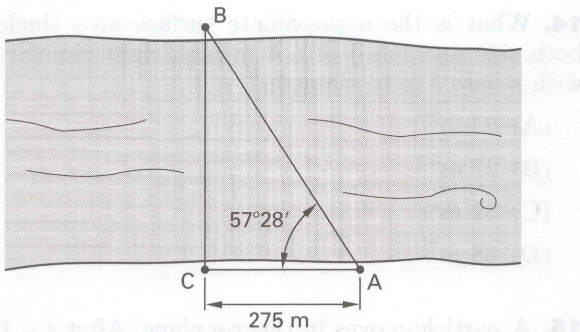
1 Analytic Geometry and Trigonometry

PRACTICE PROBLEMS

1. To find the width of a river, a surveyor sets up a transit at point C on one river bank and sights directly across to point B on the other bank. The surveyor then walks along the bank for a distance of 275 m to point A. The angle CAB is 57° 28′.



What is the approximate width of the river?

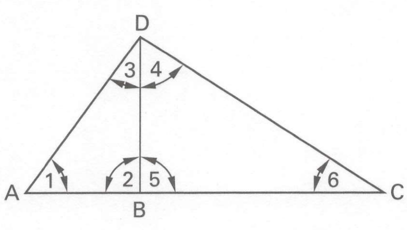
(A) 150 m

(B) 230 m

(C) 330 m

(D) 430 m

2. In the following illustration, angles 2 and 5 are 90°, AD = 15, DC = 20, and AC = 25.



What are the lengths BC and BD, respectively?

(A) 12 and 16

(B) 13 and 17

(C) 16 and 12

(D) 18 and 13

3. What is the length of the line segment with slope 4/3 that extends from the point $(6,4)$ to the y-axis?

(A) 10

(B) 25

(C) 50

(D) 75

4. Which of the following expressions is equivalent to $\sin 2\theta$?

(A) $2\sin\theta\cos\theta$

(B) $\cos^2\theta - \sin^2\theta$

(C) $\sin\theta\cos\theta$

(D) $\dfrac{1-\cos 2\theta}{2}$

5. Which of the following equations describes a circle with center at $(2,3)$ and passing through the point $(-3,-4)$?

(A) $(x+3)^2+(y+4)^2=85$

(B) $(x+3)^2+(y+2)^2=\sqrt{74}$

(C) $(x-3)^2+(y-2)^2=74$

(D) $(x-2)^2+(y-3)^2=74$

6. The equation for a circle is $x^2+4x+y^2+8y=0$. What are the coordinates of the circle's center?

(A) $(-4,-8)$

(B) $(-4,-2)$

(C) $(-2,-4)$

(D) $(2,-4)$

7. Which of the following statements is FALSE for all noncircular ellipses?

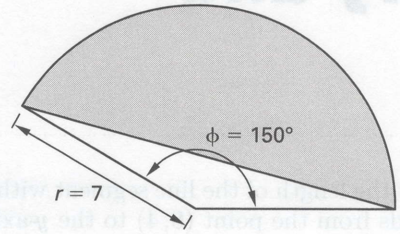
(A) The eccentricity, e, is less than one.

(B) The ellipse has two foci.

(C) The sum of the two distances from the two foci to any point on the ellipse is $2a$ (i.e., twice the semimajor distance).

(D) The coefficients A and C preceding the x^2 and y^2 terms in the general form of the equation are equal.

8. What is the area of the shaded portion of the circle shown?



(A) $\frac{5\pi}{6} - 1$

(B) $\left(\frac{49}{12}\right)(5\pi - 3)$

(C) $\frac{50\pi}{3}$

(D) $49\pi - \sqrt{3}$

9. A pipe with a 20 cm inner diameter is filled to a depth equal to one-third of its diameter. What is the approximate area in flow?

(A) 33 cm^2

(B) 60 cm^2

(C) 92 cm^2

(D) 100 cm^2

10. The equation $y = a_1 + a_2x$ is an algebraic expression for which of the following?

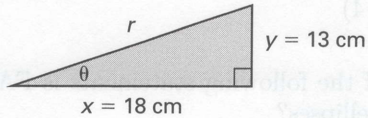
(A) a cosine expansion series

(B) projectile motion

(C) a circle in polar form

(D) a straight line

11. For the right triangle shown, $x = 18$ cm and $y = 13$ cm.



Most nearly, what is $\csc\theta$?

(A) 0.98

(B) 1.2

(C) 1.7

(D) 15

12. A circular sector has a radius of 8 cm and an arc length of 13 cm. Most nearly, what is its area?

(A) 48 cm^2

(B) 50 cm^2

(C) 52 cm^2

(D) 60 cm^2

13. The equation $-3x^2 - 4y^2 = 1$ defines

(A) a circle

(B) an ellipse

(C) a hyperbola

(D) a parabola

14. What is the approximate surface area (including both side and base) of a 4 m high right circular cone with a base 3 m in diameter?

(A) 24 m^2

(B) 27 m^2

(C) 32 m^2

(D) 36 m^2

15. A particle moves in the x-y plane. After t s, the x- and y-coordinates of the particle's location are $x = 8\sin t$ and $y = 6\cos t$. Which of the following equations describes the path of the particle?

(A) $36x^2 + 64y^2 = 2304$

(B) $36x^2 - 64y^2 = 2304$

(C) $64x^2 + 36y^2 = 2304$

(D) $64x^2 - 36y^2 = 2304$

SOLUTIONS

1. Use the formula for the tangent of an angle in a right triangle.

$$\begin{aligned}\tan\theta &= BC/AC\\ BC &= AC\tan\theta = (275\text{ m})\tan 57°28'\\ &= 431.1\text{ m}\quad (430\text{ m})\end{aligned}$$

The answer is (D).

