free body diagram

Engineering application of Newton's laws of motion

1^{st} law – particle's acceleration is 0 unless acted upon by a force

2^{Nd} law – $a \propto F$

$a \propto \frac{1}{mass}$

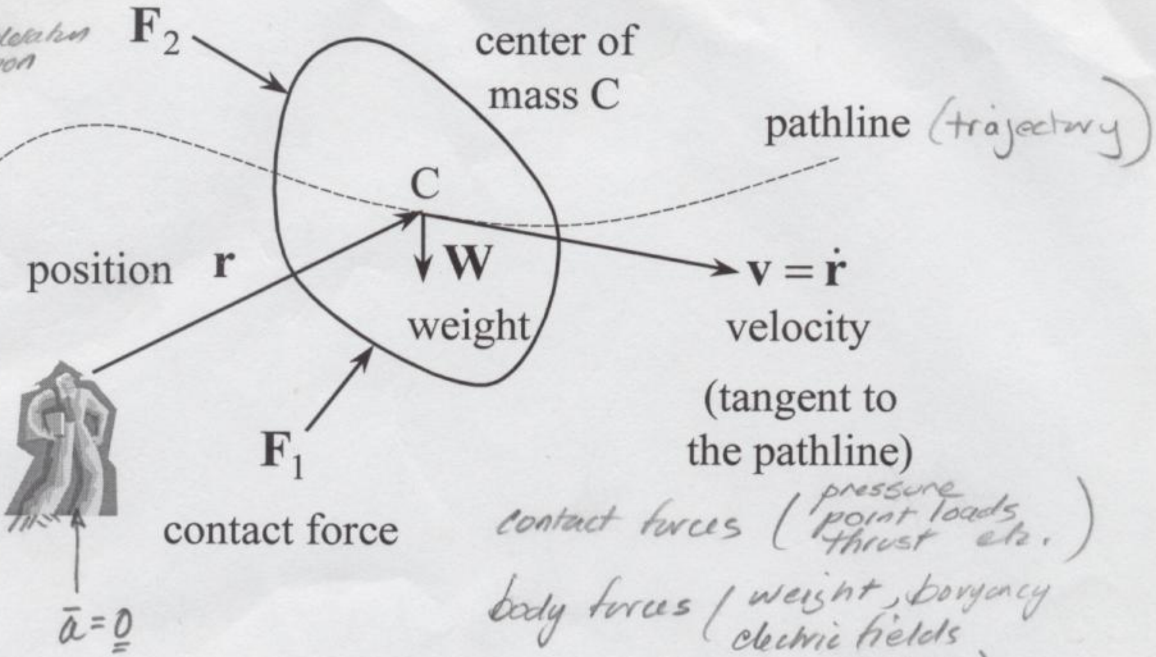
$\vec{a} \parallel \vec{F}$

3^{rd} action & reaction are equal in magnitude, opposite in sense, same line of action

gravity

$F_g \propto m_1 m_2$

$F_g \propto \frac{1}{r^2}$



contact forces (pressure, point loads, thrust etc.)

body forces (weight, boyancy, electric fields, magnetic fields)

equations of motion

$$\mathbf{F} = m\mathbf{a}$$

total force → $\mathbf{F}$; mass → m $\left(= \int \rho \, d\forall \right)$ — density (ρ), d volume ($d\forall$); $\mathbf{a}$ ← acceleration of the body's center of mass

- equation only valid in an inertial frame – inertial reference frame is a coordinate system with zero acceleration
- force obtained using a free body diagram
  - body forces (weight)
  - surface or contact forces

acceleration

$$\mathbf{a}=\dot{\mathbf{v}}=\ddot{\mathbf{r}}$$

velocity   position

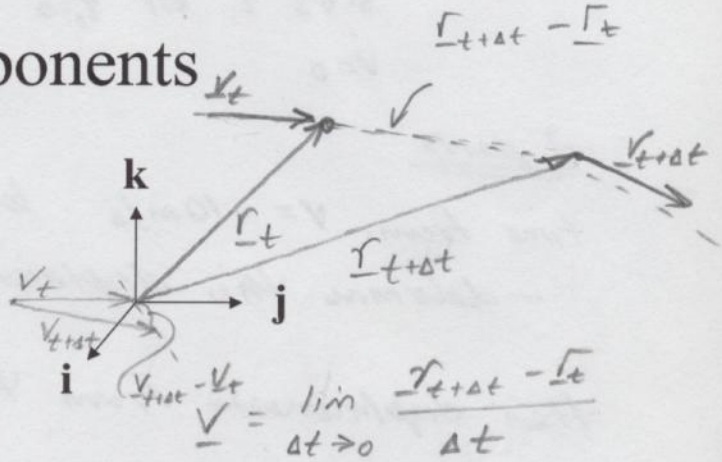
$\underline{r}$ position vector
- from inertial origin to object

