5 Probability and Statistics

PRACTICE PROBLEMS

1. What is the approximate probability that no people in a group of seven have the same birthday?

(A) 0.056

(B) 0.43

(C) 0.92

(D) 0.94

2. A study gives the following results for a total sample size of 12.

3, 4, 4, 5, 8, 8, 8, 10, 11, 15, 18, 20

What is most nearly the mean?

(A) 8.9

(B) 9.5

(C) 11

(D) 12

3. A study gives the following results for a total sample size of 8.

2, 3, 5, 8, 8, 10, 10, 12

The mean of the sample is 7.25. What is most nearly the standard deviation?

(A) 2.5

(B) 2.9

(C) 3.3

(D) 3.7

4. A study gives the following results for a total sample size of 6.

10, 12, 13, 14, 14, 15

The mean of the sample is 13. What is most nearly the sample standard deviation?

(A) 0.85

(B) 0.90

(C) 1.6

(D) 1.8

5. A study has a sample size of 5, a standard deviation of 10.4, and a sample standard deviation of 11.6. What is most nearly the variance?

(A) 46

(B) 52

(C) 110

(D) 130

6. A study has a sample size of 9, a standard deviation of 4.0, and a sample standard deviation of 4.2. What is most nearly the sample variance?

(A) 16

(B) 18

(C) 34

(D) 36

7. A bag contains 100 balls numbered 1 to 100. One ball is drawn from the bag. What is the probability that the number on the ball selected will be odd or greater than 80?

(A) 0.1

(B) 0.5

(C) 0.6

(D) 0.7

8. Measurements of the water content of soil from a borrow site are normally distributed with a mean of 14.2% and a standard deviation of 2.3%. What is the probability that a sample taken from the site will have a water content above 16% or below 12%?

(A) 0.13

(B) 0.25

(C) 0.37

(D) 0.42

9. What is the probability that either exactly two heads or exactly three heads will be thrown if six fair coins are tossed at once?

(A) 0.35

(B) 0.55

(C) 0.59

(D) 0.63

10. Which of the following properties of probability is NOT valid?

(A) The probability of an event is always positive and within the range of zero and one.

(B) The probability of an event which cannot occur in the population being examined is zero.

(C) If events A and B are mutually exclusive, then the probability of either event occurring in the same population is zero.

(D) The probability of either of two events, A and B, occurring is $P(A+B) = P(A) + P(B) - P(A, B)$.

11. One fair die is used in a dice game. A player wins \$10 if he rolls either a 1 or a 6. He loses \$5 if he rolls any other number. What is the expected winning for one roll of the die?

(A) \$0.00

(B) \$3.30

(C) \$5.00

(D) \$6.70

12. A simulation model for a transportation system is run for 30 replications, and the mean percentage utilization of the transporter used by the system is recorded for each replication. Those 30 data points are then used to form a confidence interval on mean transporter utilization for the system. At a 95% confidence level, the confidence interval is found to be 37.2% ± 3.4%.

Given this information, which of the following facts can be definitively stated about the system?

(A) At 95% confidence, the sample mean of transporter utilization lies in the range 37.2% ± 3.4%.

(B) At 95% confidence, the population mean of transporter utilization lies in the range 37.2% ± 3.4%.

(C) At 95% confidence, the population mean of transporter utilization lies outside of the range of 37.2% ± 3.4%.

(D) At 5% confidence, the population mean of transporter utilization lies inside of the range of 37.2% ± 3.4%.

13. What is the approximate probability of exactly two people in a group of seven having a birthday on April 15?

(A) 1.2×10^{-18}

(B) 2.4×10^{-17}

(C) 7.4×10^{-6}

(D) 1.6×10^{-4}

14. What are the arithmetic mean and sample standard deviation of the following numbers?

$$71.3, 74.0, 74.25, 78.54, 80.6$$

(A) 74.3, 2.7

(B) 74.3, 3.7

(C) 75.0, 2.7

(D) 75.7, 3.8

15. Four fair coins are tossed at once. What is the probability of obtaining three heads and one tail?

(A) 1/4 (0.25)

(B) 3/8 (0.375)

(C) 1/2 (0.50)

(D) 3/4 (0.75)

16. Set A and set B are subsets of set Q. The values within each set are shown.

$$A = (4, 7, 9)$$
$$B = (4, 5, 9, 10)$$
$$Q = (4, 5, 6, 7, 8, 9, 10)$$

What is the union of the complement of set A and set B, $\overline{A} \cup B$?

(A) (4, 5, 6, 7, 8, 9, 10)

(B) (4, 5, 7, 9, 10)

(C) (4, 5, 6, 8, 9, 10)

(D) (5, 10)

17. Set A consists of elements (1, 3, 6), and set B consists of elements (1, 2, 6, 7). Both sets come from the universe of (1, 2, 3, 4, 5, 6, 7, 8). What is the intersection, $\overline{A} \cap B$?

