CHAPTER

TWELVE

USE OF LINEAR PROGRAMMING IN AQUIFER MANAGEMENT

In Chap. 1 we have defined the problem of aquifer management and referred to it as our primary objective. We have also emphasized that in order to manage an aquifer system, we must be able to define it, by solving the *identification problem*, and to predict its response to our implemented decisions, by solving the *forecasting problem*. The first eleven chapters of the book are devoted to these two problems. In the present chapter, we wish to demonstate how information derived from a forecasting problem indeed serves as an essential input to a management one.

We have selected linear programming (LP) as an example of a decision making technique to be used in our demonstration, assuming that this very useful tool is better known to the reader. This should not be interpreted as an indication of the merits or superiority of this technique with respect to other ones used in the management of water resources. The basic manner in which the forecasting problem and the management one are interrelated is the same also for other techniques.

Furthermore, no attempt is made here to present this technique in any depth beyond that required for achieving our objective as stated above. The reader is referred to the literature for further information related to the management of water resources in general and to optimization techniques in particular; e.g., Maass *et al.* (1962), Hall and Dracup (1970), Buras (1972), Biswas (1976), and many articles in professional journals. A large number of publications also exists on the application of linear programming to the management of water resources, including some, e.g., Lynn *et al.* (1962), Deininger (1965), and Futagami *et al.* (1976), that focus on the management of water quality.

Of special interest as far as the present chapter is concerned should be publications which apply linear programming to groundwater systems, or to water resource systems in which aquifers play an important role, e.g., Dracup (1966) and Schwartz (1971).

At this point, the reader should return to Sec. 1-2 for a brief review of goals and objectives of water resources management, objective functions and constraints, especially those related to the management of a groundwater basin.

12-1 BRIEF REVIEW OF LINEAR PROGRAMMING

Linear programming is one of the methods used for solving the problem of allocating limited resources among competing users in an optimal manner. This means that among the various possible ways of allocating the given resources, we are to select the one which will minimize or maximize a specified objective function (Sec. 12-2).

The principles and techniques of solving linear programming problems are well described in the literature, e.g., Hadley (1962); Dantzing (1963); and Gass (1969). The brief review presented here is intended as a refresher for those who are already familiar with linear programming as an optimization technique, and as a brief introduction for those who are not familiar with it, indicating that here is another tool that can be used for making management decisions.

Hadley (1962) describes the general linear programming problem as follows: given a set of m linear inequalities or equations in r variables, we wish to find nonnegative values of these variables which will maximize or minimize some linear function of the variables, while satisfying the linear constraints. Mathematically, the problem may be stated as follows.

Determine the values of the r decision variables x_i ($i = 1, \ldots, r$) which will maximize (or minimize) the objective function z

$$z = c_1 x_1 + c_2 x_2 + \cdots + c_r x_r \qquad (12\text{-}1)$$

subject to the m constraints

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{ir} x_r \ \{\geq, =, \leq\}\ b_i, \qquad i = 1, \ldots, m \qquad (12\text{-}2)$$

where for each constraint only one of the signs $\leq, =, \geq$ holds, but the sign may vary from one constraint to another. We also require that all decision variables be nonnegative, i.e., that the nonnegativity restriction

$$x_j \geq 0, \qquad j = 1, \ldots, r \qquad (12\text{-}3)$$

be satisfied: a_{ij}, b_i, and c_j are assumed to be known constants.

Any set of x_j's which satisfies the constraints (12-2) is a solution of the linear programming problem. If, also, all the nonnegativity requirements (12-3) are satisfied, it is called a feasible solution. The feasible solution which yields the optimal value of the objective function is called the optimal solution (Hadley, 1962). Each set of decisions, x_j, is also called a policy. Our objective is to select that particular policy which will optimize the objective function, subject to the nonnegativity restrictions and the specified constraints.

When the x_j's are interim or final outputs of a production system, and the

a_{ij}'s are the use that is made of a certain resource for producing a unit of x_j, then the values b_i on the right-hand side of (12-2), with a "smaller or equal" sign, are referred to as the limited available quantities of the resources.

The x_j's are the decision variables of the problem: they may represent, for example, *activities*. When the c_j's are costs or prices associated with the x_j's, then z represents the *total cost* (or *benefit*) from operating the system at the activities x_j. The matrix composed of the coefficients a_{ij} is often referred to as the *technological matrix*.

Although the problem as stated by (12-1) through (12-3) may at first sight seem complicated to the planner who is not familiar with the solution of linear programming problems, he need not worry. His primary task should be to analyze his groundwater (or water resources) system, and state his management problem as an optimization problem (i.e., involving a minimization, or maximization of an objective function—or several objective functions in a multi-objective problem—and constraints which have to be satisfied). He should then be able to recognize a linear programming problem from the fact that the objective function as well as the constraints are linear expressions of the decision variables (that is, the x_j's).

Once a problem has been cast into the standard linear programming form, say (12-1) through (12-3), the common algebraic procedure for solving it is the Simplex method, developed by Dantzig in the late 40's (Dantzig, 1963). This procedure is well suited for solution by digital computers. The MPSX program (available in most IBM library programs) is an example of a standard procedure for solving the linear programming problem. Most computer manufacturers supply packages for solving linear programming by modified and advanced Simplex methods. Problems with hundreds of constraints can be solved by a medium-size computer. Large computers can handle thousands of constraints. When larger problems are encountered, decomposition techniques (see, for example, Lasdon, 1970; Hadley, 1962) permit a solution, although they usually require more computer time.

There is no difficulty in changing a problem from minimizing an objective function to maximizing one, noting that Min z = Max$(-z)$. If a decision variable, x_j, is unrestricted in sign, we may replace it by the difference between two (new) nonnegative decision variables, say, x_j^+ and x_j^- such that $x_j = x_j^+ - x_j^-$.

Each "less than or equal" constraint, e.g.

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kr}x_r \le b_k \tag{12-4}$$

can be written as an equality constraint

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kr}x_r + x_{r+1} = b_k \tag{12-5}$$

by adding a slack variable x_{r+1}. In a similar way, a "greater than or equal" constraint can be changed to an equality constraint by substracting an excess variable.

Hence any linear programming problem can be stated in the following form.

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the m constraints

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (12\text{-}6)$$

and

$$x_j \geq 0, \quad j = 1, 2, \ldots, n$$

which is called the standard linear programming form.

Before leaving these general introductory remarks, it may be of interest to present also the so called dual problem.

Associated with the (primal) problem as stated by (12-6), there exists another problem, referred to as the dual of the original (primal) problem. For a primal stated as

$$\text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_rx_r$$

subject to the m constraints

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1r}x_r &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2r}x_r &\leq b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mr}x_r &\leq b_m \end{aligned}\right\} \quad (12\text{-}7)$$

and

$$x_j \geq 0, \quad j = 1, \ldots, r$$

the dual problem is stated as

$$\text{Minimize } w = b_1y_1 + b_2y_2 + \cdots + b_my_m$$

subject to the r constraints

$$\left.\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\geq c_2 \\ a_{1r}y_1 + a_{2r}y_2 + \cdots + a_{mr}y_m &\geq c_r \end{aligned}\right\} \quad (12\text{-}8)$$

and

$$y_j \geq 0, \quad j = 1, \ldots, m$$

It is of interest to note (a) that the dual problem (12-8) contains the same constants as the primal (12-7), but in a rearranged (transposed) order, (b) that the dual of the dual problem is the primal, (c) that if the optimal values of x_j ($= x_j^*$) and of y_j ($= y_j^*$) are used to compute the optimal values of the objective functions z^* of z and w^* of w, then we shall find that $z^* = w^*$ (we may think of z^* as the maximum profit, and of w^* as the minimum value of the resources), and (d) that if $r < m$ then the formulation (12-7) involves more constraints than the formulation (12-8), so that the latter may be more convenient for computer solution. Actually, once one solution is obtained, the other can be obtained from it.