$\underline{V}$ velocity vector
(vector of velocity components in inertial reference frame, at location $\underline{r}$)

$\underline{a}$ acceleration vector
(vector of accelerations components in inertial frame at location $\underline{r}$)

Cartesian components

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$
$$\mathbf{v}=\dot{x}\mathbf{i}+\dot{y}\mathbf{j}+\dot{z}\mathbf{k}$$
$$\mathbf{a}=\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k}$$



$$\underline{V}=\lim_{\Delta t\to 0}\frac{\underline{r}_{t+\Delta t}-\underline{r}_t}{\Delta t}$$

$$\underline{a}=\lim_{\Delta t\to 0}\frac{\underline{V}_{t+\Delta t}-\underline{V}_t}{\Delta t}$$

92. An object with initial velocity –10 m/s accelerates at 2 m/s². What is the total distance traveled in 15 s?

$\underline{V}_0 = -10 m/s\ \underline{i}$

$\underline{a} = 2 m/s^2\ \underline{i}$

x at 15 sec?

$\underline{a}\underline{i} = \ddot{x}\,\underline{i}$

$\frac{d\dot{x}}{dt}\underline{i} = \underline{a}\,\underline{i}$

$\int d\dot{x} = \int a\,dt$

$\dot{x} = at + c_1$

$\frac{dx}{dt} = at + c_1$

$dx = \int at + c_1\,dt$

$$\boxed{x = \frac{at^2}{2} + c_1t + c_2}$$

alternate for rectilinear motion

$x(t) = \frac{1}{2}at^2 + v_0t + x_0$

a is constant

Evaluate constants

@ $t=0$ $x=0$ (defn.)

$\dot{x} = -10 m/s$

$0 = c_2$

$-10m^2 = c_1$

displacement

$x(t) = \frac{at^2}{2} - 10 m/s\ t$

$x(15) = \frac{a}{2}(15)^2 - 10 m/s\,(15 s)$

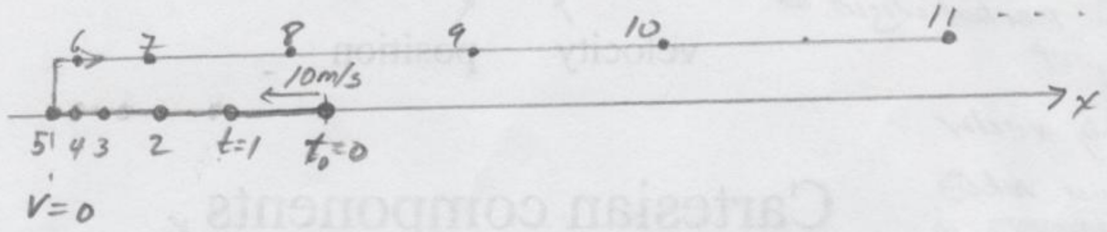
$x(15) = \frac{2m}{s^2 2}(15)(15) - (10)(15) = 75$ meters

225 – 150

$a = \ddot{x}$

$v = \dot{x}$

distance is both $\pm$ displacements



2 parts

time from $v = -10\,m/s$ to $v = 0$

- determine this displacement

then displacements from this new "t_0" to total time 15 sec.

① $x = \frac{1}{2}at^2 + v_0 t + x_0$ $\qquad x_0 = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad v_0 = -10\,m/s$

$\dot{x} = at + v_0$ $\qquad\qquad a = 2\,m/s$

Solve $\dot{x} = at + v_0$ for $\dot{x} = 0$

$0 = (2\,m/s^2)(t) - 10\,m/s$ $\qquad \frac{10\,m}{s} = 2t$

$\qquad\qquad\qquad\qquad\qquad \therefore t = 5\ sec.$

$\therefore$ Part 1 is 5 sec.

Part 2 is 10 sec.