2. For right triangle ABD,

$$\begin{aligned}(BD)^2 + (AB)^2 &= (15)^2\\ (BD)^2 &= (15)^2 - (AB)^2\end{aligned}$$

For right triangle DBC,

$$\begin{aligned}(BD)^2 + (25 - AB)^2 &= (20)^2\\ (BD)^2 &= (20)^2 - (25 - AB)^2\end{aligned}$$

Equate the two expressions for $(BD)^2$.

$$\begin{aligned}(15)^2 - (AB)^2 &= (20)^2 - (25)^2 + 50(AB) - (AB)^2\\ AB &= \frac{(15)^2 - (20)^2 + (25)^2}{50} = 9\\ BC &= 25 - AB = 25 - 9 = 16\\ (BD)^2 &= (15)^2 - (9)^2\\ BD &= 12\end{aligned}$$

Alternatively, this problem can be solved using the law of cosines.

The answer is (C).

3. The equation of the line is of the form

$$y = mx + b$$

The slope is $m = 4/3$, and a known point is $(x, y) = (6, 4)$. Find the y-intercept, b.

$$\begin{aligned}4 &= \left(\tfrac{4}{3}\right)(6) + b\\ b &= 4 - \left(\tfrac{4}{3}\right)(6) = -4\end{aligned}$$

The complete equation is

$$y = \tfrac{4}{3}x - 4$$

b is the y-intercept, so the intersection with the y-axis is at point $(0, -4)$. The distance between these two points is

$$\begin{aligned}d &= \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}\\ &= \sqrt{(4 - (-4))^2 + (6 - 0)^2}\\ &= 10\end{aligned}$$

The answer is (A).

4. The double angle identity is

$$\sin 2\theta = 2\sin\theta\cos\theta$$

The answer is (A).

5. Substitute the known points into the center-radius form of the equation of a circle.

$$\begin{aligned}r^2 &= (x - h)^2 + (y - k)^2\\ &= (-3 - 2)^2 + (-4 - 3)^2\\ &= 74\end{aligned}$$

The equation of the circle is

$$(x - 2)^2 + (y - 3)^2 = 74$$

($r^2 = 74$. The radius is $\sqrt{74}$.)

The answer is (D).

6. Use the standard form of the equation of a circle to find the circle's center.

$$\begin{aligned}x^2 + 4x + y^2 + 8y &= 0\\ x^2 + 4x + 4 + y^2 + 8y + 16 &= 4 + 16\\ (x + 2)^2 + (y + 4)^2 &= 20\end{aligned}$$

The center is at $(-2, -4)$.

The answer is (C).

7. The coefficients preceding the squared terms in the general equation are equal only for a straight line or circle, not for a noncircular ellipse.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The answer is (D).

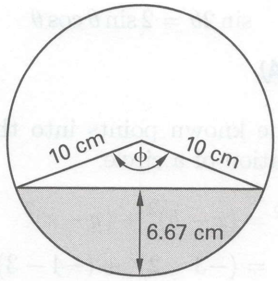
8. The area of the circle is

$$\begin{aligned}\phi &= (150°)\left(\frac{2\pi\text{ rad}}{360°}\right) = \frac{5\pi}{6}\text{ rad}\\ A &= \frac{r^2(\phi - \sin\phi)}{2}\\ &= \frac{(7)^2\left(\frac{5\pi}{6} - \sin\frac{5\pi}{6}\right)}{2}\\ &= \left(\frac{49}{2}\right)\left(\frac{5\pi}{6} - \frac{1}{2}\right)\\ &= \left(\frac{49}{12}\right)(5\pi - 3)\end{aligned}$$

The answer is (B).

Mathematics

9. Find the angle ϕ.



$$\begin{aligned}\phi &= 2\{\arccos[(r-d)/r]\} \\ &= 2\arccos\left(\frac{10\text{ cm}-6.67\text{ cm}}{10\text{ cm}}\right) \\ &= 2.462\text{ rad}\end{aligned}$$

Find the area of flow.

$$\begin{aligned}A &= [r^2(\phi-\sin\phi)]/2 \\ &= \frac{(10\text{ cm})^2\Big(2.46\text{ rad}-\sin(2.462\text{ rad})\Big)}{2} \\ &= 91.67\text{ cm}^2 \quad (92\text{ cm}^2)\end{aligned}$$

The answer is (C).

10. $y=mx+b$ is the slope-intercept form of the equation of a straight line. a_1 and a_2 are both constants, so $y=a_1+a_2x$ describes a straight line.

The answer is (D).

11. Find the length of the hypotenuse, r.

$$r=\sqrt{x^2+y^2}=\sqrt{(18\text{ cm})^2+(13\text{ cm})^2}=22.2\text{ cm}$$

Find csc θ.

$$\csc\theta=r/y=\frac{22.2\text{ cm}}{13\text{ cm}}=1.7$$

The answer is (C).

12. Find the area of the circular sector.

$$A=sr/2=\frac{(13\text{ cm})(8\text{ cm})}{2}=52\text{ cm}^2$$

The answer is (C).

13. The general form of the conic section equation is

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

$A=-3$, $C=-4$, $F=-1$, and $B=D=E=0$. A and C are different, so the equation does not define a circle. Calculate the discriminant.

$$B^2-4AC=(0)^2-(4)(-3)(-4)=-48$$

This is less than zero, so the equation defines an ellipse.

The answer is (B).

14. Find the total surface area of a right circular cone. The radius is $r=d/2=3\text{ m}/2=1.5\text{ m}$.

$$\begin{aligned}A &= \text{side area}+\text{base area}=\pi r\left(r+\sqrt{r^2+h^2}\right) \\ &= \pi(1.5\text{ m})\left(1.5\text{ m}+\sqrt{(1.5\text{ m})^2+(4\text{ m})^2}\right) \\ &= 27.2\text{ m}^2 \quad (27\text{ m}^2)\end{aligned}$$

The answer is (B).