Finally, it may be of interest to note that the dual decision variables y_i are interpreted as marginal values of the resources b_i, or their shadow prices. When z is being maximized, y_i is a measure of the rate of increase of z with respect to b_i

$$y_i = \partial z / \partial b_i \Big|_{z=z^*} \tag{12-9}$$

12-2 APPLICATION OF LINEAR PROGRAMMING TO AQUIFER MANAGEMENT

Before stating the management problem, we have to establish the link between the management problem and the forecasting one, as presented in Chaps 5–9. This link is the balance equations which have always and everywhere to be satisfied. They act as (equality) constraints in the management problem (obviously in addition to any other constraints of the problem, such as economic and technical ones). Constraints may also be imposed on water levels, gradients, subsidence and solute concentration at selected points and times, spring discharge, etc.

In certain management procedures, we speak of the transition which a given system undergoes from one state to the next as a result of executing operational decisions. We may then visualize the balance equations as state transition equations.

In this and the following paragraph, for the sake of simplicity, we shall deal with flow in confined aquifers and/or phreatic ones with relatively small water level fluctuations. The fundamental equations governing the flow of groundwater in these aquifers are presented in Sec. 5-4. These are linear equations, in which the dependent variable, whether $\phi(x, y, t)$ or $h(x, y, t)$, varies in space and time in a continuous fashion. In Sec. 5-6 the numerical approximations of these equations are presented. Discrete aquifer models are presented in Chap. 10.

It is rather difficult to use a partial differential equation as a constraint (or as an equation of state transition), since this would require the solution of a partial differential equation each time we wish to check whether a constraint is violated (or to determine the new state of the system). It is much easier to make use of the linear algebraic equations which approximate the partial differential equations in the numerical methods of solution. As we shall see below, in many cases it is more convenient to use solutions of the equation in conjunction with superposition (Sec. 5-7), which is permitted as we deal with linear flow problems (or approximated as such), instead of the differential equations themselves, or their numerical representation. This approach will be discussed in details below.

To simplify the discussion, we shall consider henceforth pumping as the only controlled activity (= decision variable), regarding artificial recharge as negative pumping.

Use of Technological Functions

Let us demonstrate this method by applying it to a multicell model of an aquifer (Schwarz, 1971; Maddock, 1972). The following notation will be used: h_k^t is the water level or piezometric head at point (or aquifer cell) k and time t resulting from all activities, both controlled and uncontrolled; $k = 1, 2, \ldots, N$. h_{k0}^t is the water level at point (or aquifer cell) k and time t resulting only from the uncontrolled factors (including pumping which is beyond the control of the planner). P_j^i is the controlled pumping rate at a well (or cell) located at point j (position vector x_j), $j = 1, 2, \ldots, M$; duration of pumping is Δt from time $(i-1)\Delta t$ to time $i\,\Delta t$ (see Fig. 8-15c, with Δt = constant, independent of i). Artificial recharge is visualized as negative pumping. $a_{k,j}^{n-i,\Delta t}$ is the net change of water level (= residual drawdown) produced at point (or cell) k by pumping at a rate of one unit of discharge from a well (or cell) j; duration of pumping is Δt, starting at time $(i-1)\Delta t$; the drawdown is observed at time $t = n\,\Delta t$; $n - i \geq 0$.

The drawdown $a_{k,j}^{n-1,\Delta t}$ is obtained for the considered aquifer with homogeneous initial and boundary conditions; that is, $h = 0$ everywhere at $t = 0$, and $h = 0$ at $t \geq 0$ on the aquifer's boundaries.

By employing the principle of superposition (Sec. 5-7, especially Ex. 4), we obtain the combined drawndown in cell k at time $t = n\,\Delta t$ produced by M wells (or cells) in the aquifer, each with its pumping schedule P_j^i

$$s(k, t) = h_{k0}^t - \sum_{i=1}^{n} \sum_{j=1}^{M} P_j^i a_{k,j}^{n-i,\Delta t} \tag{12-10}$$

The factor $a_{k,j}^{n-i,\Delta t}$ is called the *influence function* (or *coefficient*). It is also called the *technological function* (e.g., Maddock, 1972). In (12-10) we have summed for all M wells (or cells) and n time intervals (see Fig. 8-15c). If $k = 1, 2, \ldots, N$ (that is, N points at which changes in water level are measured) and M wells are operating in the considered aquifer, the array of $N \times M$ coefficients form the influence matrix (or technological matrix) $\boldsymbol{a}^{n-1,\Delta t}$. For the selected Δt, an $N \times M$ matrix has to be determined for different times $(n - i)$. Δt.

Let h_{k0}^s denote the steady state water levels in an aquifer with steady boundary conditions and natural replenishment. Pumping takes place at points (or cells) j at constant rates P_j^s and a new steady state is established at points (or cells) k, with the new water levels denoted by h_k^s. Water is drawn only from boundaries or natural replenishment, but not from storage within the aquifer. We then have

$$h_k^s = h_{k0}^s - \sum_{j=1}^{M} P_j^s a_{k,j}^s \tag{12-11}$$

where $a_{k,j}^s$ is the steady state influence (or technological) function. The values of h_{k0}^s and h_{k0}^t already include the effects of boundaries, natural replenishment, and all uncontrollable pumping or recharge activities.

We have to emphasize again that we deal here with situations in which superposition is permitted because of the linearity of the system and the homogeneity of the boundary conditions. If, for example, during the operations, boundary conditions vary, the technological function also varies and superposition as employed above does not describe the resulting water levels.

Thus, instead of using the balance equations as constraints, we use their solutions in the forms of the technological functions, $a^s_{k,i}$ and $a^{n-i,\Delta t}_{k,j}$, which give drawdowns produced by pumping. Since these functions are related to unit pumping rates, they are independent of the decision variables P^s_j or P^i_j.

If minimal or maximal water levels, which may be time dependent, are now introduced as constraints of the problem, e.g.

$$h_{k,\min} \le h^i_k \le h_{k,\max} \quad \text{or} \quad h^i_{k,\min} \le h^i_k \le h^i_{k,\max}$$
$$h^s_{k,\min} \le h^s_k \le h^s_{k,\max} \tag{12-12}$$

we may use (12-10) and (12-11) to express them in the forms

$$\sum_{i=1}^{n} \sum_{j=1}^{M} P^i_j a^{n-i,\Delta t}_{k,j} \le b^i_k; \qquad b^i_k = h^i_{k0} - h^i_{k,\min}; \qquad k = 1, 2, \ldots, N \tag{12-13}$$

$$\sum_{j=1}^{M} P^s_j a^s_{k,j} \le b^s_k; \qquad b^s_k = h^s_{k0} - h^s_{k,\min}; \qquad k = 1, 2, \ldots, N \tag{12-14}$$

and similar expressions for maximal water levels. The b_k's represent permissible drawdowns at points k resulting from implementing the planned activities; we may think of water levels as the limited resources at our disposal.

In (12-13), or in (12-14), we have N linear constraints, one for each critical point at which a minimal water level is specified during the planned operation of the groundwater system. The constraints (12-13) and (12-14) are written in terms of the known technological functions and the set of activities, P^s_j, or P^i_j, which constitute the decision variables of the problem. This is the standard form of the constraints in (12-2). Obviously, all P's are positive (or can easily be made so in the case of artificial recharge).

The determination of the limiting values $h_{k,\min}$ and $h_{k,\max}$ is affected by various considerations. Among them, we may mention the following.

