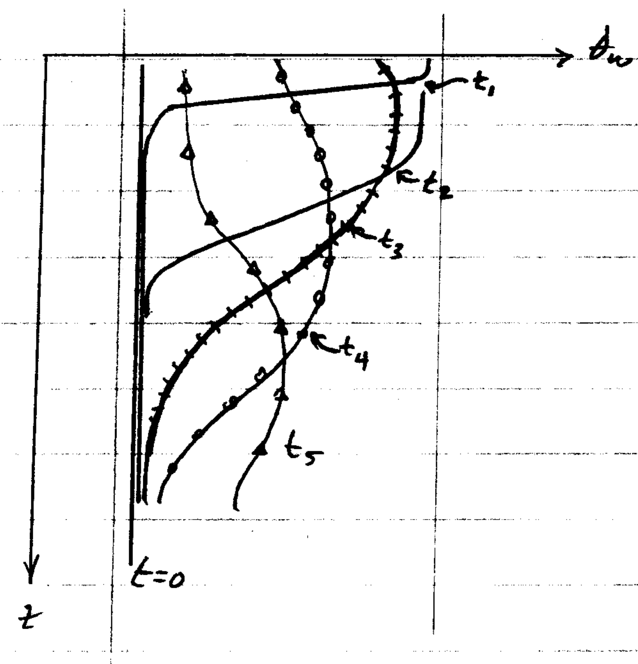
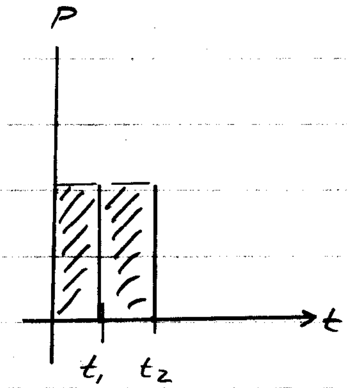
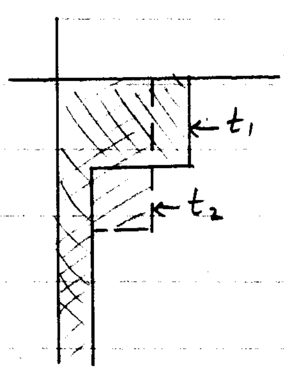
Soil water distribution
(Wave progression during rainfall)





Simple models
- Rectangular profile
- Kinematic wave model

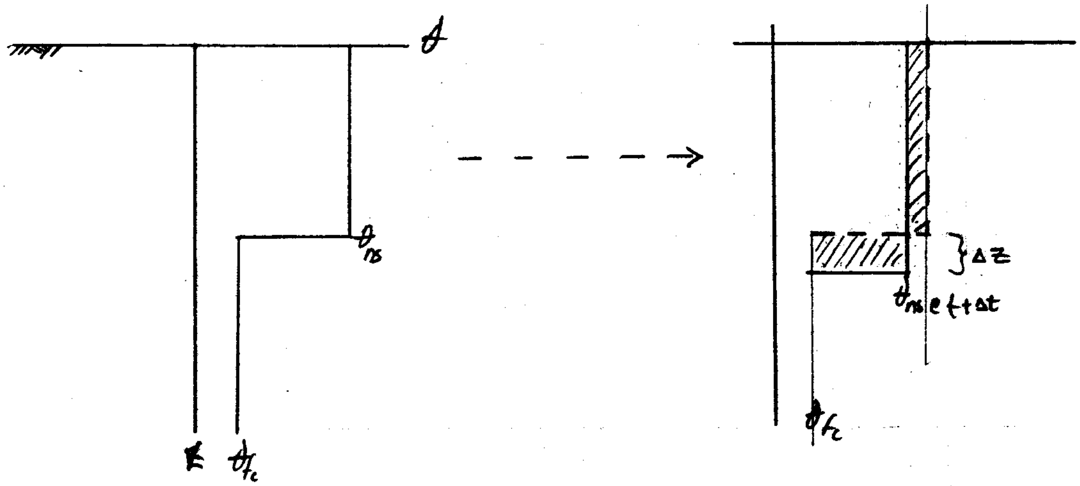
- Rectangular profile



Volume conserved in profile (mass balance)