Now determine displacements (or positions) $\qquad dist = |x - x_0|$

$x_1(5) = \frac{1}{2}(2)(5)^2 - 10(5) + 0$ $\qquad s = x_0 - x_5$

$= 25 - 50 = -25$ $\quad$ 25 meters $\qquad = 0 - (-25) = |25|$

$x_2(15) = \frac{1}{2}(2)(15)^2 - 10(15) + 0$

$= 225 - 150 = 75$ meters $\qquad s = x_5 - x_{15}$

$\qquad\qquad\qquad\qquad = -25 - 75 = |-100|$

total = 125 meters

constant acceleration

initial velocity

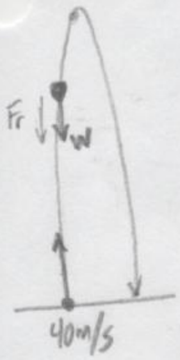
$$a = a_0$$

$$v = a_0 t + v_0 \; ; \; t = \frac{v - v_0}{a_0}$$

$$s - s_0 = \frac{a_0 t^2}{2} + v_0 t = \frac{v^2 - {v_0}^2}{2a_0} = \frac{(v + v_0)}{2} t$$

initial position

mean velocity



An object is launched upward at an initial velocity of 40 m/s. How high will it go?

$x = \frac{1}{2}at^2 + v_0 t + x_0$ $\qquad a = -g$

Solve for t when $\dot{x} = 0$

$\dot{x} = at + v_0$

$0 = (-9.8 m/s^2)(t) + 40 m/s$

$\frac{40}{9.8} = t \approx 4 sec$

$x = \frac{1}{2}at^2 + v_0 t + \cancel{x_0}^{=0}$

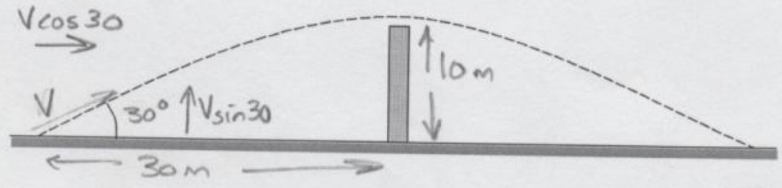
$= \frac{1}{2}(9.8)(4)^2 + (40)(4)$

$\approx 5(16) + 160$

$80 + 160 \approx 240$ meters

93. A cannon is fired at 30° above the horizon. What is the minimum exit velocity needed to clear a 10 m wall?

$U_0 = V\cos 30$
$V_0 = V\sin 30$



2 eqn.

$x = \frac{1}{2}\cancel{a_x}^{=0} t^2 + U_0 t + \cancel{x_0}^{0} = 30 \Rightarrow V\cos 30\, t = 30 \Rightarrow V = \frac{30}{\cos 30\, t}$

$y = \frac{1}{2}a_y t^2 + V_0 t + \cancel{y_0}^{0} = 10 \Rightarrow -\frac{10\,m/s^2}{5} t^2 + V\sin 30 t = 10$

9.8 is precise, simpler arithmetic usually close enough

$10 = -5t^2 + \frac{30\sin 30\, t}{\cos 30\, t} = -5t^2 + 30\tan 30$

$\therefore t^2 = \frac{10 - 30\tan 30}{-5} = 1.464 \qquad t = 1.21$

$V = \frac{30}{\cos 30\,(1.21)}$
$= 28.62\,m/sec$

polar components

– relative to cartesian system
- normal & tangential (path is known)
- radial & transverse (path unknown, angular velocities, angular momentum specified)

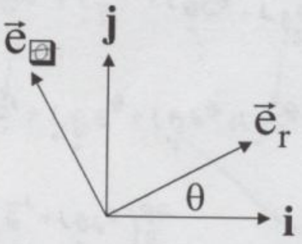
angular velocity (ω)

$$\mathbf{r} = r\mathbf{e}_r$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

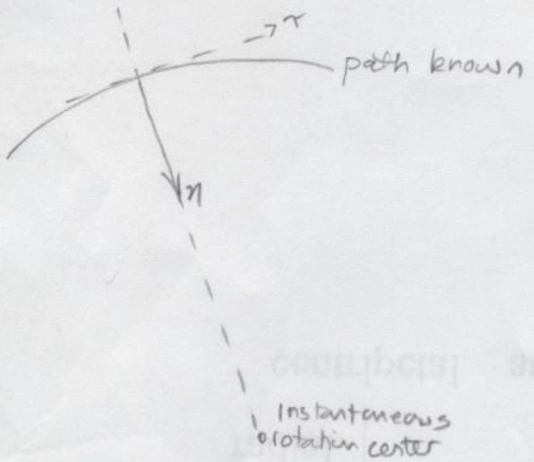
$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

radial　centripetal　angular acceleration (α)　Coriolis

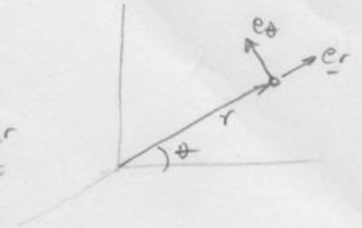


over

path unknown



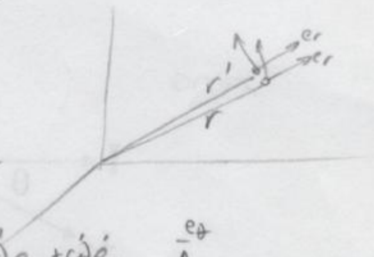
$\underline{r} = r\,\underline{e}_r$



$\underline{v} = \dot{r} = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt}$