(*a*) Water levels cannot drop to below the physical bottom of an aquifer.
(*b*) When artificial recharge is implemented, it is undesirable to let water levels rise to the ground surface (and cause inundation) or above some maximum depth below ground surface (to prevent damage to foundations).
(*c*) Rising water levels may cause subsidence due to compaction by wetting of certain unconsolidated, moisture deficient soils (e.g., loess and other eolian deposits).
(*d*) Subsidence may be caused by dewatering certain formations. In general, subsidence is a function of the drop in water levels.
(*e*) Changes in water levels (and gradients) may produce an undesirable flow pattern (e.g., movement of water bodies of inferior quality towards pumping wells).
(*f*) Water levels may not drop below the elevations of the screens of pumping wells. Pump characteristics may also limit drawdown.
(*g*) Considerations of energy (availability and/or cost) required to lift the water to the ground surface.

Obviously, when values of $h^i_{k,\min}$ and $h^i_{k,\max}$ are not determined by natural conditions they may be considered as decision variables to be determined as part of the optimization anlysis.

In addition to legal, hydrologic, physical, or economic constraints imposed on water levels we have those imposed on the decision variables P^i_j (e.g., due to size of available pumping installations).

Sometimes constraints are imposed on the hydraulic gradients, in order to control the flow. We may then express the gradients in terms of water levels at points (or cells). For example

$$L_{mk}\, J_{mk,\min} \geq h^i_m - h^i_k \geq L_{mk}\, J_{mk,\max} \tag{12-15}$$

where m and k denote a pair of adjacent points (or cells) in the aquifer, with a distance L_{mk} and gradient J_{mk} between them.

The above constraints may be imposed at all points in a considered aquifer, or only at certain critical points.

We have shown above how, rather than introduce the balance equations as constraints (in addition to minimum and maximum water levels), we have introduced their solution through the technological functions. We have still to discuss how these functions can be determined. Let us review several procedures for achieving this goal.

(a) Procedures based on well drawdown equations These equations were developed in Chap. 8 for aquifers for various types. For example, for steady flows in a confined aquifer, we have from (8-6)

$$a^s_{k,j} \equiv s^s_{k,j}/Q_j = \frac{1}{2\pi T}\ln(R/r_{kj}), \qquad r_{kj} \leq R \tag{12-16}$$

where R is the radius of influence of the well.

For unsteady flow in an infinite confined aquifer, we have from (8-68b)

$$\begin{aligned} a^{n-i,\Delta t}_{k,j} = s^{n-i,\Delta t}_{k,j}/Q^i_j &= \frac{1}{4\pi T}\left[W\left\{\frac{Sr^2_{kj}}{4T(n-i+1)\,\Delta t}\right\} - W\left\{\frac{Sr^2_{kj}}{4T(n-i)\,\Delta t}\right\}\right] \\ &\equiv \beta(k, j, n-i) \end{aligned} \tag{12-17}$$

where $\beta(k, j, n-1)$ is the value of $\beta(n-i)$ as defined by (8-68b) corresponding to the distance $r_{kj} = \left[(x_k - x_j)^2 + (y_k - y_j)^2\right]^{1/2}$ from the well at $\mathbf{x}_j$ to the point $\mathbf{x}_k$, for $k \neq j$ and $r_{kj} \equiv (r_w)_j$ for $k = j$; Q^i_j is the pumping at well j during Δt between $(i-1)\,\Delta t$ and $i\,\Delta t$, $s^{n-i,\Delta t}_{k,j}$ is the drawdown at point k produced by Q^i_j.

When Q^i_j ($\equiv P^i_j$ in (12-10)) varies in a stepwise fashion, say, as represented by Fig. 8-15c with Δt = constant, the resultant drawdown at point k at time $t = n\,\Delta t$ is obtained from (8-68b)

$$s(\mathbf{x}_k, t) \equiv s^n_k \equiv \sum_{i=1}^{n}\sum_{j=1}^{M} Q^i_j\,\beta(k, j, n-i) \equiv a^{n-i,\Delta t}_{k,j} \tag{12-18}$$

The conditions under which (12-16) and 12-17) are applicable are given in

Secs 8-2 and 8-5, respectively. In a similar way, the technological function can be obtained for other types of aquifers (e.g., leaky), by introducing the appropriate well function in the definition of β.

In most cases, however, we are interested in a bounded aquifer which is also inhomogeneous. If the region influenced by the pumping is homogeneous, isotropic, and sufficiently removed from the boundaries, we may still use (12-17) as a good approximation. Sometimes the geometry of the boundaries is such that we can modify (12-17) using the method of images as discussed in Sec. 8-10. If all this is not possible, we have to derive the technological function by solving the appropriate partial differential equation.

(b) Procedures based on the solution of the partial differential equation The technological function, for cases of practical interest, may be obtained by solving the forecasting problem for the considered aquifer with its boundaries by any of the methods discussed in Sec. 5-6. The more common method nowadays is a numerical solution, using a digital computer, which can be applied to inhomogeneous and anisotropic aquifers having boundaries of any shape. With homogeneous boundary and initial conditions, we introduce pumping during Δt at one unit rate of discharge at a point j and determine the resulting response (= drawdowns) at all points k of interest at a times $t = (n - i)\,\Delta t$ and obtain $a_{k,j}^{n-i,\Delta t}$. The procedure is repeated for all points and times of interest.

The technological function $a_{k,j}^{n-i,\Delta t}$ can also be obtained by solving the forecasting problem using an RC-network analog (Bear, 1972, Chap. 11) instead of a digital computer.

Maddock (1974) derived nonlinear technological functions for aquifers whose transmissivities vary with drawdown.

The procedure described above is also applicable to the case where we use a multicell model to represent the aquifer. The mathematical statement is similar in this case to that of the finite difference technique.

We may now return to the linear programming problem of determining the optimal utilization of an aquifer. A typical problem may be stated in the following way.

Given an aquifer with a known influence matrix ($\boldsymbol{a}^{n-1,\Delta t}$ or $\boldsymbol{a}^s$), determine the pumping (and/or artificial recharge) rates (P_j^i, or P_j^s) at M wells (or M aquifer cells) so as: (1) to maximize the net benefits (or minimize costs), (2) not to produce at N critical points drawdowns (and/or water level rises) in excess of certain permissible values, (3) meet specified demands, or supply at least a specified demand, if the latter is given, and (4) do not exceed the capacities of pumping (and/or recharge) installations, if these already exist.

Other objective functions and additional constraints may also be used. Following are several examples of possible additional constraints.

(a) Seasonally varied demand In a quasi-steady state problem, the annual pumpage P_j^s may be made up of seasonal demands P_j^m for the mth season. The additional constraint is then $\sum_{(m)} P_j^m \geq P_j^s$ for all wells $j = 1, 2, \ldots, M$.

(b) Maximum installed pumping (or artificial recharge) capacities If MP_j^m denotes the maximum pumping capacity available in the mth season, then, $P_j^m \leq MP_j^m$ for all seasons and all wells.

(c) Meeting demand in a cell from different sources The demand in a typical cell, k, can be supplied by water pumped within the same cell, by water pumped in another cell and transported to the considered cell, or by water (both groundwater and surface water) imported into the area from a source external to the considered aquifer.

(d) Staged development Planned pumping rates and/or installed capacities may vary with the stage of development of a regional water resource system.