Rectangular Profile Model

At end of infiltration period (Green-Ampt), water then redistributes



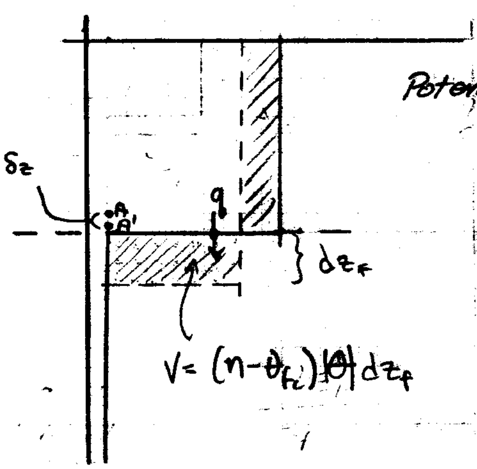
Volume conserved in profile

$$V = I = (\theta - \theta_{fc}) z_f \qquad \Theta = \frac{\theta - \theta_{fc}}{n - \theta_{fc}} \qquad \therefore\ I = (n - \theta_{fc})\,\Theta\, z$$

Neglect evaporation

$$\frac{dV}{dt} = (n - \theta_{fc})\, z\, \frac{d\Theta}{dt} + (n - \theta_{fc})\,\Theta\,\frac{dz}{dt} = 0 \quad \text{(Mass balance after infiltration stops)}$$

Examine the front



Potentials: $\psi_A = \psi_f - z$

$\psi_{A'} = \psi_f - z - \delta z$

∴ Darcy's law

$$q = K(\Theta)\frac{\psi_A - \psi_{A'}}{\delta z} = K\,\frac{\cancel{\psi_f} - \cancel{z} - \cancel{\psi_f} + \cancel{z} + \delta z}{\delta z}$$

$q = K(\Theta)$ at front

∴ By volume balance

$$(n - \theta_{fc})\,\Theta\, dz_f = K(\Theta)\, dt$$

Substitute into $\frac{dV}{dt} = 0 = (n - \theta_{fc})\, z_f \frac{d\Theta}{dt} + K(\Theta) = \frac{V}{\Theta}\frac{d\Theta}{dt} + K(\Theta) = 0$

Use Brooks & Corey relative permeability model

$$K(|\Theta|) = K_{ws}\,|\Theta|^{\varepsilon}$$

So now have

$$\frac{\mathcal{V}}{|\Theta|}\frac{d|\Theta|}{dt} + K_{ws}|\Theta|^{\varepsilon} = 0$$

OR

$$\frac{\mathcal{V}}{|\Theta|^{\varepsilon+1}}\frac{d|\Theta|}{dt} + K_{ws} = 0$$ separate & integrate

$$\int \frac{\mathcal{V}}{|\Theta|^{\varepsilon+1}}\,d|\Theta| = \int -K_{ws}\,dt$$

$$\int \mathcal{V}\,|\Theta|^{-\varepsilon-1}\,d|\Theta| = -K_{ws}t (+ C$$

$$\frac{\mathcal{V}}{-\varepsilon}|\Theta|^{-\varepsilon} = -K_{ws}t + C$$ at $t=0,\ |\Theta| = |\Theta|_0$

X^2 $$\frac{\mathcal{V}|\Theta|^{-\varepsilon}}{-\varepsilon} + K_{ws}t - \frac{V|\Theta|_0^{-\varepsilon}}{-\varepsilon} = 0$$ $$C = \frac{\mathcal{V}|\Theta|_0^{-\varepsilon}}{-\varepsilon-1}$$

solve for $|\Theta|$

$$\frac{\mathcal{V}|\Theta|^{-\varepsilon}}{\varepsilon} - K_{ws}t - \frac{V|\Theta|_0^{-\varepsilon}}{\varepsilon} = 0$$

$$\frac{\cancel{\mathcal{V}}|\Theta|^{-\varepsilon}}{\cancel{\varepsilon}} = \frac{\cancel{\mathcal{V}}|\Theta|_0^{-\varepsilon}}{\cancel{\varepsilon}} + \frac{K_{ws}t(\varepsilon))}{\mathcal{V}}$$

$$|\Theta|^{-\varepsilon} = |\Theta|_0^{-\varepsilon} + \frac{K_w t(\varepsilon))}{\mathcal{V}}$$