$\underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$

$\lim \frac{\Delta\theta}{\Delta t} = \dot{\theta}$



pendulums

wheels

pulleys

roller coaster

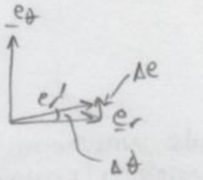
curves on roads (radius specified)

$\underline{a} = \ddot{\underline{r}} = (\dot{r}\underline{e}_r + r\dot{\theta}e_\theta)\frac{d}{dt}$

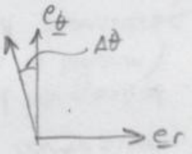
$= \ddot{r}\underline{e}_r + \dot{r}\dot{\underline{e}}_r + \dot{r}\dot{\theta}e_\theta + r\ddot{\theta}e_\theta + r\dot{\theta}\dot{e}_\theta$

$= \ddot{r}\underline{e}_r + \dot{r}\dot{\theta}e_\theta + \dot{r}\dot{\theta}e_\theta + r\ddot{\theta}e_\theta - r\dot{\theta}^2\underline{e}_r$

$= (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta$

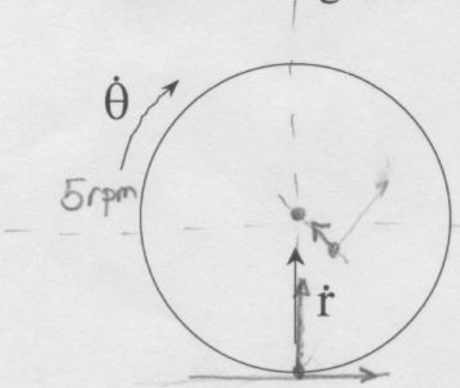


$\Delta e_r = \sin\Delta\theta \approx \Delta\theta\,\underline{e}_\theta$



$\Delta e_\theta = -\sin\Delta\theta \approx -\Delta\theta\,\underline{e}_r$

A boy walks at 4 m/s toward the center of a merry-go-round rotating clockwise at 5 rpm. Find his acceleration.



$\dot{r} = -4\,m/s\ \underline{e}_r$

$\ddot{r} = \cancel{0}$ (boy walks toward center at constant speed)

$\dot{\theta} = \frac{5(2\pi)}{60 sec}$

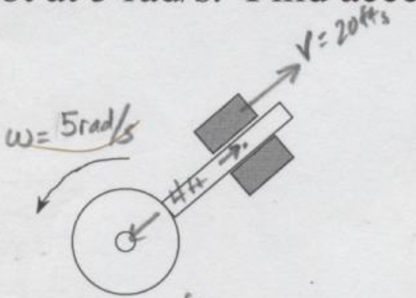
$\ddot{\theta} = 0$, merry-go-round has constant angular velocity

$$\underline{a} = \left(0 - r\left(\frac{-10\pi}{60}\right)^2\right)\underline{e_r} + \left(r\,0 + 2\left(-4\tfrac{m}{s}\right)\left(-\frac{10\pi}{60}\right) e_\theta\right)$$

$$= -r\omega^2\,\underline{e_r} + 2\dot{r}\omega\,\underline{e_\theta}$$

magnitude $|\underline{a}| = \sqrt{(r\omega^2)^2 + (2\dot{r}\omega)^2}$

A slider moves at 20 ft/s along a rod rotating about a pivot at 5 rad/s. Find acceleration 4 ft from the pivot?



$\dot{r} = 20ft/s$ $\quad\ddot{r} = 0$ (constant slider velocity)

$r = 4ft$

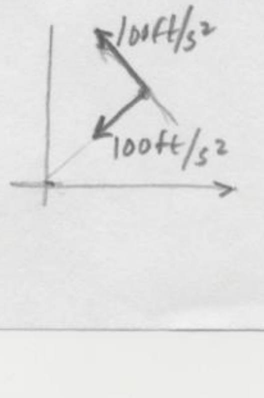
$\omega = 5\,rad/sec$

$\dot{\omega} = \ddot{\theta} = 0$ (constant rpm)