Coupling the Management and the Forecasting Problems

By using the technological (or influence) functions, we separate the forecasting problem from the management one. First, the forecasting problem is solved a number of times, yielding, as results, the technological functions. These, in turn, are used in the constraints of the management problem. However, the two problems may be solved simultaneously as a single management problem, with the balance equations serving as constraints. This approach is especially advantageous when the balance equation takes the form of a set of linear algebraic equations such as (10-7), representing simple cell balances in a multicell model of an aquifer, or any of the numerical representation schemes (finite difference or finite element) of the partial differential equation as discussed in Sec. 5-6. A general matrix form of the algebraic equations for solving the continuity equation, say (5-58), is

$$[A]\{h\} + [B]\left\{\frac{\partial h}{\partial t}\right\} = \{N\} \tag{12-19}$$

where $[A]$ is a conductance (or stiffness) matrix depending on aquifer transmissivities (and leakage factors, if leakage exists) and element configuration, $[B]$ is a capacitance matrix depending on aquifer storativity and element configuration, $\{h\}$ is a vector made up of piezometric heads at the nodes, and $\{N\}$ is a vector representing the discharge rates at the nodes. If h is taken as the average piezometric head at time t and $t - \Delta t$, (12-19) becomes

$$[C]\{h^t\} = [D]\{h^{t-\Delta t}\} + 2\{N\} \tag{12-20}$$

where: $[C] = [A] + (2/\Delta t)[B]$, $[D] = -[A] + (2/\Delta t)[B]$. It is also possible to use a certain weight $\frac{1}{2} \leq \theta \leq 1$ between h^t and $h^{t-\Delta t}$ (see Sec. 5-6). Our interest here is only in the end result, namely, that the continuity equation can be expressed as a set of linear algebraic equations which, in turn, can serve as linear equality constraints in the management problem.

If the drawdowns are limited by some minimum water levels and we add the constraints $h_k^t \geq h_{k,\min}$, it is sometimes convenient to introduce a new variable $h_k'^t = h^t - h_{k,\min}$ (rather than the water level h_k^t) as decision variable. Each constraint is then expressed as $h_k'' \geq 0$, which is an intrinsic nonnegativity constraint of the linear programming model. Substituting h_k'' for h_k' in the algebraic equations results in constant terms which are transferred to the right-hand side of the equations.

12-3 EXAMPLES

Let us consider several examples of applying the linear programming technique to the management problem.

Example 1: Pumping in a two-cell aquifer This example serves as a demonstration of a graphical method for solving LP problems when only two decision variables are involved. All other problems have to be solved numerically, usually by computers.

Consider a rectangular aquifer represented by a two-cell model (Fig. 12-1). Steady state conditions prevail.

Given the geometrical dimension $(2L, W)$ and area $(A = 2LW)$ of the aquifer, its transmissivity (T) and the rate of its natural replenishment (N; dims. L/T), it is required to determine the pumping rates (dims. L^3/T), P_1 in cell 1 and P_2 in cell 2, so as to supply a specified total demand D at the lowest possible total cost. The pumping rates should be such that water levels in the cells do not drop below certain specified minimum levels, $h_{1,\min}$ and $h_{2,\min}$ in cells 1 and 2, respectively.

The costs of pumping and supply per unit of P are C_1 and C_2 for water pumped from cells 1 and 2, respectively. In the discussion here we do not specify any particular unit for T, P, N, L, W, C. and h. However, one should be careful to employ a consistent system of units. For example: P in cubic meters per year, N in meters per year, T in square meters per year, W, L, and h in meters, and C in monetary units per cubic meter per year.

The undisturbed water levels are h_{10} and h_{20} in cells 1 and 2, respectively. The drawdown in cell k produced by pumping one unit in cell j is denoted by a_{kj}.

The influence matrix $\boldsymbol{a}$ (elements a_{kj}) is obtained by solving the balance equations for the two cells for unit rates of pumping.

(i) *Cell 1*, $P_1 = 0, P_2 = 0$

$$NA + (h_{20} - h_{10})\frac{WT}{L} - (h_{10} - 0)\frac{WT}{L/2} = 0 \tag{12-21}$$

(ii) *Cell 2*, $P_1 = 0, P_2 = 0$

$$NA - (h_{20} - h_{10})\frac{WT}{L} = 0 \tag{12-22}$$

(iii) *Cell 1*, $P_1 = 1, P_2 = 0$

$$NA - 1 + [(h_{20} - a_{21}) - (h_{10} - a_{11})]\frac{WT}{L} - (h_{10} - a_{11})\frac{WT}{L/2} = 0 \tag{12-23}$$

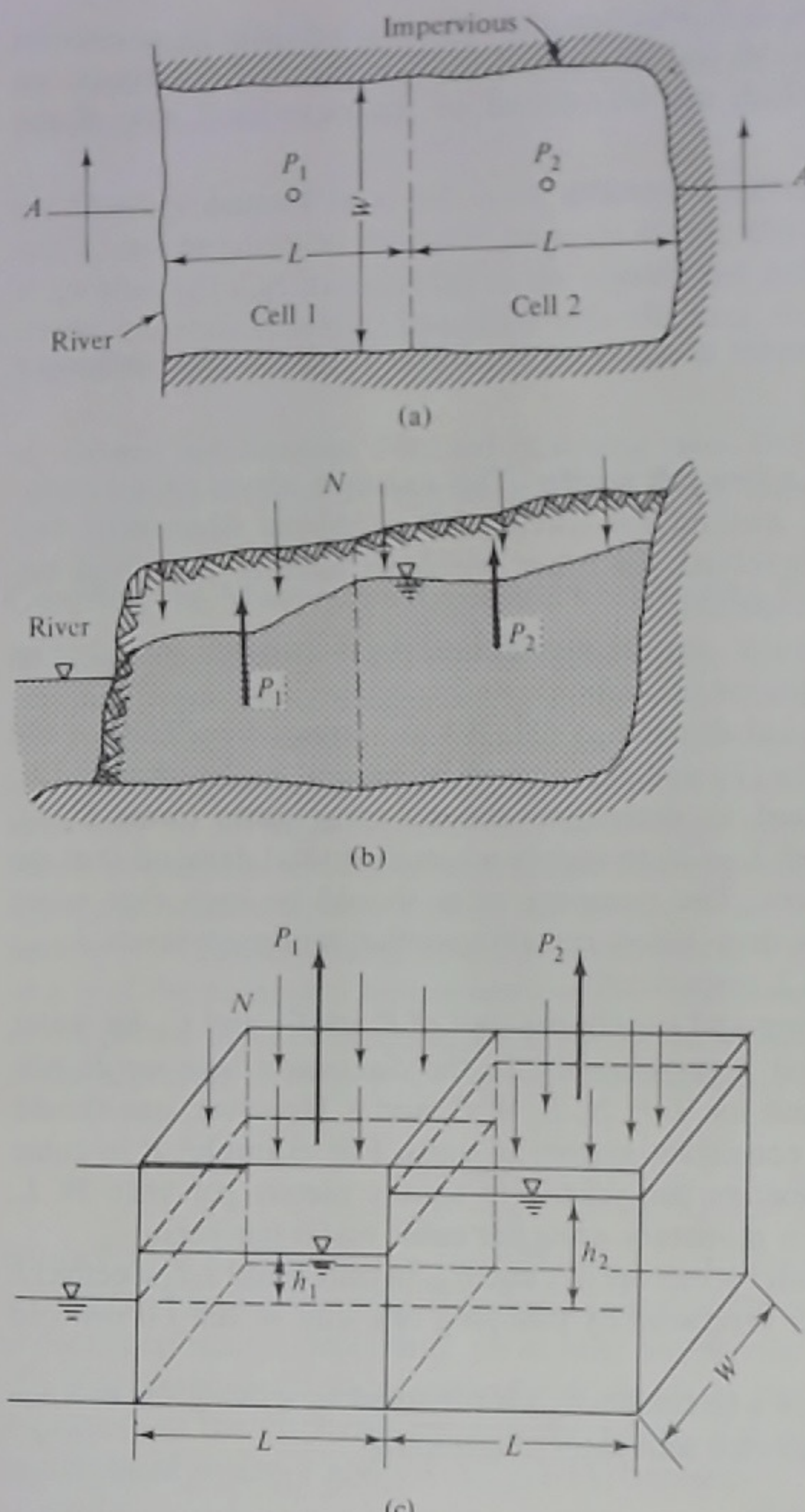

Figure 12-1 Management of an aquifer represented as a two-cell model. (a) Map. (b) Cross section A–A. (c) Two-cell aquifer model.