$$\frac{1}{|\Theta|^{\varepsilon}} = \frac{1}{|\Theta|_0^{\varepsilon}} + \frac{K_w t(\varepsilon))}{\mathcal{V}}$$

$$|\Theta| = \frac{1}{\left[\left(\frac{1}{|\Theta|_0}\right)^{\varepsilon} + \frac{K_w t\,\varepsilon}{\mathcal{V}}\right]^{1/\varepsilon}}$$ } same as 4.6.4 in text

$$z_f = \frac{\mathcal{V}}{(n-\theta_{fc})|\Theta|}$$

Rectangular model also provides a flux estimate from

$$q(z,t) = -z(n-\theta_{fc})\frac{d\Theta}{dt} = \frac{(n-\theta_{fc})K_{w_s} z}{\psi\left(\left(\frac{1}{\Theta_b}\right)^{\varepsilon} + \frac{\varepsilon K t}{\psi}\right)^{1+1/\varepsilon}}$$

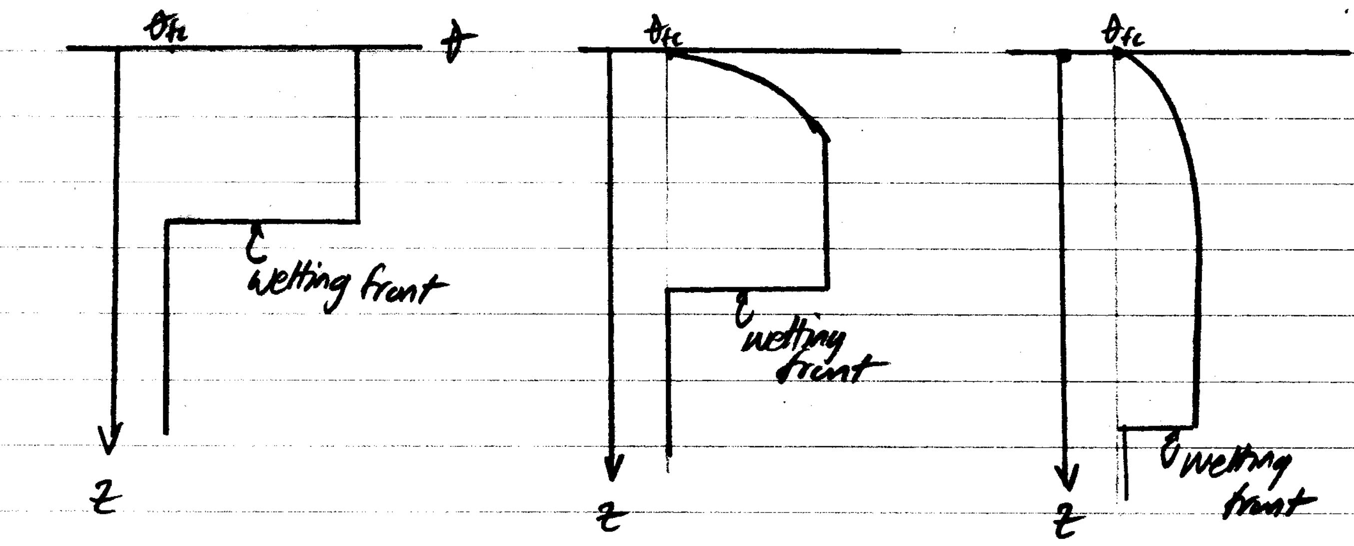
(Obtained from

first part of $\frac{d\theta}{dt}$ and Darcy's law at front.

Cumulative recharge G is obtained by integration of $q(z_{wt},t)$

$$G(t) = \int_{t_{wt}}^{t} q(z_{wt},\tau)\,d\tau = (n-\theta_{fc})\,z_{wt}\left(\Theta(t_{wt}) - \Theta(t)\right)$$

Kinematic Wave Model



Assume pressure gradients are negligible

$$\therefore \quad \frac{\partial \theta_w}{\partial t} - \frac{\partial K_w(\theta_w)}{\partial z} = 0 \qquad \text{express as reduced saturation}$$

$$(n-\theta_{fc})\frac{\partial \Theta}{\partial t} + \frac{\partial K_w(\Theta)}{\partial \Theta} \cdot \frac{\partial \Theta}{\partial z} = 0$$

$$d\Theta = \frac{\partial \Theta}{\partial t} dt + \frac{\partial \Theta}{\partial z} dz$$