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta$

$$\underline{a} = \left(-(4ft)(5rad/s)^2\right)\underline{e}_r + \left((4ft)(0) + 2\left(\tfrac{20ft}{s}\right)(5rad/sec)\right)e_\theta$$

$$= -100\ ft/sec^2\ \underline{er} + 100\ ft/sec\ \underline{e_\theta}$$



impact & momentum

− Newtons law

$\frac{dL}{dt} = \Sigma F$

impact

- momentum constant for a central impact

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$$

system momentum in config. 1 = system momentum in config. 2 + losses of energy

- dissipation affects the approach velocity

separation, approach

$$v_{1n}' - v_{2n}' = -e\left(v_{1n} - v_{2n}\right)$$

coefficient of restitution

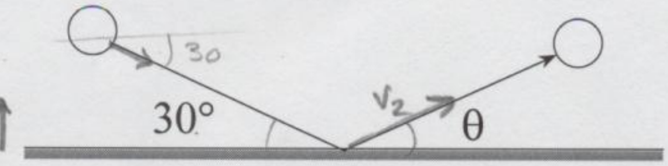
if $e = 1$ collision is elastic (No energy loss)

$e = 0$ plastic (Maximum energy loss)

99. A ball strikes a flat, horizontal surface at 30°. Find the reflection angle if the coefficient of restitution is 0.8.

$v_{1n}' = -e(v_{1n})$

approach, separation



$v_2 \sin\theta = -(-v_1 \sin 30)(0.8)$

$(-v_2 \cos\theta = v_1 \cos 30$

Solve for $v_1 = \frac{v_2 \cos\theta}{\cos 30}$

$\cancel{v_2} \sin\theta = \frac{\cancel{v_2}\cos\theta}{\cos 30} \sin 30\,(0.8)$

$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{\sin 30}{\cos 30}(0.8)$

$= 0.8 \tan 30$

$\theta = \tan^{-1}(0.8 \tan 30)$

24.79°

Mechanical problems
Work done by/against gravity.
- potential energy
Work done by/against a change in momentum
- kinetic energy
- spring
- pressure

work and kinetic energy

$F \cdot d = \text{work}$
$F \cdot \frac{d}{t} = F \cdot v = \frac{\text{work}}{t} = \text{power}$
$\int_t \text{Power}\, dt = \text{Work}$

$$W(t_1, t_2) = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v}\, dt \quad \text{work}$$

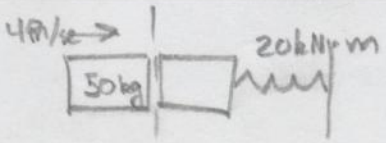
examples: $W = -mg\Delta h$ (gravitational) $\quad W = -\frac{k}{2}\left(x_2^{\,2} - x_1^{\,2}\right)$ (spring)

$$KE(t) = \frac{1}{2} m\, \mathbf{v} \cdot \mathbf{v} \quad \text{kinetic energy}$$

$$W(t_1, t_2) = KE(t_2) - KE(t_1) \quad \text{work-energy relation}$$

98. A 50 kg object moving at 40 m/s strikes a spring (k = 20kN/m). Determine the maximum deflection.

KE system just before impact

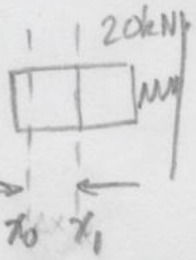


$x_0 = 0$

Spring Energy just before rebound

$KE_1 = \frac{1}{2}(50\,kg)(40\,m/s)^2 = 40\,000\ N \cdot m$

$KE_2 = 0$

W = 

$= -\frac{20kN}{2\,m}\left(x_1^2 - \cancel{x_0^2}\right)$ $=0$

$-40\,kN \cdot m = -\frac{10kN}{m} x^2$

$x^2 = 4 \quad x = \sqrt{4} = 2\,m$

A bullet of mass m strikes a stationary pendulum of mass M. Find the bullet velocity v in terms of the pendulum length L and the maximum angle θ.