(iv) *Cell 2*, $P_1 = 1,\ P_2 = 0$

$$NA + [(h_{20} - a_{21}) - (h_{10} - a_{11})]\frac{WT}{L} = 0 \tag{12-24}$$

(v) *Cell 1*, $P_1 = 0,\ P_2 = 1$

$$NA + [(h_{20} - a_{22}) - (h_{10} - a_{11})]\frac{WT}{L} - (h_{10} - a_{11})\frac{WT}{L/2} = 0 \tag{12-25}$$

(vi) *Cell 2*, $P_1 = 0$, $P_2 = 1$

$$NA - 1 - [(h_{20} - a_{22}) - (h_{10} - a_{12})]\frac{WT}{L} = 0 \qquad (12\text{-}26)$$

The first two equations yield the steady, undisturbed water levels

$$h_{10} = NL^2 T \qquad h_{20} = 2NL^2/T \qquad (12\text{-}27)$$

By solving (12-23) through (12-26), we obtain

$$a_{11} = a_{12} = a_{21} = L/2WT; \qquad a_{22} = 3L/2WT \qquad (12\text{-}28)$$

The water levels with pumping taking place in the two cells are therefore

$$\left.\begin{aligned} h_1 &= h_{10} - (P_1 a_{11} + P_2 a_{12}) \\ h_2 &= h_{20} - (P_1 a_{21} + P_2 a_{22}) \end{aligned}\right\} \qquad (12\text{-}29)$$

The objective function is the total cost of supply

$$z = C_1 P_1 + C_2 P_2 \qquad (12\text{-}30)$$

Here the costs are independent of the water levels, h_1 and h_2, but could be made a function of them if necessary. In the latter case $C_2 = C_2(h)$, as cost of pumping depends on the depth to water table and the problem is nonlinear. It can be linearized by assuming some average depth $\bar{h}$, neglecting the effect of changes in h due to drawdown.

We can now state the complete LP problem: Determine P_1 and P_2 such that

$$\text{Minimize } z = C_1 P_1 + C_2 P_2 \qquad (12\text{-}31)$$

subject to the constraints:

$$P_1 + P_2 \geq D \qquad (12\text{-}32)$$

$$P_1 a_{11} + P_2 a_{12} \leq h_{10} - h_{1,\min} (\equiv b_1) \qquad (12\text{-}33)$$

$$P_1 a_{21} + P_2 a_{22} \leq h_{20} - h_{2,\min} (\equiv b_2) \qquad (12\text{-}34)$$

$$P_1 \geq 0, \qquad P_2 \geq 0 \qquad (12\text{-}35)$$

Equations (12-32) through (12-35) define the domain of feasible solutions (Fig. 12-2). For $C_1 < C_2$, we draw the lines z = const. and seek the line of smallest z which still intersects the domain of feasible solution, at least at one point. We find this point to be B. This means that the optimal solution of our problem in this case is $P_1^* = D$ (that is, to pump the entire demand in cell 1), $P_2^* = 0$.

For $C_1 > C_2$ the solution is given by point A in Fig. 12-2, with optimal values P_1^* and P_2^*.

Note that Fig. 12-2 is drawn for the case $b_1/a_{12} \leq D$ and $b_1/a_{21} \leq D$.

For example, for $D = 45$ Mm^3/yr (1 Mm^3 = 10^6 m^3), $C_1 = 0.2$ MU/m^3/yr (MU ≡ monetary unit), $C_2 = 0.1$ MU/m^3/yr, $W = 10$ km, $L = 10$ km, $T = 10{,}000$ m^2/d, $N = 360$ mm/yr, $h_{1,\min} = 2.5$ m, $h_{2,\min} = 5$ m, we obtain: $a_{11} = 0.189$ m/Mm^3/yr $= a_{12} = a_{21}$, $a_{22} = 0.417$ m/Mm^3/yr, $h_{10} = 10.0$ m,

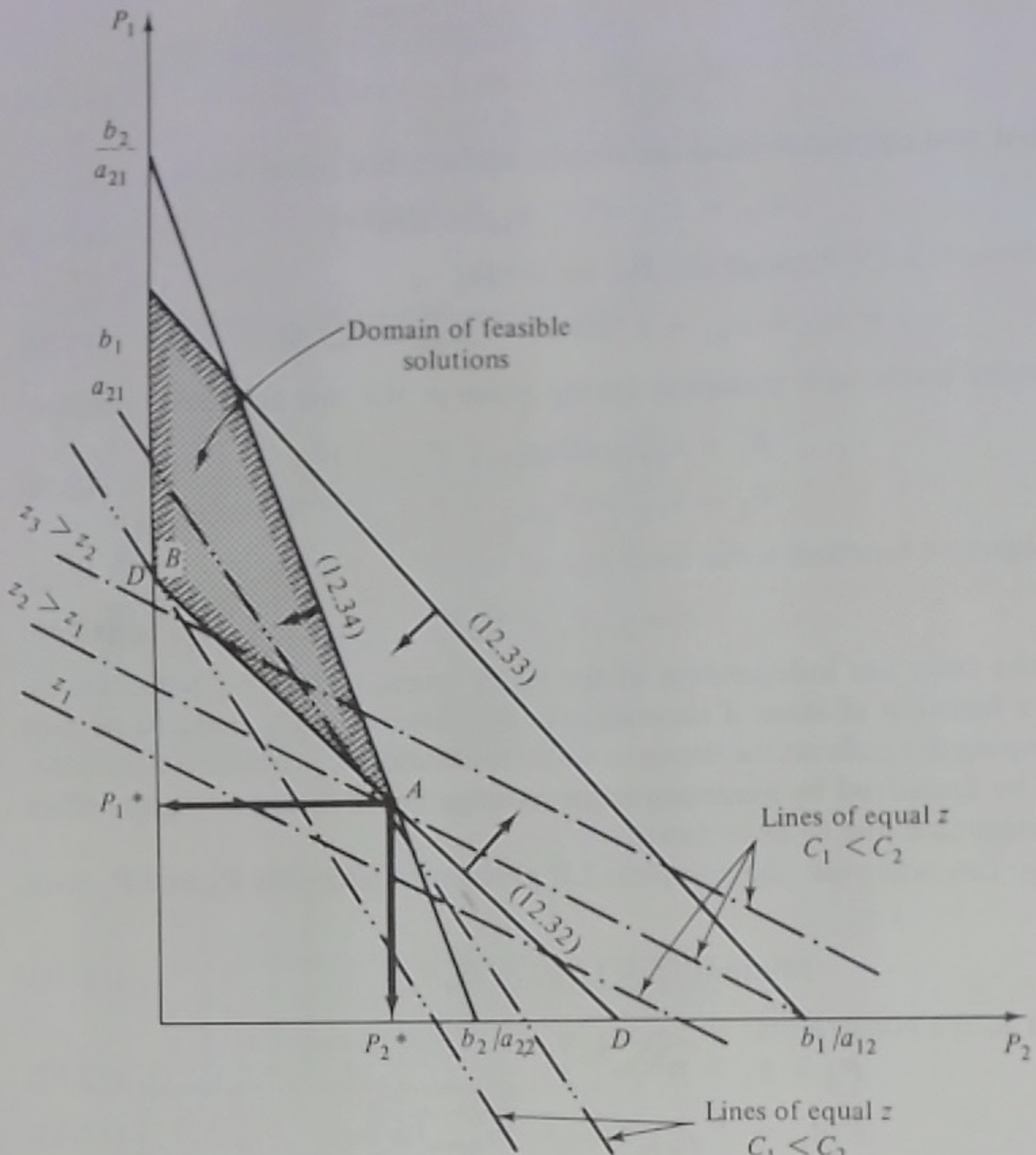

Figure 12-2 Graphical solution of management problem for a two-cell aquifer model (Example 1).