15. Rearrange the two coordinate equations.

$$\sin t=\frac{x}{8}$$
$$\cos t=\frac{y}{6}$$

Use the following trigonometric identity.

$$\sin^2\theta+\cos^2\theta=1$$
$$\left(\frac{x}{8}\right)^2+\left(\frac{y}{6}\right)^2=1$$

To clear the fractions, multiply both sides by $(8)^2\times(6)^2=2304$.

$$36x^2+64y^2=2304$$

The answer is (A).

2 Algebra and Linear Algebra

PRACTICE PROBLEMS

1. What is the name for a vector that represents the sum of two vectors?

(A) scalar

(B) resultant

(C) tensor

(D) moment

2. The second and sixth terms of a geometric progression are 3/10 and 243/160, respectively. What is the first term of this sequence?

(A) 1/10

(B) 1/5

(C) 3/5

(D) 3/2

3. Using logarithmic identities, what is most nearly the numerical value for the following expression?

$$\log_3 \tfrac{3}{2} + \log_3 12 - \log_3 2$$

(A) 0.95

(B) 1.33

(C) 2.00

(D) 2.20

4. Which of the following statements is true for a power series with the general term $a_i x^i$?

I. An infinite power series converges for $x < 1$.

II. Power series can be added together or subtracted within their interval of convergence.

III. Power series can be integrated within their interval of convergence.

(A) I only

(B) II only

(C) I and III

(D) II and III

5. What is most nearly the length of the resultant of the following vectors?

$$\begin{aligned} 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \\ 7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \\ -16\mathbf{i} - 14\mathbf{j} + 2\mathbf{k} \end{aligned}$$

(A) 3

(B) 4

(C) 10

(D) 14

6. What is the solution to the following system of simultaneous linear equations?

$$\begin{aligned} 10x + 3y + 10z &= 5 \\ 8x - 2y + 9z &= 3 \\ 8x + y - 10z &= 7 \end{aligned}$$

(A) $x=0.326$; $y=-0.192$; $z=0.586$

(B) $x=0.148$; $y=1.203$; $z=0.099$

(C) $x=0.625$; $y=0.186$; $z=-0.181$

(D) $x=0.282$; $y=-1.337$; $z=-0.131$

7. What is the inverse of matrix $\mathbf{A}$?

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

(A) $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$

(D) $\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

8. If the determinant of matrix **A** is −40, what is the determinant of matrix **B**?

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1.5 & 1 & 0.5 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

(A) −80

(B) −40

(C) −20

(D) 0.5

9. Given the origin-based vector $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, what is most nearly the angle between **A** and the x-axis?

(A) 22°

(B) 24°

(C) 66°

(D) 80°

10. Which is a true statement about these two vectors?

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{B} = \mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$$

(A) Both vectors pass through the point $(0, -1, 6)$.

(B) The vectors are parallel.

(C) The vectors are orthogonal.

(D) The angle between the vectors is 17.4°.

11. What is most nearly the acute angle between vectors $\mathbf{A} = (3, 2, 1)$ and $\mathbf{B} = (2, 3, 2)$, both based at the origin?

(A) 25°

(B) 33°

(C) 35°

(D) 59°

12. Force vectors **A**, **B**, and **C** are applied at a single point.

$$\mathbf{A} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$
$$\mathbf{B} = 2\mathbf{i} + 7\mathbf{j} - \mathbf{k}$$
$$\mathbf{C} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

What is most nearly the magnitude of the resultant force vector, **R**?

(A) 13

(B) 14

(C) 15

(D) 16

13. What is the sum of $12 + 13j$ and $7 - 9j$?

(A) $19 - 22j$

(B) $19 + 4j$

(C) $25 - 22j$

(D) $25 + 4j$

14. What is the product of the complex numbers $3 + 4j$ and $7 - 2j$?

(A) $10 + 2j$

(B) $13 + 22j$

(C) $13 + 34j$

(D) $29 + 22j$

SOLUTIONS

1. By definition, the sum of two vectors is known as the resultant.

The answer is (B).

2. The common ratio is

$$l = ar^{n-1}$$

$$\frac{l_6}{l_2} = \frac{ar^{6-1}}{ar^{2-1}} = r^4$$

$$r = \sqrt[4]{\frac{l_6}{l_2}}$$

$$= \sqrt[4]{\frac{\frac{243}{160}}{\frac{3}{10}}}$$

$$= 3/2$$

The term before 3/10 is

$$a_1 = \frac{\frac{3}{10}}{\frac{3}{2}} = 1/5$$

The answer is (B).

3. Use the logarithmic identities.

$$\log xy = \log x + \log y$$

$$\log x/y = \log x - \log y$$

$$\log_3 \frac{3}{2} + \log_3 12 - \log_3 2 = \log_3 \frac{\left(\frac{3}{2}\right)(12)}{2}$$

$$= \log_3 9$$

Since $(3)^2 = 9$,

$$\log_3 9 = 2.00$$

The answer is (C).

4. Power series can be added together, subtracted from each other, differentiated, and integrated within their interval of convergence. The interval of convergence is $-1 < x < 1$.

The answer is (D).

5. The resultant is produced by adding the vectors.

$$\begin{array}{r} 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \\ 7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \\ -16\mathbf{i} - 14\mathbf{j} + 2\mathbf{k} \\ \hline -6\mathbf{i} - 8\mathbf{j} + 0\mathbf{k} \end{array}$$

The length of the resultant vector is

$$|\mathbf{R}| = \sqrt{(-6)^2 + (-8)^2 + (0)^2}$$

$$= 10$$

The answer is (C).

6. There are several ways of solving this problem.

$$\mathbf{AX} = \mathbf{B}$$

$$\begin{bmatrix} 10 & 3 & 10 \\ 8 & -2 & 9 \\ 8 & 1 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$$

$$\mathbf{AA}^{-1}\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\mathbf{X} = \begin{bmatrix} \frac{11}{806} & \frac{20}{403} & \frac{47}{806} \\ \frac{76}{403} & \frac{-90}{403} & \frac{-5}{403} \\ \frac{12}{403} & \frac{7}{403} & \frac{-22}{403} \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} (5)\left(\frac{11}{806}\right) + (3)\left(\frac{20}{403}\right) + (7)\left(\frac{47}{806}\right) \\ (5)\left(\frac{76}{403}\right) + (3)\left(\frac{-90}{403}\right) + (7)\left(\frac{-5}{403}\right) \\ (5)\left(\frac{12}{403}\right) + (3)\left(\frac{7}{403}\right) + (7)\left(\frac{-22}{403}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0.625 \\ 0.186 \\ -0.181 \end{bmatrix}$$

(Direct substitution of the four answer choices into the original equations is probably the fastest way of solving this type of problem.)