(A) (2, 7)

(B) (2, 3, 7)

(C) (2, 4, 5, 7, 8)

(D) (4, 5, 8)

SOLUTIONS

1. This is the classic "birthday problem." The problem is to find the probability that all seven people have distinctly different birthdays. The solution can be found from simple counting.

The first person considered can be born on any day, which means the probability they will not be born on one of the 365 days of the year is 0.

$$P(1) = 1 - P(\text{not } 1) = 1 - 0 = 1 \quad (365/365)$$

The probability the second person will be born on the same day as the first person is 1 in 365. (The second person can be born on any other of the 364 days.) The probability that the second person is born on any other day is

$$P(2) = 1 - P(\text{not } 2) = 1 - \frac{1}{365} = \frac{364}{365}$$

The third person cannot have been born on either of the same days as the first and second people, which has a 2 in 365 probability of happening. The probability that the third person is born on any other day is

$$P(3) = 1 - P(\text{not } 3) = 1 - \frac{2}{365} = \frac{363}{365}$$

This logic continues to the seventh person. The probability that all seven conditions are simultaneously satisfied is

$$\begin{aligned} &P(7 \text{ distinct birthdays}) \\ &\quad = P(1) \times P(2) \times P(3) \times P(4) \times P(5) \\ &\qquad \times P(6) \times P(7) \\ &\quad = \left(\frac{365}{365}\right)\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\left(\frac{362}{365}\right)\left(\frac{361}{365}\right) \\ &\qquad \times \left(\frac{360}{365}\right)\left(\frac{359}{365}\right) \\ &\quad = 0.9438 \quad (0.94) \end{aligned}$$

The answer is (D).

2. The mean is

$$\begin{aligned} \overline{X} &= (1/n)\sum_{i=1}^{n} X_i \\ &= \left(\frac{1}{12}\right)\begin{pmatrix} 3+4+4+5 \\ \quad +8+8+8+10 \\ \quad +11+15+18+20 \end{pmatrix} \\ &= 9.5 \end{aligned}$$

The answer is (B).

3. The standard deviation is calculated using the sample mean as an unbiased estimator of the population mean.

$$\begin{aligned} \sigma &= \sqrt{(1/N)\sum(X_i - \mu)^2} \approx \sqrt{(1/N)\sum(X_i - \overline{X})^2} \\ &= \sqrt{\left(\frac{1}{8}\right)\begin{pmatrix} (2-7.25)^2 + (3-7.25)^2 \\ \quad + (5-7.25)^2 + (8-7.25)^2 \\ \quad + (8-7.25)^2 + (10-7.25)^2 \\ \quad + (10-7.25)^2 + (12-7.25)^2 \end{pmatrix}} \\ &= 3.34 \quad (3.3) \end{aligned}$$

The answer is (C).

4. The sample standard deviation is

$$\begin{aligned} s &= \sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i - \overline{X})^2} \\ &= \sqrt{\left(\frac{1}{6-1}\right)\begin{pmatrix} (10-13)^2 + (12-13)^2 \\ \quad + (13-13)^2 + (14-13)^2 \\ \quad + (14-13)^2 + (15-13)^2 \end{pmatrix}} \\ &= 1.79 \quad (1.8) \end{aligned}$$

The answer is (D).

5. The variance is

$$\sigma^2 = (10.4)^2 = 108 \quad (110)$$

The answer is (C).

6. The sample variance is

$$s^2 = (4.2)^2 = 17.64 \quad (18)$$

The answer is (B).

7. There are 50 odd-numbered balls. Including ball 100, there are 20 balls with numbers greater than 80.

$$P(A) = P(\text{ball is odd}) = \frac{50}{100} = 0.5$$

$$P(B) = P(\text{ball} > 80) = \frac{20}{100} = 0.2$$

It is possible for the number on the selected ball to be both odd and greater than 80. Use the law of total probability.

$$\begin{aligned} P(A+B) &= P(A) + P(B) - P(A,B) \\ &= P(A) + P(B) - P(A)P(B) \end{aligned}$$

$$P(\text{odd or} > 80) = 0.5 + 0.2 - (0.5)(0.2) = 0.6$$

The answer is (C).

8. Find the standard normal values for the two points of interest.

$$Z_{16\%} = \frac{x-\mu}{\sigma} = \frac{16\% - 14.2\%}{2.3\%} = 0.78 \quad \text{[use 0.80]}$$

$$Z_{12\%} = \frac{x-\mu}{\sigma} = \frac{12\% - 14.2\%}{2.3\%} = -0.96 \quad \text{[use } -1.00\text{]}$$

Use the unit normal distribution table. The probabilities being sought can be found from the values of $R(x)$ for both standard normal values. $R(0.80) = 0.2119$ and $R(1.00) = 0.1587$. The probability that the sample will fall outside these values is the sum of the two values.

$$P(x < 12\% \text{ or } x > 16\%) = 0.2119 + 0.1587 = 0.3706 \quad (0.37)$$

The answer is (C).