$h_{20} = 20.0$ m, $b_1 = 7.5$ m, $b_2 = 15$ m, $P_1^* = 12.5$ Mm^3/yr, $P_2^* = 32.5$ Mm^3/yr, $z^* = 5.75 \times 10^6$ MU/yr. It may be of interest to note that in this case, the constraint (12-33) does not affect the result.

Since only two decision variables are involved, we could obtain the solution graphically. When a larger number of variables is involved, the *Simplex method* of solution is employed, using a digital computer (see any text on Linear Programming).

Instead of determining the influence matrix $\boldsymbol{a}$ prior to solving the LP problem, let us combine the two problems. Assuming, as is usually the case, that h_{10} and h_{20} are known, the balance equations to be satisfied are

$$NA - P_1 + (h_2 - h_1)\frac{WT}{L} - (h_1 - h_0)\frac{WT}{L/2} = 0$$
$$NA - P_2 - (h_2 - h_1)\frac{WT}{L} = 0 \qquad (12\text{-}36)$$

or

$$P_1 + \frac{3WT}{L}h_1 - \frac{WT}{L}h_2 = NA + h_0\frac{WT}{L/2}$$
$$P_2 - \frac{WT}{L}h_1 + \frac{WT}{L}h_2 = NA \qquad (12\text{-}37)$$

which serve as equality constraints of the management problem. These equations are actually identical to (12-23) and (12-26) written for P_1 instead of $P_1 = 1$. We can always select a reference level such that

$$h_1 \geq 0, \qquad h_2 \geq 0 \qquad (12\text{-}38)$$

In addition, we have to satisfy

$$h_1 \geq h_{1,\min}, \qquad h_2 \geq h_{2,\min} \qquad (12\text{-}39)$$

The constraints (12-32) and (12-35) remain unchanged. We may thus visualize the problem as having four decision variables P_1, P_2, h_1, h_2, where h_1 and h_2 have to satisfy (12-37). The LP problem now consists of minimizing the objective function, subject to the constraints (12-32), (12-35), and (12-37) through (12-39).

It is also possible to have an objective function affected also by the values of h, for example, when we take into account the cost of lifting the water. We may thus have in general

$$z = C_1P_1 + C_2P_2 + C_3h_1 + C_4h_2 \qquad (12\text{-}40)$$

In (12-31), $C_3 = C_4 = 0$. Note that if C_3 and C_4 are functions of h_3 and h_4, respectively, the problem becomes nonlinear.

Example 2: Allocation of pumping in a 25-cell aquifer (Schwarz, 1971) The investigated aquifer has the shape of a square 10 km × 10 km (Fig. 12-3). On three sides the aquifer's boundaries are impervious. Its fourth side is a lake into which the aquifer is draining. However, because of the poor quality of the water in the lake, it is important to keep certain minimum water levels in the aquifer along the lake so as to prevent lake water from encroaching into the aquifer. Let these water levels be +0.64 m (above lake level used as datum level) and +0.95 m at distances of 1 km and 3 km from the lake, respectively.

The aquifer is replenished from precipitation at a rate $N = 100$ mm/yr, uniformly distributed over the entire area. Under natural conditions the entire replenishment is drained through the aquifer to the lake.

The aquifer is homogeneous with transmissivity $T = 1000$ m^2/d. Since we assume steady flow, the knowledge of storativity is not required.

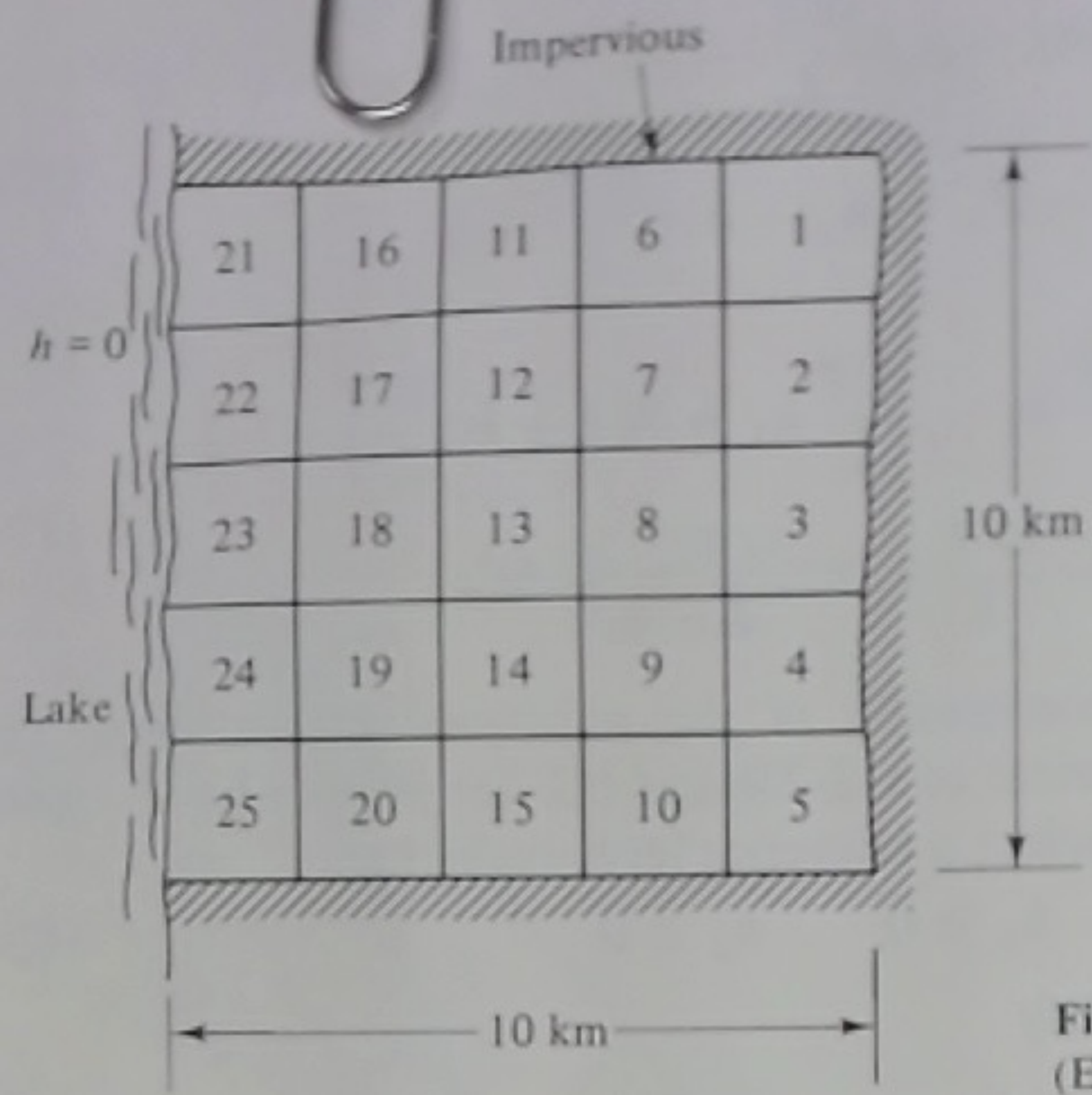

Figure 12-3 Layout of a 25-cell aquifer (Example 2).