The answer is (C).

7. Find the determinant.

$$|\mathbf{A}| = 2 \times 1 - 1 \times 3 = -1$$

The inverse of a 2×2 matrix is

$$\mathbf{A}^{-1} = \frac{\mathrm{adj}(\mathbf{A})}{|\mathbf{A}|} = \frac{\begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}}{|\mathbf{A}|}$$

$$= \frac{\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}}{-1}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

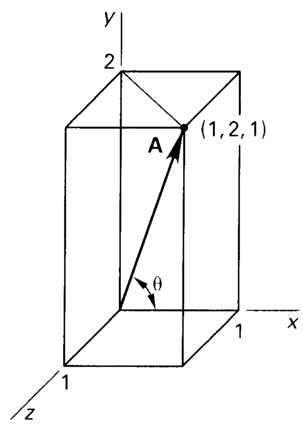
The answer is (D).

8. The first row of matrix **B** is half that of **A**, and the other rows are the same in **A** and **B**, so the determinant of **B** is half the determinant of **A**.

The answer is (C).

9. The magnitude of vector **A** is

$$|\mathbf{A}| = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$$



The x-component of the vector is 1, so the direction cosine is

$$\cos\theta_x = \frac{1}{\sqrt{6}}$$

The angle is

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 65.9^\circ \quad (66^\circ)$$

The answer is (C).

10. The magnitudes of the two vectors are

$$|\mathbf{A}| = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$$

$$|\mathbf{B}| = \sqrt{(1)^2 + (3)^2 + (-7)^2} = \sqrt{59}$$

The angle between them is

$$\begin{aligned}\phi &= \cos^{-1}\left(\frac{a_x b_x + a_y b_y + a_z b_z}{|\mathbf{A}||\mathbf{B}|}\right)\\ &= \cos^{-1}\left(\frac{(1)(1) + (2)(3) + (1)(-7)}{\sqrt{6}\sqrt{59}}\right)\\ &= 90^\circ\end{aligned}$$

The vectors are orthogonal.

The answer is (C).

11. The angle between the two vectors is

$$\begin{aligned}\theta &= \arccos\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right)\\ &= \arccos\left(\frac{a_x b_x + a_y b_y + a_z b_z}{|\mathbf{A}||\mathbf{B}|}\right)\\ &= \arccos\left(\frac{(3)(2) + (2)(3) + (1)(2)}{\sqrt{(3)^2 + (2)^2 + (1)^2}\sqrt{(2)^2 + (3)^2 + (2)^2}}\right)\\ &= 24.8^\circ \quad (25^\circ)\end{aligned}$$

The answer is (A).

12. The magnitude of **R** is

$$\begin{aligned}|\mathbf{R}| &= \sqrt{(1 + 2 - 1)^2 + (3 + 7 + 4)^2 + (4 - 1 + 2)^2}\\ &= \sqrt{4 + 196 + 25}\\ &= \sqrt{225}\\ &= 15\end{aligned}$$

The answer is (C).

13. Add the real parts and the imaginary parts of each complex number.

$$\begin{aligned}(a + jb) + (c + jd) &= (a + c) + j(b + d)\\ (12 + 13j) + (7 - 9j) &= (12 + 7) + j(13 + (-9))\\ &= 19 + 4j\end{aligned}$$

The answer is (B).

14. Use the algebraic distributive law and the equivalency $j^2 = -1$.

$$\begin{aligned}(a + jb)(c + jd) &= (ac - bd) + j(ad + bc)\\ (3 + 4j)(7 - 2j) &= 21 - 8j^2 + 28j - 6j\\ &= 21 + 8 + 28j - 6j\\ &= 29 + 22j\end{aligned}$$

The answer is (D).

3 Calculus

PRACTICE PROBLEMS

1. Which of the following is NOT a correct derivative?

(A) $\dfrac{d}{dx}\cos x = -\sin x$

(B) $\dfrac{d}{dx}(1-x)^3 = -3(1-x)^2$

(C) $\dfrac{d}{dx}\dfrac{1}{x} = -\dfrac{1}{x^2}$

(D) $\dfrac{d}{dx}\csc x = -\cot x$

2. What is the derivative, dy/dx, of the expression $x^2y - e^{2x} = \sin y$?

(A) $\dfrac{2e^{2x}}{x^2 - \cos y}$

(B) $\dfrac{2e^{2x} - 2xy}{x^2 - \cos y}$

(C) $2e^{2x} - 2xy$

(D) $x^2 - \cos y$

3. What is the approximate area bounded by the curves $y = 8 - x^2$ and $y = -2 + x^2$?

(A) 22

(B) 27

(C) 30

(D) 45

4. What are the minimum and maximum values, respectively, of the equation $f(x) = 5x^3 - 2x^2 + 1$ on the interval $[-2, 2]$?

(A) $-47, 33$

(B) $-4, 4$

(C) $0.95, 1$

(D) $0, 0.27$

5. In vector calculus, a gradient is a

I. vector that points in the direction of a general rate of change of a scalar field

II. vector that points in the direction of the maximum rate of change of a scalar field

III. scalar that indicates the magnitude of the rate of change of a vector field in a general direction

IV. scalar that indicates the maximum magnitude of the rate of change of a vector field in any particular direction

(A) I only

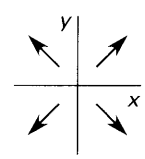
(B) II only

(C) I and III

(D) II and IV

6. Which of the illustrations shown represents the vector field, $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$, for nonzero values of x and y?