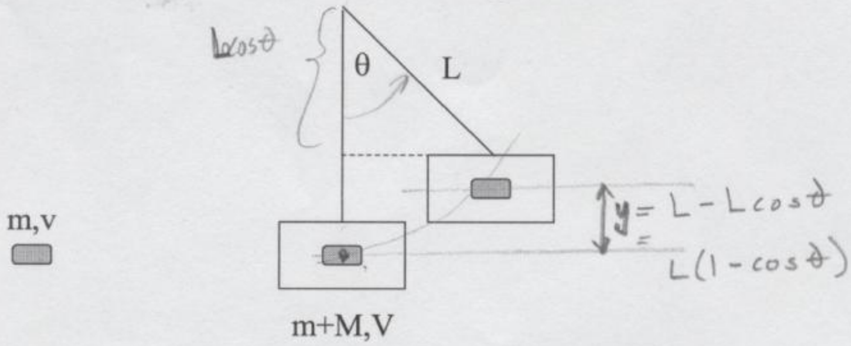
K.E. before impact

K.E. = $\frac{1}{2}mv^2$

K.E. at impact

K.E. = $\frac{1}{2}(m+M)V^2$

plastic collision all energy to combined system



K.E. at top of arc = 0

P.E. at top of arc $= (m+M)gy = (m+M)gL(1-\cos\theta)$

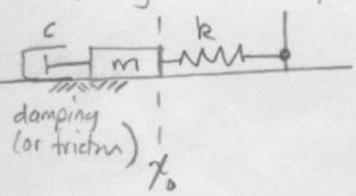
KE before impact = P.E. at top of arc

$$\cancel{\tfrac{1}{2}}\cancel{m}v^2 = \frac{2(m+M)gL(1-\cos\theta)}{m}$$

$$v^2 = 2\left(\frac{m+M}{m}\right)gL(1-\cos\theta)$$

$$v = \sqrt{2gL\left(\frac{m+M}{m}\right)(1-\cos\theta)}$$

spring-mass systems



free vibrations

damping — damping ratio ($0 \le \zeta < 1$)

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

mass — spring — natural frequency

$$x(t) = \exp(-\zeta\omega_n t)\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

Solution by Fourier methods or Undetermined coefficients

$$\omega_n = \sqrt{k/m}\,;\ \zeta = \frac{c}{2\sqrt{mk}}\,;\ \omega = \omega_n\sqrt{1-\zeta^2}$$

If No damping then
c = 0
ζ = 0
$x(t) = A\cos\omega t + B\sin\omega t$

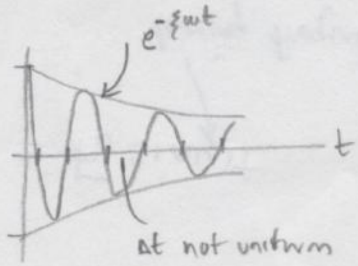
if no spring then no vibration

ζ=1 critically damped, ζ>1 over-damped

critical damped
— Vibration stops

overdamped
— Vibration never starts

underdamped
—



↓ Forced on back

8. Find the damping ratio and natural frequency of a system described by the following equation.

Forced vibration

$$\ddot{y}+8\dot{y}+25y=16\sin(\Omega t)$$

$\xi = \frac{c}{2\sqrt{mk}}$ $\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = 16\sin\Omega t$

$2\xi\omega_n = 8$ $\xi = \frac{8}{2\omega_n} = \frac{4}{5} = \frac{c}{2\sqrt{1\cdot 25}} = \frac{c}{2\cdot 5} = \frac{c}{10}$

$\omega_n^2 = 25$

$\omega_n = \sqrt{k/m} = 5$ $\xi = 4/5$

$\omega_n = 5$

$m = 1$

5-8

$\sqrt{k} = 5$

$k = 25$

Over

planar rigid body motion

$\Sigma F = m\underline{a}$

$\Sigma M = I\alpha$

Use tables for I_c (centroidal moment of inertia)

(Table 8.2, pg 344 review manual)

mass moment of inertia

$$M_c = I_c\alpha \; ; \; \alpha = \dot{\omega}$$

moment angular acceleration

This equation applies at the center of mass or a fixed pivot.

5 $\ddot{y} + 25y = 16\sin\omega t$

$y(0) = 0$

$\dot{y}(0) = 8$

homogeneous

$\ddot{y} + 25y = 0$ $\qquad r^2 + 25 = 0$

$y = A\cos\omega t + B\sin\omega t$ $\qquad \cancel{(r-5)}(r+5)$

$\omega = \sqrt{\frac{25}{1}} = \sqrt{25} = 5$

$\therefore\ y = A\cos 5t + B\sin 5t \qquad (D)$

6) if $\omega = 3$ (forced vibration)

$x =$

8) Amplitude $\qquad \ddot{y} + 8\dot{y} + 25y = 16\sin\omega t \qquad \omega = 5$ (natural frequency)

$y = A\sin 5t + B\cos 5t$

$\cancel{-25A\sin 5t} - 25B\cos 5t + 40A\cos 5t - 40B\sin 5t + \cancel{25A\sin 5t} + \cancel{25B\cos 5t} = 16\sin 5t$