After analyzing the areal distribution of pumping, the aquifer is divided into 25 cells of size 2 km × 2 km. Each such cell is considered as a single point at which pumping takes place. Pumping is planned in 15 cells (6 through 20) only.

A digital computer is used to derive the components of the steady state influence matrix $\boldsymbol{a}^s$ for all the 15 cells in which pumping is planned ($j = 15, 16, \ldots, 25$, assuming pumping to be concentrated at the center of each cell). The components of $\boldsymbol{a}^s$ are obtained by solving (10-7) written for all 25 cells. However, since we are interested in the water levels in only 10 critical cells ($k = 16, 17, \ldots, 25$), the matrix $\boldsymbol{a}^s$ in this case consists only of 10 rows and columns. The components $a^s_{k,j}$, $k = 16, 17, \ldots, 25$, $j = 6, 7, \ldots, 20$, are given in Table 12-1.

Initial values h_{i0} (with no pumping anywhere) are computed to be

$$h_{i0} = 13.88 \text{ m}, \qquad i = 1, \ldots, 5$$
$$h_{i0} = 12.77 \text{ m}, \qquad i = 6, \ldots, 10$$
$$h_{i0} = 10.55 \text{ m}, \qquad i = 11, \ldots, 15$$
$$h_{i0} = 7.22 \text{ m}, \qquad i = 16, \ldots, 20$$
$$h_{i0} = 2.79 \text{ m}, \qquad i = 21, \ldots, 25$$

In this example we have only a single consumer (or center of consumption) in the region, located in cell 18. The total demand of this consumer is $D = 7.0 \text{ Mm}^3/\text{yr}$. The cost of pumping and conveyance of water from cell 18 to this consumer is 1.0 MU/m^3 (MU = monetary unit); the cost increases with distance from the pumping area to this consumer at a rate of $0.5 \text{ MU/m}^3/1000$ m distance.

Table 12-1 contains all this information, including the permissible draw-

Table 12-1 LP model for steady state allocation of pumping in a 25-cell aquifer (Example 2)

Activity															Constraint		
P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	Type	Level	Designation
1.10	0.98	0.82	0.72	0.68	1.32	1.00	0.78	0.60	0.55	1.88	1.00	0.60	0.42	0.38	≤	6.27	Level-16
0.97	0.92	0.85	0.78	0.72	1.01	1.10	0.88	0.70	0.62	0.98	1.52	0.82	0.53	0.43	≤	6.27	Level-17
0.82	0.87	0.95	0.87	0.81	0.78	0.93	1.01	0.93	0.78	0.60	0.81	1.43	0.81	0.60	≤	6.27	Level-18
0.72	0.78	0.85	0.92	0.97	0.62	0.70	0.88	1.10	1.01	0.43	0.53	0.82	1.52	0.98	≤	6.27	Level-19
0.68	0.72	0.82	0.98	1.10	0.55	0.60	0.78	1.00	1.32	0.38	0.42	0.60	1.00	1.88	≤	6.27	Level-20
0.37	0.32	0.28	0.24	0.21	0.42	0.35	0.27	0.21	0.20	0.60	0.47	0.22	0.18	0.12	≤	2.15	Level-21
0.32	0.30	0.29	0.27	0.25	0.35	0.35	0.30	0.23	0.21	0.37	0.45	0.30	0.20	0.15	≤	2.15	Level-22
0.28	0.30	0.30	0.30	0.28	0.25	0.30	0.31	0.30	0.25	0.21	0.29	0.41	0.29	0.21	≤	2.15	Level-23
0.25	0.27	0.29	0.30	0.32	0.21	0.23	0.30	0.35	0.35	0.15	0.20	0.30	0.45	0.37	≤	2.15	Level-24
0.21	0.24	0.28	0.32	0.37	0.20	0.21	0.27	0.35	0.42	0.12	0.18	0.22	0.47	0.60	≤	2.15	Level-25
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	=	7.0	Demand
3.9	3.2	3.0	3.2	3.9	3.2	2.4	2.0	2.4	3.2	3.0	2.0	1.0	2.0	3.0	Free	Min.	Cost

down in cells 16 through 25, the components of the influence matrix, the total to be supplied (row before last), and the costs associated with each P_j (last row).

The first row gives the vector of activities, or decision variables (here the P_j's). Then come the water level constraints and the demand constraint. The last row specifies the costs associated with the P_j's. It is very convenient to summarize the information in such a table.

The objective function is the total costs associated with supplying the specified demand

$$z = C_6 P_6 + C_7 P_7 + \cdots + C_{20} P_{20}; \qquad P_j\text{'s in Mm}^3\text{/yr} \tag{12-41}$$

It is required to determine the values of P_j, $j = 6, 7, \ldots, 20$, which will supply the required demand at a minimum cost. The constraints are

$$\left.\begin{aligned} P_6 + P_7 + P_8 + \cdots + P_{20} &= 7.0 \\ \begin{cases} 1.10\,P_6 + 0.98\,P_7 + 0.82\,P_8 + \cdots + 0.38\,P_{20} \le 6.27 \\ \vdots \\ 0.21\,P_6 + 0.24\,P_7 + 0.28\,P_8 + \cdots + 0.60\,P_{20} \le 2.15 \end{cases} \\ P_j \ge 0, \qquad j = 6, 7, \ldots, 20 \end{aligned}\right\} \tag{12-42}$$

Table 12-2 presents the solution obtained by using one of the available library programs for solving linear programming problems.

As in Example 1, we can use algebraic balance equations written for the cells as constraints, rather than expressing the latter in terms of components of the influence matrix which have to be determined separately.

Table 12-2 Results of Example 2

Cell No.	Optimal pumping (Mm^3/yr)	Drawdown (m)	Permissible drawdown (m)
13	1.06		
14	—		
15	—		
16	0.80	5.59	6.27
17	1.59	6.27	6.27
18	1.16	6.27	6.27
19	1.59	6.27	6.27
20	0.80	5.59	6.27
21	—	2.15	6.27
22	—	2.11	2.15
23	—	2.06	2.15
24	—	2.11	2.15
25	—	2.15	2.15
Total	7.00	—	—

No pumping in cells 6–12.

The total cost $z^* = 14.43 \times 10^6$ MU/yr.

Example 3: Staged development in a two-cell aquifer (Schwarz, 1971) This example is presented to illustrate the effect of time. The studied region is the two-cell aquifer described in Ex. 1 (Fig. 12-1). The water supply system is to be based on both groundwater pumped from within the two cells and on water imported from outside into the region. The project is to be built in four stages, at $t = 0$, 5, 10, and 15 years. The total available budget is limited and serves as a constraint. The investments at the various stages constitute the decision variables. The planning horizon of the project is 20 years.

Investments in the pumping installations are smaller than those required for the conveyance system for the imported water. Initially, the aquifer contains a relatively large volume of water in storage; this may serve as a temporary source of water if we wish to postpone investments in constructing the import installations.

In the last stage, pumping rates may not exceed the steady state (or long term average) yield of the aquifer. At the end of the planning horizon, the drawdowns in the aquifer may not exceed the specified (tolerable) maximum values of 7.5 m in cell 1 and 15.0 m in cell 2. These constraints on ultimate drawdowns will ensure a state of equilibrium at the end of the planning horizon, and prevent the tendency to mine water from the aquifer, leaving it empty for future generations. There is no limit, however, on temporary mining during the first three stages of the project.

The minimum demand which has to be supplied in each cell is 30 Mm^3/yr. Water may be supplied at a higher rate if this will produce an increased net benefit. However, a reduction in supply from one stage to the next is not permitted. Furthermore, in the final stage not more than 55 Mm^3/yr from groundwater and 45 Mm^3/yr of imported water may be supplied.

The interest rate on all costs and benefits is 8%.