(A)



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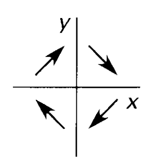

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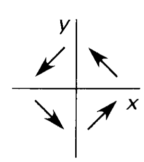
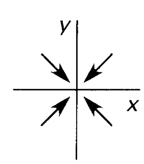


(D)



7. A peach grower estimates that if he picks his crop now, he will obtain 1000 lugs of peaches, which he can sell at \$1.00 per lug. However, he estimates that his crop will increase by an additional 60 lugs of peaches for each week that he delays picking, but the price will drop at a rate of \$0.025 per lug per week. In addition, he will experience a spoilage rate of approximately 10 lugs for each week he delays. In order to maximize his revenue, how many weeks should he wait before picking the peaches?

(A) 2 weeks

(B) 5 weeks

(C) 7 weeks

(D) 10 weeks

8. Determine the following indefinite integral.

$$\int \frac{x^3 + x + 4}{x^2} dx$$

(A) $\frac{x}{4} + \ln|x| - \frac{4}{x} + C$

(B) $\frac{-x}{2} + \log x - 8x + C$

(C) $\frac{x^2}{2} + \ln|x| - \frac{2}{x^2} + C$

(D) $\frac{x^2}{2} + \ln|x| - \frac{4}{x} + C$

9. Find dy/dx for the parametric equations given.

$$x = 2t^2 - t$$
$$y = t^3 - 2t + 1$$

(A) $3t^2$

(B) $3t^2/2$

(C) $4t - 1$

(D) $(3t^2 - 2)/(4t - 1)$

10. A two-dimensional function, $f(x, y)$, is defined as

$$f(x, y) = 2x^2 - y^2 + 3x - y$$

What is the direction of the line passing through the point $(1, -2)$ that has the maximum slope?

(A) $4\mathbf{i} + 2\mathbf{j}$

(B) $7\mathbf{i} + 3\mathbf{j}$

(C) $7\mathbf{i} + 4\mathbf{j}$

(D) $9\mathbf{i} - 7\mathbf{j}$

11. Evaluate the following limit.

$$\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2} \right)$$

(A) 0

(B) 2

(C) 4

(D) ∞

12. If $f(x, y) = x^2y^3 + xy^4 + \sin x + \cos^2 x + \sin^3 y$, what is $\partial f/\partial x$?

(A) $(2x + y)y^3 + 3\sin^2 y \cos y$

(B) $(4x - 3y^2)xy^2 + 3\sin^2 y \cos y$

(C) $(3x + 4y^2)xy + 3\sin^2 y \cos y$

(D) $(2x + y)y^3 + (1 - 2\sin x)\cos x$

13. What is dy/dx if $y = (2x)^x$?

(A) $(2x)^x(2 + \ln 2x)$

(B) $2x(1 + \ln 2x)^x$

(C) $(2x)^x(\ln 2x^2)$

(D) $(2x)^x(1 + \ln 2x)$

SOLUTIONS

1. Determine each of the derivatives.

$$\frac{d}{dx}\cos x = -\sin x \quad \text{[OK]}$$

$$\frac{d}{dx}(1-x)^3 = (3)(1-x)^2(-1) = (-3)(1-x)^2 \quad \text{[OK]}$$

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = (-1)(x^{-2}) = \frac{-1}{x^2} \quad \text{[OK]}$$

$$\frac{d}{dx}\csc x = -\cot x \quad \text{[incorrect]}$$

The answer is (D).

2. Since neither x nor y can be extracted from the equality, rearrange to obtain a homogeneous expression in x and y.

$$x^2y - e^{2x} = \sin y$$

$$f(x,y) = x^2y - e^{2x} - \sin y = 0$$

Take the partial derivatives with respect to x and y.

$$\frac{\partial f(x,y)}{\partial x} = 2xy - 2e^{2x}$$

$$\frac{\partial f(x,y)}{\partial y} = x^2 - \cos y$$

Use implicit differentiation.

$$\frac{\partial y}{\partial x} = \frac{\dfrac{-\partial f(x,y)}{\partial x}}{\dfrac{\partial f(x,y)}{\partial y}} = \frac{2e^{2x} - 2xy}{x^2 - \cos y}$$

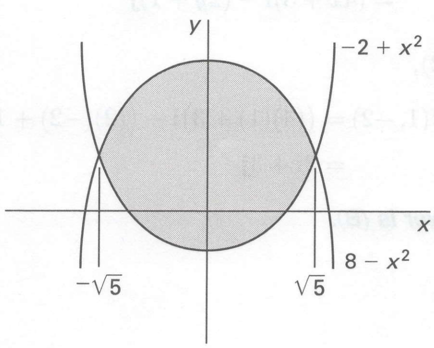
The answer is (B).

3. Find the intersection points by setting the two functions equal.

$$-2 + x^2 = 8 - x^2$$

$$2x^2 = 10$$

$$x = \pm\sqrt{5}$$



The integral of $f_1(x) - f_2(x)$ represents the area between the two curves between the limits of integration.

$$A = \int_{x_1}^{x_2} \left(f_1(x) - f_2(x)\right) dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \left((8 - x^2) - (-2 + x^2)\right) dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} (10 - 2x^2)\, dx$$

$$= \left(10x - \tfrac{2}{3}x^3\right)\Big|_{-\sqrt{5}}^{\sqrt{5}}$$

$$= 29.8 \quad (30)$$

The answer is (C).