$-40B\sin 5t = 16\sin 5t \qquad \therefore B = -0.4$

$40A\cos 5t = 0 \qquad A = 0$

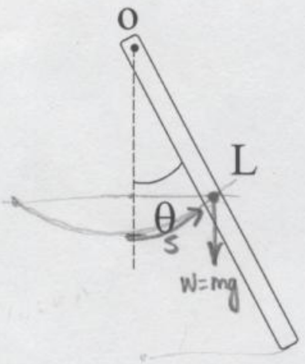
$y_p(t) = -0.4\sin 5t$

Amp $= 0.4$

What is the natural frequency of a slender rod of length L and mass m that is pinned at one end?

skip

$L(1-\cos\theta)$



$\Sigma F_\tau = -mg\sin\theta = ma_t$

$a_t = \frac{d^2 s}{dt^2}$

$s = \frac{L}{2}\theta \quad \dot{s} = \frac{L}{2}\dot{\theta} \quad \ddot{s} = \frac{L}{2}\ddot{\theta}$

FBD

$\therefore -\cancel{m}g\sin\theta = \cancel{m}\frac{L}{2}\ddot{\theta}$

$\ddot{\theta} + \frac{2g}{L}\sin\theta = 0$

for small displacements $\sin\theta \approx \theta$

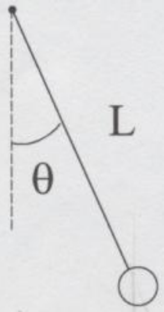
$\ddot{\theta} + \frac{2g}{L}\theta = 0$

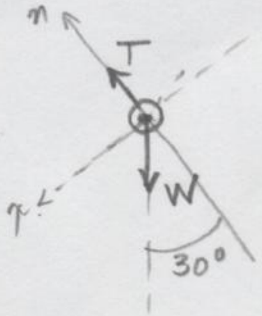
compare to free vibration

$m=1, \quad c=0, \quad k=\frac{2g}{L}$

$\omega_n = \sqrt{\frac{2g}{L}}$

97. The maximum acceleration of a simple pendulum occurs at the top of a swing. Find this acceleration for a 2 m pendulum with a maximum angle of 30°.





V = tangential velocity = 0

$\Sigma F_N = T - W\cos 30 = ma_n = \frac{mV^2}{r}$

$\Sigma F_\tau = W\sin 30 = ma_\tau = mr\alpha$

$\cancel{m}g\sin 30 = \cancel{m}a_\tau$

$a_t = g\sin 30$

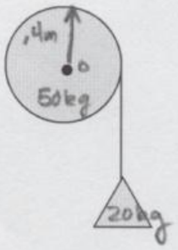
$\approx 5\ m/s^2$

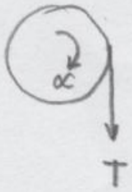
$\cancel{m}g\sin 30 = \cancel{m}r\alpha$

$\alpha = \frac{g}{r}\sin 30 = \frac{g}{L}\sin 30$

$\alpha = \frac{5 m/s^2}{L}$

96. A string is wrapped around a 50 kg cylinder of radius 0.4 m and attached to a 20 kg mass. Find the tension in the string if the cylinder rotates freely?





$+\circlearrowleft \sum M_o = I_c \alpha$

$-T(.4) = -\frac{(50)(.4)^2}{2}\alpha$

$T = \frac{50(.4)}{2}\alpha$

$\downarrow a = -r\alpha$

$\sum F = mA$

$T - mg = -r\alpha m$

$T - (20)(10) = -(.4)\alpha(20)$

solve for α

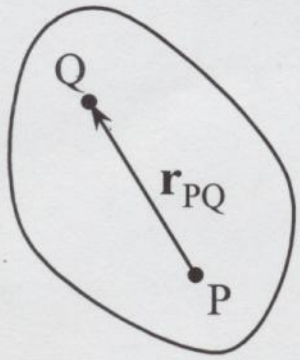
$50(\frac{.4}{2})\alpha - 200 + 8\alpha = 0$

$18\alpha = 200$

$\alpha = 11.11 \, rad/s^2$

Solve for $T = 111.1 \, N$

velocity of a rigid body



position vector

$$\mathbf{v}_Q = \mathbf{v}_P + \omega \times \mathbf{r}_{PQ}$$

velocity — angular velocity

*point on a rigid body = displacement of another point on body + angular displacement relative to reference point.

Use kinematics & trigonometry if possible

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94. Find the angular velocity of link AB if the angles are both 30°. Both links are 50 cm long, and the slider moves at velocity 20 m/s to the right.