With an existing installed pumping capacity of 5 Mm^3/yr in each cell, it is required to plan the project by determining: (1) rates of supply to each cell and during each stage from groundwater and from imported water, and (2) investment in pumping and conveyance installations (for supplying the imported water) at each stage. The objective is to maximize the present worth of the flow of net benefits to be derived from the operation of the project. All decisions are subject to the hydrologic, economic, and demand constraints listed above. Table 12-3 summarizes costs, benefits and present worth of net benefits.

Table 12-4 summarizes the LP model

Activities are divided into two main groups: Development (columns 1–16) and Operations (columns 17–32). Each of these is further subdivided into those related to pumping (columns 1–8, 17–24) and those related to import (columns 9–16, 25–32). Finally, each activity is considered separately for each cell.

Rows 1–2 give the constraints on groundwater levels in the cells at the end of stage 4 (i.e., at the end of 20 years). The elements of these two rows

Table 12-3 Costs, benefits and present worth of net benefits (MU/m^3) for Ex. 3

Cell	1				2			
Stage	1	2	3	4	1	2	3	4
Present worth of investments								
In pumping installations	20.00	13.62	9.26	6.30	20.00	13.62	9.26	6.30
In import installations†	10.00	6.81	4.63	3.15	40.00	27.24	18.52	12.60
Benefits and costs								
Annual benefits from water supplied	9.0	9.0	9.0	9.0	10.0	10.0	10.0	10.0
Annual costs of pumping	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0
Annual net benefits from pumping	7.0	7.0	7.0	7.0	8.0	8.0	8.0	8.0
Present worth of flow of net benefits from pumping	27.95	19.03	12.94	8.80	31.95	21.75	14.79	10.06
Annual costs of import†	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
Annual net benefits from import	8.6	8.6	8.6	9.6	9.6	9.6	9.6	9.6
Present worth of flow of net benefits from import	34.33	23.38	15.90	10.82	38.33	26.10	17.75	12.07

† Import to cell 1 is possible only through cell 2. The cost is only that required for transporting the water from cell 1 to cell 2.

are components of the influence matrices $\boldsymbol{a}^{5,20}$ (columns 17–18), $\boldsymbol{a}^{5,15}$ (columns 19–20), $\boldsymbol{a}^{5,10}$ (columns 21–22) and $\boldsymbol{a}^{5,5}$ (columns 23–24), which give the drawdowns (in cm/Mm^3/yr) due to 5 years of pumping, observed after 20, 15, 10, and 5 years from the beginning of pumping, respectively.

The constraints given by rows 3, 4 limit the pumping in the last stage to the long term sustained yield by limiting the drawdowns to the maximum steady state ones.

The constraints which limit the total import to the cells to the total available import of 45 Mm^3/yr are given by rows 5–8.

Rows 9–16 show the constraints on water demands. During the first stage, water supply may not drop below 30 Mm^3/yr to each cell. In each of the following stages the supply during each stage should not be reduced with respect to the preceding one.

The next two groups of constraints (rows 17–24 and 25–32) specify the availability of pumping and import installations in each stage and in each cell. Pumping and import in each cell and in each stage cannot exceed (i.e., should be equal to or smaller than) the capacity installed in that cell in the current stage and in the previous ones. The initial capacity of the pumping installations is 5.0 Mm^3/yr. The import capacity to cell 2 should be capable of handling also conveyance to cell 1.

Rows 33–34 specify the budget constraints. The total budget in all stages cannot exceed 3000 MU, while in the first stage the available budget amounts to 1500 MU only.

Table 12-4 Linear programming model for Ex. 3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
	Development																Operations																Constraints			
	Pumping								Import								Pumping								Import								Type	Level		
Stage	1		2		3		4		1		2		3		4		1		2		3		4		1		2		3		4					
Cell	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	J	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2				
1																	0.72	1.72	1.08	2.44	2.09	3.27	8.33	2.45									≤	7.50	20 yrs.	Groundwater levels
2																	1.72	4.17	2.44	5.96	3.27	8.62	2.45	13.24									≤	15.00		
3																							13.90	13.90									≤	7.50	Steady state	
4																							13.90	41.70									≤	15.00		
5																									1.0	1.0							≤	45.00	Import	
6																											1.0	1.0					≤	45.00		
7																													1.0	1.0			≤	45.00		
8																															1.0	1.0	≤	45.00		
9																	1.0								1.0								≥	30.00	Present	Water demands
10																		1.0								1.0							≥	30.00		
11																	-1.0		1.0						-1.0		1.0						≥	0	Future increases	
12																		-1.0		1.0						-1.0		1.0					≥	0		
13																			-1.0		1.0						-1.0		1.0				≥	0		
14																				-1.0		1.0						-1.0		1.0			≥	0		
15																					-1.0		1.0						-1.0		1.0		≥	0		
16																						-1.0		1.0						-1.0		1.0	≥	0		
17	-1.0																1.0																≤	5.00	Pumping	Available installations
18		-1.0																1.0															≤	5.00		
19	-1.0		-1.0																1.0														≤	5.00		
20		-1.0		-1.0																1.0													≤	5.00		
21	-1.0		-1.0		-1.0																1.0												≤	5.00		
22		-1.0		-1.0		-1.0																1.0											≤	5.00		
23	-1.0		-1.0		-1.0		-1.0																1.0										≤	5.00		
24		-1.0		-1.0		-1.0		-1.0																1.0									≤	5.00		
25									-1.0																1.0								≤	0	Import	
26										-1.0															1.0	1.0							≤	0		
27									-1.0		-1.0																1.0						≤	0		
28										-1.0		-1.0															1.0	1.0					≤	0		
29									-1.0		-1.0		-1.0																1.0				≤	0		
30										-1.0		-1.0		-1.0															1.0	1.0			≤	0		
31									-1.0		-1.0		-1.0		-1.0																1.0		≤	0		
32										-1.0		-1.0		-1.0		-1.0															1.0	1.0	≤	0		
33	20	20							10	10																								1500	Stage 1	Budget
34	20	20	20	20	20	20	20	20	10	10	10	40	10	40	10	40																		3000	Total	
35	-20.0		-13.6		-9.3		-6.3		-10.8		-6.8		-4.6		-3.1		27.95		19.03		12.94		8.80		34.33		23.38		15.90		10.82		Free	Max	Total net benefit	
		-20.0		-13.6		-9.3		-6.3		-40.0		-6.8		-4.6		-3.1		31.95		21.75		14.79		10.0		38.33		26.10		17.57		12.07				

Table 12-5 Results of Ex. 3

Cell	1				2			
Stage	1	2	3	4	1	2	3	4
Planned supply (Mm^3/yr)	30.0	30.0	30.0	30.0	43.5	69.0	69.0	69.0
Additional pumping installations (Mm^3/yr)	25	0	0	0	27	0	0	0
Additional import installations (Mm^3/yr)	0	0	0	0	11.5	25.5	0	8.0
Pumping (Mm^3/yr)	30.0	30.0	30.0	30.0	32.0	32.0	32.0	24.0
Import (Mm^3/yr)	0	0	0	0	11.5	37.0	37.0	45.0
Water level drawdown at end of stage (m)				6.78				13.0

Present worth of total net benefits: 4787 MU
Total investments: 2840 MU
Investments in stage 1: 1500 MU

The objective function is the present worth of all net benefits. The elements of this function are shown in row 35.

Table 12-5 summarizes the results obtained by using a standard library program for solving the linear programming problem. From this table it follows that the present demand in cell 1 of 30 Mm^3/yr should continue to be supplied from groundwater. The ultimate supply to cell 2 is 69 Mm^3/yr; part of it is supplied by temporary mining at a rate of 8 Mm^3/yr. This temporary overdraft is replaced by import in the last stage. The import to cell 2 is developed only partly in stage 1 due to the budget constraint.