4. The critical points are located where the first derivative is zero.

$$f(x) = 5x^3 - 2x^2 + 1$$

$$f'(x) = 15x^2 - 4x$$

$$15x^2 - 4x = 0$$

$$x(15x - 4) = 0$$

$$x = 0 \quad \text{or} \quad x = 4/15$$

Test for a maximum, minimum, or inflection point.

$$f''(x) = 30x - 4$$

$$f''(0) = (30)(0) - 4$$

$$= -4$$

$$f''(a) < 0 \quad \text{[maximum]}$$

$$f''\left(\frac{4}{15}\right) = (30)\left(\frac{4}{15}\right) - 4$$

$$= 4$$

$$f''(a) > 0 \quad \text{[minimum]}$$

These could be a local maximum and minimum. Check the endpoints of the interval and compare with the function values at the critical points.

$$f(-2) = (5)(-2)^3 - (2)(-2)^2 + 1 = -47$$

$$f(2) = (5)(2)^3 - (2)(2)^2 + 1 = 33$$

$$f(0) = (5)(0)^3 - (2)(0)^2 + 1 = 1$$

$$f\left(\frac{4}{15}\right) = (5)\left(\frac{4}{15}\right)^3 - (2)\left(\frac{4}{15}\right)^2 + 1$$

$$= 0.95$$

The minimum and maximum values of the equation, −47 and 33, respectively, are at the endpoints.

The answer is (A).

5. A gradient (gradient vector) at some point P is described by use of the gradient (del, grad, nabla, etc.) function, $\nabla f_{\mathrm{P}} \cdot \mathbf{a}$, where $\mathbf{a}$ is a unit vector. In three-dimensional rectangular coordinates, the gradient is equivalent to the partial derivative vector $\nabla f \cdot \mathbf{a} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$. This is a vector that points in the direction of the maximum rate of change (i.e., maximum slope).

The answer is (B).

6. From $-y\mathbf{i}$, it can be concluded that for

(a) positive values of y, the vector field points to the left, and

(b) negative values of y, the vector field points to the right.

From $+x\mathbf{j}$, it can be concluded that for

(a) positive values of x, the vector field points upward, and

(b) negative values of x, the vector field points downward.

The answer is (C).

7. Let x represent the number of weeks.

The equation describing the price as a function of time is

$$\frac{\text{price}}{\text{lug}} = \$1 - \$0.025x$$

The equation describing the yield is

$$\begin{aligned}\text{lugs sold} &= 1000 + (60 - 10)x \\ &= 1000 + 50x\end{aligned}$$

The revenue function is

$$\begin{aligned}R &= \left(\frac{\text{price}}{\text{lug}}\right)(\text{lugs sold}) \\ &= (1 - 0.025x)(1000 + 50x) \\ &= 1000 + 50x - 25x - 1.25x^2 \\ &= 1000 + 25x - 1.25x^2\end{aligned}$$

To maximize the revenue function, set its derivative equal to zero.

$$\begin{aligned}\frac{dR}{dx} &= 25 - 2.5x = 0 \\ x &= 10 \text{ weeks}\end{aligned}$$

The answer is (D).

8. Separate the fraction into parts and integrate each one.

$$\begin{aligned}\int \frac{x^3 + x + 4}{x^2}\,dx &= \int \frac{x^3}{x^2}\,dx + \int \frac{x}{x^2}\,dx + \int \frac{4}{x^2}\,dx \\ &= \int x\,dx + \int \frac{1}{x}\,dx + 4\int \frac{1}{x^2}\,dx \\ &= \frac{x^2}{2} + \ln|x| + 4\left(\frac{x^{-1}}{-1}\right) + C \\ &= \frac{x^2}{2} + \ln|x| - \frac{4}{x} + C\end{aligned}$$

The answer is (D).

9. Calculate the derivatives of x and y with respect to t.

$$\begin{aligned}\frac{dy}{dt} &= 3t^2 - 2 \\ \frac{dx}{dt} &= 4t - 1\end{aligned}$$

The derivative of y with respect to x is

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{3t^2 - 2}{4t - 1}\end{aligned}$$

The answer is (D).

10. The direction of the line passing through $(1, -2)$ with maximum slope is found by inserting $x = 1$ and $y = -2$ into the gradient vector function.

The gradient of the function is

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f(x, y, z)}{\partial x}\mathbf{i} + \frac{\partial f(x, y, z)}{\partial y}\mathbf{j} + \frac{\partial f(x, y, z)}{\partial z}\mathbf{k} \\ &= \frac{\partial(2x^2 - y^2 + 3x - y)}{\partial x}\mathbf{i} \\ &\quad + \frac{\partial(2x^2 - y^2 + 3x - y)}{\partial y}\mathbf{j} \\ &= (4x + 3)\mathbf{i} - (2y + 1)\mathbf{j}\end{aligned}$$

At $(1, -2)$,

$$\begin{aligned}\nabla f(1, -2) &= ((4)(1) + 3)\mathbf{i} - ((2)(-2) + 1)\mathbf{j} \\ &= 7\mathbf{i} + 3\mathbf{j}\end{aligned}$$

The answer is (B).

11. The expression approaches 0/0 at the limit.

$$\frac{(2)^2-4}{2-2}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)=\lim_{x\to 2}\left(\frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)}\right)=\lim_{x\to 2}\left(\frac{2x}{1}\right)$$
$$=\frac{(2)(2)}{1}$$
$$=4$$

This could also be solved by factoring the numerator.

The answer is (C).

12. The partial derivative with respect to x is found by treating all other variables as constants. Therefore, all terms that do not contain x have zero derivatives.

$$\frac{\partial f}{\partial x}=2xy^3+y^1+\cos x+2\cos x(-\sin x)$$
$$=(2x+y)y^3+(1-2\sin x)\cos x$$

The answer is (D).

13. From the table of derivatives,

$$\mathbf{D}\big(f(x)\big)^{g(x)}=g(x)\big(f(x)\big)^{g(x)-1}\mathbf{D}f(x)$$
$$+\ln\big(f(x)\big)\big(f(x)\big)^{g(x)}\mathbf{D}g(x)$$
$$f(x)=2x$$
$$g(x)=x$$
$$\frac{d(2x)^x}{dx}=x(2x)^{x-1}(2)+(\ln 2x)(2x)^x(1)$$
$$=(2x)^x+(2x)^x\ln 2x$$
$$=(2x)^x(1+\ln 2x)$$

The answer is (D).