(Since Θ is a function of t, z we can write its characteristic equation, and examine the PDE using the characteristic equation (recall stream & potential functions)

$$\frac{\partial \Theta}{\partial t}(n-\theta_{fc}) + \frac{\partial \Theta}{\partial z}\frac{\partial K(\Theta)}{\partial \Theta} = 0$$

$$\frac{\partial \Theta}{\partial t} dt + \frac{\partial \Theta}{\partial z} dz = d\Theta$$

For the two functionals to be equivalent then

$$\frac{(n-\theta_{fc})}{dt} = \frac{\frac{\partial K(\Theta)}{\partial \Theta}}{dz} = \frac{0}{d\Theta}$$

(Thus each equation is parallel in z, t, Θ space)

With these ratios preserved then

$$\frac{dz}{dt} = \frac{1}{n - \theta_{fc}} \cdot \frac{dK(\Theta)}{d\Theta} \quad \Big\} \text{ integration of this expression relates } z, t \;\&\; \Theta$$

front velocity

If $\Theta \to 0$ at end of constant infiltration Θ then

$$\frac{z}{t} = \frac{1}{n - \theta_{fc}} \frac{dK_w}{d\Theta} = \frac{\varepsilon K_{ws} (\Theta)^{\varepsilon - 1}}{n - \theta_{fc}}$$

And one can solve for $\Theta = \left(\frac{(n - \theta_{fc}) z}{\varepsilon K_w t} \right)^{\frac{1}{\varepsilon - 1}}$

Text has additional results for behavior at wetting front

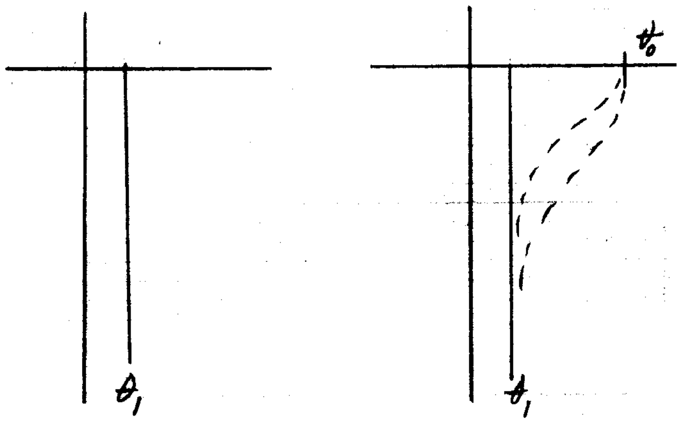
Both models are based on
1. volume (mass) balance
2. Darcian flux laws
3. Power-law models for K, θ

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta) \frac{\partial \theta}{\partial z} \right)$$

Richards' equation
(Non-linear diffusion equation)

If we linearize $D(\theta)$ by assuming $D = \text{const.}$ and specify

$\theta = \theta_1 \quad z>0, t=0$ and $\theta = \theta_0$ at $z=0, t>0$



The equation is now:

$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial z^2}$ which is a linear diffusion equation in θ

The solution for the prescribed boundary conditions are

$$\frac{\theta - \theta_0}{\theta - \theta_1} = \operatorname{erf}\left(\frac{z}{\sqrt{4Dt}}\right)$$

The flux at the surface is

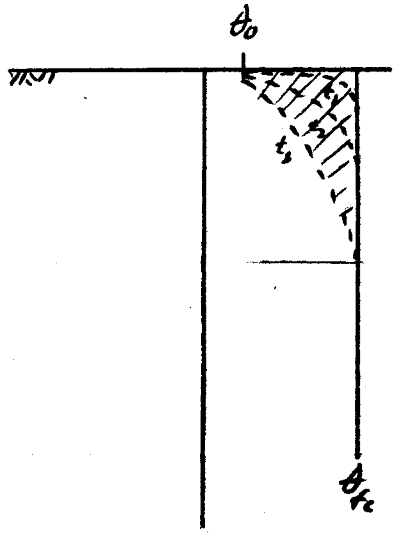
$$q = -D\frac{\partial \theta}{\partial z}\Big|_{z=0} = (\theta_0 - \theta_1)\sqrt{\frac{D}{\pi t}}$$