$V_{B/A}$ must $\perp$ AB

V_C must be horizontal &

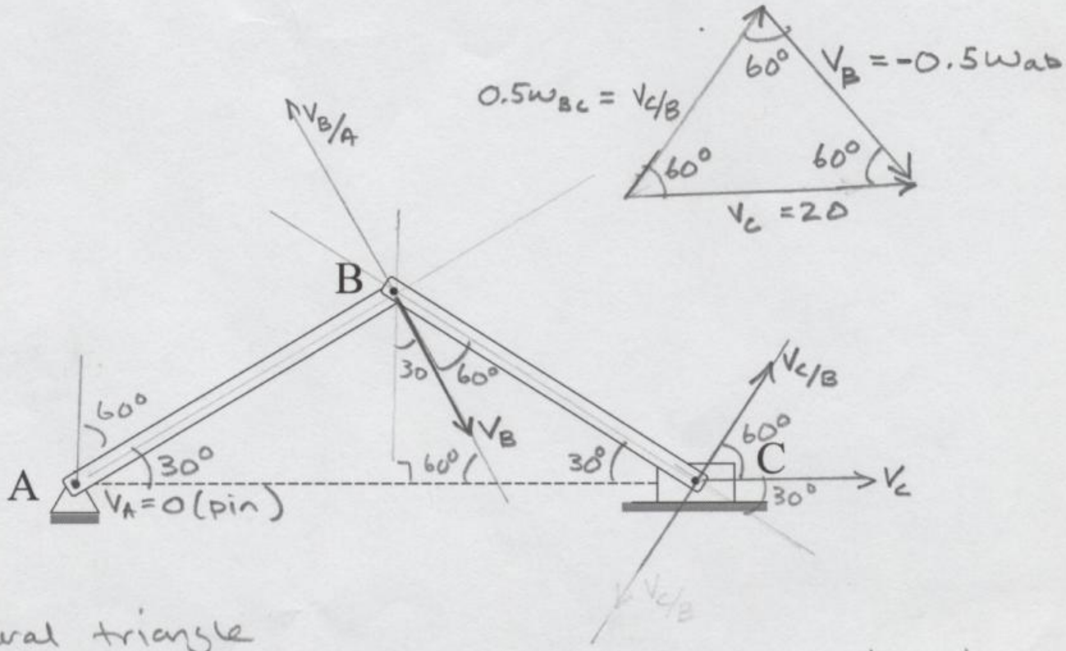
$V_{C/B}$ $\perp$ BC

$V_B = \underbrace{V_A}_{0} + V_{B/A}$

$= -\omega_{AB} r = -0.5\omega_{AB}$

$V_C = V_B + V_{C/B}$

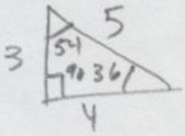
$=$



Equilateral triangle

$\therefore\ 20 = 0.5\omega_{BC} = -0.5\omega_{ab}$ $\qquad \omega_{BC} = \omega_{AB} = -40$ rad/sec

28. Find the slider velocity and the angular velocity of the 50 cm link BC. The 30 cm drive link is vertical with an angular velocity of 10 rad/s.



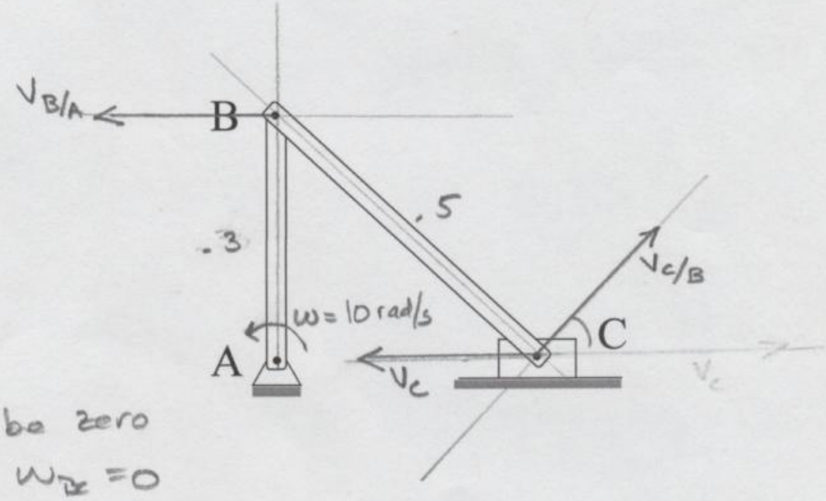
$V_B = \cancel{V_A}^{=0} + V_{B/A}$

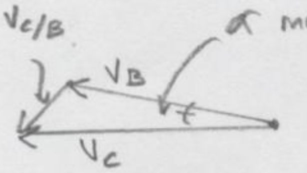
$= -0.3\omega_{AB}$

$V_C = V_B + V_{C/B}$

$=$

$-0.3\omega_{AB} + 0.5\omega_{BC}$





α must be zero

$\therefore \omega_{BC} = 0$