4 Differential Equations and Transforms

PRACTICE PROBLEMS

1. What is the solution to the following differential equation?

$$y' + 5y = 0$$

(A) $y = 5x + C$

(B) $y = Ce^{-5x}$

(C) $y = Ce^{5x}$

(D) either (A) or (B)

2. What is the solution to the following linear difference equation?

$$(k+1)\big(y(k+1)\big) - ky(k) = 1$$

(A) $y(k) = 12 - \frac{1}{k}$

(B) $y(k) = 1 - \frac{12}{k}$

(C) $y(k) = 12 + 3k$

(D) $y(k) = 3 + \frac{1}{k}$

3. What is the general solution to the following differential equation?

$$2\left(\frac{d^2y}{dx^2}\right) - 4\left(\frac{dy}{dx}\right) + 4y = 0$$

(A) $y = C_1 \cos x + C_2 \sin x$

(B) $y = C_1 e^x + C_2 e^{-x}$

(C) $y = e^{-x}(C_1 \cos x - C_2 \sin x)$

(D) $y = e^x(C_1 \cos x + C_2 \sin x)$

4. What is the general solution to the following differential equation?

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

(A) $y = C_1 \sin x - C_2 \cos x$

(B) $y = C_1 \cos x - C_2 \sin x$

(C) $y = C_1 \cos x + C_2 \sin x$

(D) $y = e^{-x}(C_1 \cos x + C_2 \sin x)$

5. What is the complementary solution to the following differential equation?

$$y'' - 4y' + \tfrac{25}{4}y = 10 \cos 8x$$

(A) $y = 2C_1 x + C_2 x - C_3 x$

(B) $y = C_1 e^{2x} + C_2 e^{1.5x}$

(C) $y = C_1 e^{2x} \cos 1.5x + C_2 e^{2x} \sin 1.5x$

(D) $y = C_1 e^x \tan x + C_2 e^x \cot x$

6. What is the general solution to the following differential equation?

$$y'' + y' + y = 0$$

(A) $y = e^{-\frac{1}{2}x}\left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right)$

(B) $y = e^{-\frac{1}{2}x}\left(C_1 \cos \frac{3}{2}x + C_2 \sin \frac{3}{2}x\right)$

(C) $y = e^{-2x}\left(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x\right)$

(D) $y = e^{-2x}\left(C_1 \cos \frac{3}{2}x + C_2 \sin \frac{3}{2}x\right)$

7. What is the solution to the following differential equation if $x = 1$ at $t = 0$, and $dx/dt = 0$ at $t = 0$?

$$\tfrac{1}{2}\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = 5$$

(A) $x = e^{-4t} + 4te^{-4t}$

(B) $x = \frac{3}{8}e^{-2t}(\cos 2t + \sin 2t) + \frac{5}{8}$

(C) $x = e^{-4t} + 4te^{-4t} + \frac{5}{8}$

(D) $x = \frac{3}{8}e^{-4t} + \frac{3}{2}te^{-4t} + \frac{5}{8}$

8. In the following differential equation with the initial condition $x(0) = 12$, what is the value of $x(2)$?

$$\frac{dx}{dt} + 4x = 0$$

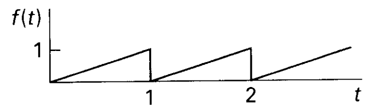
(A) 3.4×10^{-3}

(B) 4.0×10^{-3}

(C) 5.1×10^{-3}

(D) 6.2×10^{-3}

9. What are the three general Fourier coefficients for the sawtooth wave shown?



(A) $a_0 = 0,\ a_n = 0,\ b_n = \frac{-1}{\pi n}$

(B) $a_0 = \frac{1}{2},\ a_n = 0,\ b_n = \frac{-1}{\pi n}$

(C) $a_0 = 1,\ a_n = 1,\ b_n = \frac{1}{\pi n}$

(D) $a_0 = \frac{1}{2},\ a_n = \frac{1}{2},\ b_n = \frac{1}{\pi n}$

10. The values of an unknown function follow a Fibonacci number sequence. It is known that $f(1) = 4$ and $f(2) = 1.3$. What is $f(4)$?

(A) −4.1

(B) 0.33

(C) 2.7

(D) 6.6

SOLUTIONS

1. This is a first-order linear equation with characteristic equation $r + 5 = 0$. The form of the solution is

$$y = Ce^{-5x}$$

In the preceding equation, the constant, C, could be determined from additional information.

The answer is (B).

2. Since nothing is known about the general form of $y(k)$, the only way to solve this problem is by trial and error, substituting each answer option into the equation in turn. Option B is

$$y(k) = 1 - \frac{12}{k}$$

Substitute this into the difference equation.

$$\begin{aligned}
(k+1)\big(y(k+1)\big) - k\big(y(k)\big) &= 1 \\
(k+1)\left(1 - \frac{12}{k+1}\right) - k\left(1 - \frac{12}{k}\right) &= 1 \\
(k+1)\left(\frac{k+1-12}{k+1}\right) - k\left(\frac{k-12}{k}\right) &= 1 \\
k + 1 - 12 - k + 12 &= 1 \\
1 &= 1
\end{aligned}$$

$y(k) = 1 - 12/k$ solves the difference equation.

The answer is (B).

3. This is a second-order, homogeneous, linear differential equation. Start by putting it in general form.

$$\begin{aligned}
y'' + 2ay' + by &= 0 \\
2y'' - 4y' + 4y &= 0 \\
y'' - 2y' + 2y &= 0 \\
a &= -2 \\
b &= 2
\end{aligned}$$

Since $a^2 < 4b$, the form of the equation is

$$\begin{aligned}
y &= e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta_x) \\
\alpha &= \frac{-a}{2} = \frac{-2}{2} = 1 \\
\beta &= \frac{\sqrt{4b - a^2}}{2} \\
&= \frac{\sqrt{(4)(2) - (-2)^2}}{2} \\
&= 1 \\
y &= e^x(C_1 \cos x + C_2 \sin x)
\end{aligned}$$

The answer is (D).