Cumulative flux is

$$I = \int_0^t q \, d\tau = (\theta_0 - \theta_1)\sqrt{\frac{4Dt}{\pi}}$$

The non-linear case is usually solved numerically or using self-similar variable transformations (see text)

Examine soil:



Cumulative evaporation at any time is

$$E = \int_0^\infty (\theta_{fc} - \theta(z))\,dz = \int_{\theta_0}^{\theta_{fc}} z(\theta)\,d\theta = -\int_0^t q\,d\tau$$

$$\therefore \quad E = \underbrace{(\theta_i - \theta_0)\sqrt{\frac{4D}{\pi}}}_{\text{soil desorptivity} = S_e}\sqrt{t} = S_e\sqrt{t}$$

$$\frac{dE}{dt} = e \quad \text{(evaporation rate)}$$

$$\frac{dE}{dt} = \frac{1}{2}(\theta_i - \theta_0)\sqrt{\frac{4D}{\pi}}\,t^{-1/2} = \frac{(\theta_i - \theta_0)\sqrt{\frac{4D}{\pi}}}{2\sqrt{t}} = \frac{S_e}{2\sqrt{t}}$$

$$\therefore \quad e = \frac{S_e}{2\sqrt{t}} = \frac{S_e}{2\sqrt{t}} \cdot \frac{S_e\sqrt{t}}{S_e\sqrt{t}} = \frac{S_e^2}{2E}$$

Actual rate is limited by thermodynamics - the maximum possible rate is called e_p (potential evap.)

$$t_{ec} = \frac{S_e^2}{2e_p^2}$$ (time from thermodynamic limiting conditions to soil water profile conditions)

Calculations treated in same fashion as infiltration

Estimating S_e from Brooks & Corey model

$$S_e = \sqrt{\frac{16(\varepsilon - 3)K_{ns}\psi_b(n - \theta_r)}{3(\varepsilon + 3)(\varepsilon + 5)}}\left(\frac{\theta_{ep} - \theta_r}{n - \theta_r}\right)^{\frac{\varepsilon + 5}{4}}$$

Ultimately one wants to estimate recharge rates & travel times as these factors have significant impact on water resources investigations and pollutant transport

Estimating recharge

① Water balance $\underbrace{P}_{\text{precipitation}} = \overbrace{R}^{\text{runoff}} + \underbrace{ET}_{\text{evap. transpiration}} + \overbrace{G}^{\text{recharge}} + \underbrace{\Delta S}_{\text{soil moisture storage}}$

Challenge: can measure P & R, accurately and cheaply

Cannot measure ET, but can estimate

Can measure G with difficulty as can ΔS.

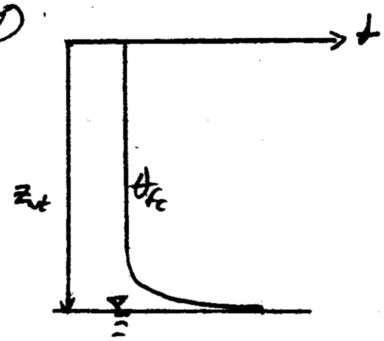
Typically assume $\Delta S \approx 0$, then can estimate G from P, R & good guess of ET.

② Tracer methods

a) Tritium (actually dates waters location in hydrologic cycle)
b) Cl-36 (radioactive tracer from hydrogen bomb testing)
c) Isotope ratios
d) Freon
e) introduced tracers

③ Computer programs (HELP)

Conclude all methods are imperfect because of $ limitations, so recharge estimates are just that – estimates. Estimates should agree with long term water budget & basin regression results

① 

$z_{wt} = v \cdot t$

$v = \dfrac{G}{\bar{\theta}}$ $\quad G$ = net recharge rate

$\therefore\ t = \dfrac{z\bar{\theta}}{G}$

② Assume unit-gradient behavior

$G \approx K_w(\bar{\theta})$

$t = \dfrac{z\,\bar{\theta}}{K_w(\bar{\theta})}$ Use Brooks & Corey for example.