9. Find the probability of exactly 2 heads being thrown. The probability will be the quotient of the total number of possible combinations of six objects taken two at a time and the total number of possible outcomes from tossing six fair coins. The total number of possible outcomes is $(2)^6 = 64$. The total number of possible combinations in which exactly two heads are thrown is

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{6!}{2!(6-2)!} = 15$$

The probability of exactly two heads out of six fair coins is

$$P(A) = P(2 \text{ heads}) = \frac{15}{64} = 0.234$$

The probability of exactly three heads being thrown is found similarly. The total number of possible combinations in which exactly three heads are thrown is

$$C(n,r) = \frac{n!}{r!(n-r)!} = \frac{6!}{3!(6-3)!} = 20$$

The probability of exactly three heads out of six fair coins is

$$P(B) = P(3 \text{ heads}) = \frac{20}{64} = 0.313$$

From the law of total probability, the probability that either of these outcomes will occur is the sum of the individual probabilities that the outcomes will occur, minus the probability that both will occur. These two outcomes are mutually exclusive (i.e., both cannot occur), so the probability of both happening is 0.

The total probability is

$$\begin{aligned} P(2 \text{ heads or } 3 \text{ heads}) &= P(A) + P(B) - P(A,B) \\ &= 0.234 + 0.313 - 0 \\ &= 0.547 \quad (0.55) \end{aligned}$$

The answer is (B).

10. If events A and B are mutually exclusive, the probability of both occurring is zero. However, either event could occur by itself, and the probability of that is nonzero.

The answer is (C).

11. For a fair die, the probability of any face turning up is $^1/_6$. There are two ways to win, and there are four ways to lose. The expected value is

$$E[X] = \sum_{k=1}^{n} x_k f(x_k) = (\$10)\left((2)\left(\frac{1}{6}\right)\right) + (-\$5)\left((4)\left(\frac{1}{6}\right)\right) = \$0.00$$

The answer is (A).

12. A 95% confidence interval on mean transporter utilization means there is a 95% chance the population (or true) mean transporter utilization lies within the given interval.

The answer is (B).

13. Use the binomial probability function to calculate the probability that two of the seven samples will have been born on April 15. $x = 2$, and the sample size, n, is 7.

The probability that a person will have been born on April 15 is 1/365. Therefore, the probability of "success," p, is 1/365, and the probability of "failure," $q = 1 - p$, is 364/365.

$$P_n(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$\begin{aligned} P_7(2) &= \left(\frac{7!}{2!(7-2)!}\right)\left(\frac{1}{365}\right)^2\left(\frac{364}{365}\right)^{7-2} \\ &= (21)\left(\frac{1}{365}\right)^2\left(\frac{364}{365}\right)^5 \\ &= 1.555 \times 10^{-4} \quad (1.6 \times 10^{-4}) \end{aligned}$$

The answer is (D).

14. The arithmetic mean is

$$\begin{aligned}\overline{X} &= (1/n)\sum_{i=1}^{n} X_i \\ &= \left(\frac{1}{5}\right)(71.3 + 74.0 + 74.25 + 78.54 + 80.6) \\ &= 75.738 \quad (75.7)\end{aligned}$$

The sample standard deviation is

$$\begin{aligned}s &= \sqrt{[1/(n-1)]\sum_{i=1}^{n}(X_i - \overline{X})^2} \\ &= \sqrt{\left(\frac{1}{5-1}\right)\begin{pmatrix}(71.3 - 75.738)^2 + (74.0 - 75.738)^2 \\ + (74.25 - 75.738)^2 \\ + (78.54 - 75.738)^2 \\ + (80.6 - 75.738)^2\end{pmatrix}} \\ &= 3.756 \quad (3.8)\end{aligned}$$

The answer is (D).

15. The binomial probability function can be used to determine the probability of three heads in four trials.

$$\begin{aligned}p &= P(\text{heads}) = 0.5 \\ q &= P(\text{not heads}) = 1 - 0.5 = 0.5 \\ n &= \text{number of trials} = 4 \\ x &= \text{number of successes} = 3\end{aligned}$$

From the binomial function,

$$\begin{aligned}P_n(x) &= \frac{n!}{x!(n-x)!}p^x q^{n-x} \\ &= \left(\frac{4!}{3!(4-3)!}\right)(0.5)^3(0.5)^{4-3} \\ &= 0.25 \quad (1/4)\end{aligned}$$

The answer is (A).

16. The complement of set A contains all of the members of set Q that are not members of set A: $(5, 6, 8, 10)$.

The union of the complement of set A and set B is the set of all members appearing in either.

$$\begin{aligned}\overline{A} \cup B &= (5, 6, 8, 10) \cup (4, 5, 9, 10) \\ &= (4, 5, 6, 8, 9, 10)\end{aligned}$$

The answer is (C).

17. Set "not A" consists of all universe elements not in set A: $(2, 4, 5, 7, 8)$.

The intersection of "not A" and B is the set of all elements appearing in both.

$$\begin{aligned}\overline{A} \cap B &= (2, 4, 5, 7, 8) \cap (1, 2, 6, 7) \\ &= (2, 7)\end{aligned}$$

The answer is (A).