4. The characteristic equation is

$$r^2 + 2r + 2 = 0$$
$$a = 2$$
$$b = 2$$

The roots are

$$\begin{aligned} r_{1,2} &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} \\ &= \frac{-2 \pm \sqrt{(2)^2 - (4)(2)}}{2} \\ &= (-1 + i), (-1 - i) \end{aligned}$$

Since $a^2 < 4b$, the solution is

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$
$$\alpha = \frac{-a}{2} = \frac{-2}{2} = -1$$
$$\begin{aligned} \beta &= \frac{\sqrt{4b - a^2}}{2} = \frac{\sqrt{(4)(2) - (2)^2}}{2} \\ &= 1 \end{aligned}$$
$$y = e^{-x}(C_1 \cos x + C_2 \sin x)$$

The answer is (D).

5. The complementary solution to a nonhomogeneous differential equation is the solution of the homogeneous differential equation.

The characteristic equation is

$$r^2 + ar + b = 0$$
$$r^2 - 4r + \frac{25}{4} = 0$$

So, $a = -4$, and $b = 25/4$.

The roots are

$$\begin{aligned} r_{1,2} &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - (4)\left(\frac{25}{4}\right)}}{2} \\ &= 2 \pm 1.5i \end{aligned}$$

Since the roots are imaginary, the homogeneous solution has the form of

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$
$$\alpha = 2$$
$$\beta = \pm 1.5$$

The complementary solution is

$$\begin{aligned} y &= e^{2x}(C_1 \cos 1.5x + C_2 \sin 1.5x) \\ &= C_1 e^{2x} \cos 1.5x + C_2 e^{2x} \sin 1.5x \end{aligned}$$

The answer is (C).

6. This is a second-order, homogeneous, linear differential equation with $a = b = 1$. This differential equation can be solved by the method of undetermined coefficients with a solution in the form $y = Ce^{rx}$. The substitution of the solution gives

$$(r^2 + ar + b)Ce^{rx} = 0$$

Because Ce^{rx} can never be zero, the characteristic equation is

$$r^2 + ar + b = 0$$

Because $a^2 = 1 < 4b = 4$, the general solution is in the form

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

Then,

$$\alpha = -a/2 = -1/2$$

$$\beta = \sqrt{\frac{4b - a^2}{2}} = \sqrt{\frac{(4)(1) - (1)^2}{2}} = \frac{\sqrt{3}}{2}$$

Therefore, the general solution is

$$y = e^{-\frac{1}{2}x}\left(C_1 \cos \tfrac{\sqrt{3}}{2}x + C_2 \sin \tfrac{\sqrt{3}}{2}x\right)$$

The answer is (A).

7. Multiplying the equation by 2 gives

$$x'' + 8x' + 16x = 10$$

The characteristic equation is

$$r^2 + 8r + 16 = 0$$

The roots of the characteristic equation are

$$r_1 = r_2 = -4$$

The homogeneous (natural) response is

$$x_{\text{natural}} = Ae^{-4t} + Bte^{-4t}$$

By inspection, $x = 5/8$ is a particular solution that solves the nonhomogeneous equation, so the total response is

$$x = Ae^{-4t} + Bte^{-4t} + \tfrac{5}{8}$$

Since $x = 1$ at $t = 0$,

$$\begin{aligned} 1 &= Ae^{0} + \tfrac{5}{8} \\ A &= \tfrac{3}{8} \end{aligned}$$

Differentiating x,

$$x' = \tfrac{3}{8}(-4)e^{-4t} + B(-4te^{-4t} + e^{-4t}) + 0$$

Since $x' = 0$ at $t = 0$,

$$\begin{aligned} 0 &= -\tfrac{3}{2} + B(0+1) \\ B &= \tfrac{3}{2} \\ x &= \tfrac{3}{8}e^{-4t} + \tfrac{3}{2}te^{-4t} + \tfrac{5}{8} \end{aligned}$$

The answer is (D).

8. This is a first-order, linear, homogeneous differential equation with characteristic equation $r + 4 = 0$.

$$\begin{aligned} x' + 4x &= 0 \\ x &= x_0 e^{-4t} \\ x(0) &= x_0 e^{(-4)(0)} \\ &= 12 \\ x_0 &= 12 \\ x &= 12e^{-4t} \\ x(2) &= 12e^{(-4)(2)} \\ &= 12e^{-8} \\ &= 4.03 \times 10^{-3} \quad (4.0 \times 10^{-3}) \end{aligned}$$

The answer is (B).

9. By inspection, $f(t) = t$, with the period $T = 1$. The angular frequency is

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

The average is

$$\begin{aligned} a_0 &= (1/T)\int_0^T f(t)\,dt = (1/T)\int_0^T t\,dt = \frac{1}{2}t^2\bigg|_0^1 = \frac{1}{2} - 0 \\ &= \frac{1}{2} \end{aligned}$$

The general a term is

$$\begin{aligned} a_n &= (2/T)\int_0^T f(t)\cos(n\omega_0 t)\,dt \\ &= 2\int_0^1 t\cos(2\pi nt)\,dt \\ &= 0 \end{aligned}$$

The general b term is

$$\begin{aligned} b_n &= (2/T)\int_0^T f(t)\sin(n\omega_0 t)\,dt \\ &= 2\int_0^1 t\sin(2\pi nt)\,dt \\ &= \frac{-1}{\pi n} \end{aligned}$$

The answer is (B).

10. The value of a number in a Fibonacci sequence is the sum of the previous two numbers in the sequence.

Use the second-order difference equation.

$$\begin{aligned} f(k) &= f(k-1) + f(k-2) \\ f(3) &= f(2) + f(1) = 1.3 + 4 \\ &= 5.3 \\ f(4) &= f(3) + f(2) = 5.3 + 1.3 \\ &= 6.6 \end{aligned}$$

The answer is (D).