulic gradient must get steeper toward the well.

Several important differences exist between the cones of depression in confined and unconfined aquifers. Withdrawals from an unconfined aquifer result in drainage of water from the rocks through which the water table declines as the cone of depression forms (1). Because the storage coefficient of an unconfined aquifer equals the specific yield of the aquifer material, the cone of depression expands very slowly. On the other hand, dewatering of the aquifer results in a decrease in transmissivity, which causes, in turn, an increase in drawdown both in the well and in the aquifer.

Withdrawals from a confined aquifer cause a drawdown in artesian pressure but do not (normally) cause a dewatering of the aquifer (2). The water withdrawn from a confined aquifer is derived from expansion of the water and compression of the rock skeleton of the aquifer. (See "Storage Coefficient.") The very small storage coefficient of confined aquifers results in a very rapid expansion of the cone of depression. Consequently, the mutual interference of expanding cones around adjacent wells occurs more rapidly in confined aquifers than it does in unconfined aquifers.

Cones of depression caused by large withdrawals from extensive confined aquifers can affect very large areas. Sketch 3 shows the overlapping cones of depression that existed in 1981 in an extensive confined aquifer composed of unconsolidated sands and interbedded silt and clay of Cretaceous age in the central part of the Atlantic Coastal Plain. The cones of depression are caused by withdrawals of about 277,000 m^3 d^{-1} (73,000,000 gal d^{-1}) from well fields in Virginia and North Carolina. (See "Source of Water Derived From Wells.")

POTENTIOMETRIC SURFACE OF THE LOWERMOST CRETACEOUS AQUIFER IN SOUTHEASTERN VIRGINIA AND NORTHEASTERN NORTH CAROLINA



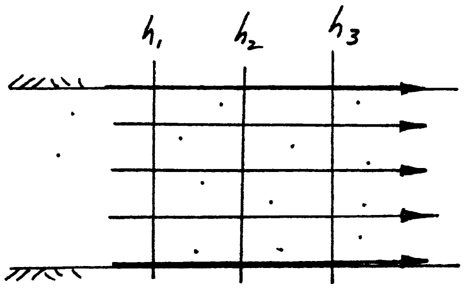
EXPLANATION

Water levels are in feet

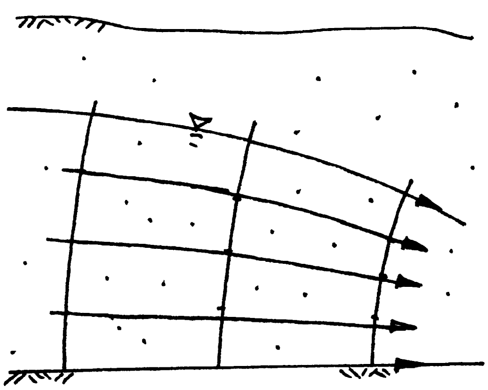
NATIONAL GEODETIC VERTICAL DATUM 1929

(3)

In a confined aquifer, streamlines are parallel and equipotentials are vertical (relative to aquifer orientation)

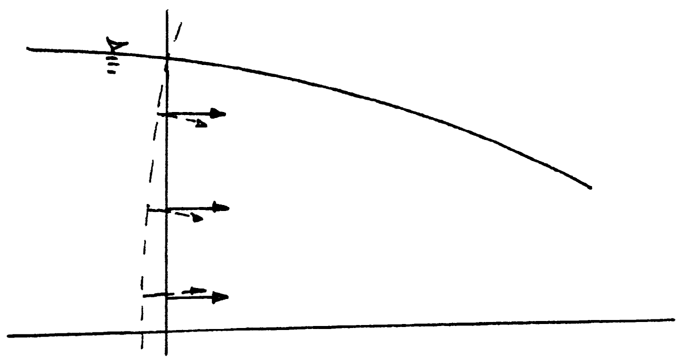


In an unconfined aquifer, such conditions no longer hold



Streamlines converge in direction of flow. Equipotentials curve in direction of flow. Darcy's law is often tedious to apply.

Dupuit Assumptions

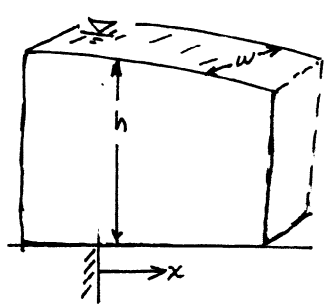


① Equipotentials are <u>assumed</u> vertical

② Streamlines are <u>assumed</u> horizontal

③ Slope of tangent line to free surface is gradient of head

Result:



Darcy's Law under these assumptions is

$$Q = -K h w \frac{\partial h}{\partial x}$$

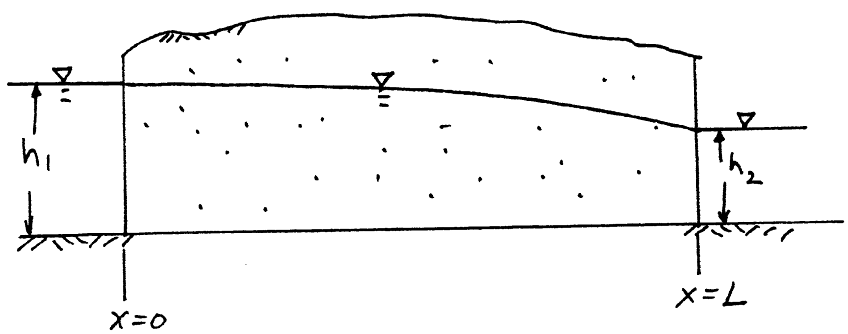
or

$$Q = K h w \frac{dh}{dl}$$

(Recall $-\frac{\partial h}{\partial x} = \frac{\partial h}{\partial l}$)

Dupuit's assumptions allow one to analyze unconfined systems in same manner as confined systems

Unconfined Flow Between Two Ditches



$$Q = -KA\frac{\partial h}{\partial x} \quad \text{(Darcy's Law)}$$

$$Q = -Khw\frac{\partial h}{\partial x} \quad \text{(Dupuit assumptions)}$$

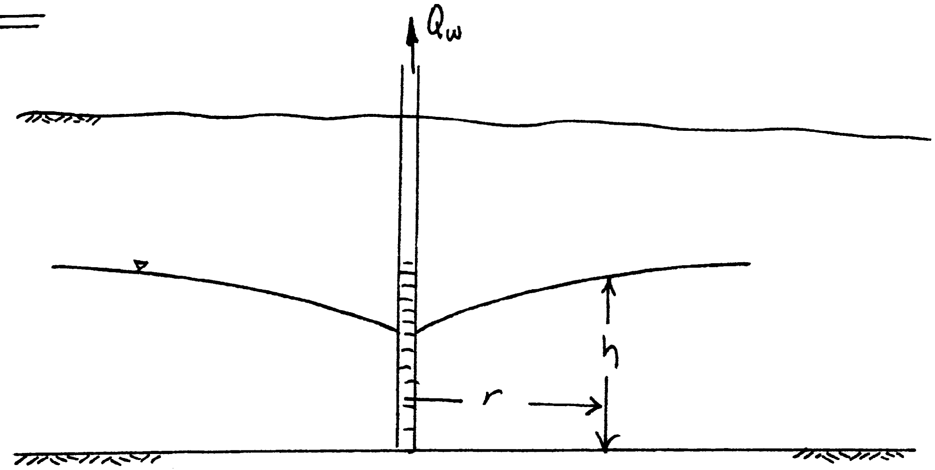
but $$h\frac{\partial h}{\partial x} = \frac{1}{2}\frac{\partial h^2}{\partial x}$$

$\therefore$ $$Q = -\frac{Kw}{2}\frac{\partial h^2}{\partial x} \quad \text{or}$$

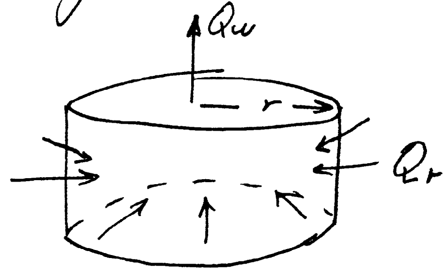
$$Q = \frac{Kw}{2}\frac{h_1^2 - h_2^2}{L}$$

Flow Geometry is not always rectlinear, often radial flow towards a well is studied

Steady Radial Flow to a Well in Unconfined Aquifer



Consider a cylinder at r



Darcy's Law at r is

$$Q_r = KA\frac{\partial h}{\partial r}$$

$$Q_r = K 2\pi r h \frac{\partial h}{\partial r}$$

$$\frac{\partial h}{\partial r} = \frac{h(r+\Delta r) - h(r)}{\Delta r}$$

r increases opposite to flow direction

By conservation of mass

$$Q_r = Q_w$$

$$\therefore \quad Q_w = K\, 2\pi r h \frac{dh}{dr}$$

Now to find head variation in r,
separate & integrate

$$\frac{dr}{r} = \frac{2\pi K}{Q_w} h\, dh = \frac{\cancel{2}\pi K}{Q_w} \frac{1}{\cancel{2}}\, dh^2$$

$$\int_{r_1}^{r_2} \frac{dr}{r} = \int_{h_1}^{h_2} \frac{\pi K}{Q}\, d(h^2)$$

$$\ln(r_2) - \ln(r_1) = \frac{\pi K}{Q_w}\, h_2^2 - h_1^2$$

$$\underline{\text{or}} \quad h_2^2 - h_1^2 = \frac{Q_w}{\pi K} \ln\left(\frac{r_2}{r_1}\right)$$