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Chapter VI
Newton-Raphson Method

Introduction

In Chapter II the Newton-Raphson method (Eq. 2-17), one of the most widely used method for solving implicit or nonlinear equations, was described. Most books dealing with numerical methods or numerical analysis provide additional treatment of the Newton-Raphson method. It is widely used because it converges rapidly to the soltuion. In review of the discussion of the Newton-Raphson method in Chapter II, a solution to the equation $F(x) = 0$ is obtained by the iterative formula $x^{(m+1)} = x^{(m)} - F(x^{(m)})/F'(x^{(m)})$. Mathematically the convergence of the Newton-Raphson method can be examined by using Taylor's formula to evaluate $F(x) = 0$ from the function at some iterative value $x^{(m)}$; or

$$0 = F(x) = F(x^{(m)}) + (x - x^{(m)}) F'(x^{(m)}) + (x - x^{(m)})^2 F''(\xi)/2$$

in which $\xi^{(m)}$ lies between $x^{(m)}$ and x. Solving for x gives

$$x = x^{(m)} - \frac{F(x^{(m)})}{F'(x^{(m)})} - (x - x^{(m)})^2 \frac{F''(\xi)}{2F'(x^{(m)})}$$

or

$$x = x^{(m+1)} - (x - x^{(m)})^2 \frac{F''(\xi)}{2F'(x^{(m)})}$$

Thus the error of the $(m+1)^{th}$ iterate is proportional to the square of the error in the m^{th} iterate. Convergence of this type is called quadratic convergence and in simple terms it means that each subsequent error reduction is proportional to the square of the previous error. Thus if the initial guess is 20 percent (i.e. 0.2) in error, successive iterations will produce errors of 4 percent, 1.6 percent, 0.026 percent, etc.

The Newton-Raphson method may be used to solve any of the three sets of equations describing flow in pipe networks which are discussed in Chapter IV, i.e. the equations considering (1) the flow rate in each pipe unknown, (2) the head at each junction unknown, or (3) the corrective flow rate around each loop unknown. The Newton-Raphson method requires an initial guess to the solution. Since the other two systems of equations are fewer in number than the Q-equations, the Newton-Raphson method is probably the best method to use for larger systems of equations. It requires less computer storage not only because of the fact that few simultaneous equations are included in the H- or ΔQ-equations, but also because it requires less storage for a given number of equations.

Before describing how the Newton-Raphson method can be used to solve either of the latter two systems of equations, it is necessary to extend this method from a single equation to a system of simultaneous equations. Notationally this extension is very simple. The iterative Newton-Raphson formula for a system of equations is,

$$\vec{x}^{(m+1)} = \vec{x}^{(m)} - D^{-1}\vec{F}(x^m) \quad \text{.............(6-1)}$$

The unknown vectors $\vec{x}$ and $\vec{F}$ replace the single varible x and function F and the inverse of the Jacobian, D^{-1}, replaces $1/dF/dx$ in the Newton-Raphson formula for solving a single equation. If solving the equation with the heads as the unknowns (i.e. the H-equations) the vector $\vec{x}$ becomes the vector $\vec{H}$ and if solving the equations containing the corrective loop flow rates (i.e. the ΔQ-equations) $\vec{x}$ becomes $\vec{\Delta Q}$. The individual elements for $\vec{H}$ and $\vec{\Delta Q}$ are

$$\vec{H} = \begin{vmatrix} H_1 \\ H_2 \\ \cdot \\ \cdot \\ \cdot \\ H \end{vmatrix} \text{ with the known H omitted from the vector} \quad \text{or} \quad \vec{\Delta Q} = \begin{vmatrix} \Delta Q_1 \\ \Delta Q_2 \\ \cdot \\ \cdot \\ \cdot \\ \Delta Q_L \end{vmatrix}$$

The Jacobian matrix D consists of derivative elements, individual rows of which are derivatives of that particular functional equation with respect to the variables making up the column headings. For the head equation the Jacobian is,

$$D = \begin{bmatrix} \frac{\partial F_1}{\partial H_1} & \frac{\partial F_1}{\partial H_2} & \cdots & \frac{\partial F_1}{\partial H_J} \\ \frac{\partial F_2}{\partial H_1} & \frac{\partial F_2}{\partial H_2} & \cdots & \frac{\partial F_2}{\partial H_J} \\ \vdots & & & \\ \frac{\partial F_J}{\partial H_1} & \frac{\partial F_J}{\partial H_2} & \cdots & \frac{\partial F_J}{\partial H_J} \end{bmatrix}$$

in which the row and column corresponding to the known head are omitted.

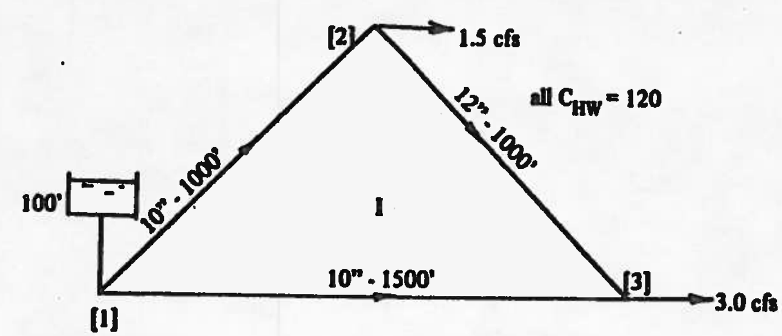
The last term $D^{-1}F$ in Eq. 6-1 contains the inverse of D, since division by a matrix is undefined. However in application of the Newton-Raphson method the inverse is never obtained and premultiplied by $\vec{F}$ as Eq. 6-1 implies. Rather the solution vector $\vec{z}$ of the linear system $D\vec{z} = \vec{F}$ is subtracted from the previous iterative vector of unknowns. Selecting the H-equations in the following notation, the Newton-Raphson iterative formula in practice becomes

$$\vec{H}^{(m+1)} = \vec{H}^{(m)} - \vec{z}^{(m)} \quad \text{..................(6-2)}$$

The equivalence of Eqs. 6-2 and 6-1 is evident since $\vec{z} = D^{-1}\vec{F}$. Since fewer computations are needed to solve the linear system $D\vec{z} = \vec{F}$ than to find the inverse D^{-1} obviously Eq. 6-2 is the form of the Newton-Raphson method used in practice. The Newton-Raphson method, therefore, obtains the solution to a system of nonlinear equations by iteratively solving a system of linear equations. In this sense it is similar to the linear theory method and can call on the same algorithm for solving a linear system of equations as does the linear theory method. It turns out, however, that the Jacobian is a symmetric matrix, and consequently an algorithm for solving a linear system of equations with a symmetric matrix might preferably be used for greater computational efficiency. The Newton-Raphson method does require a reasonably accurate initialization or it may not converge.

Head-equation

The Newton-Raphson method will be illustrated in detail by using it to solve the H-equations for the simple one loop network shown below. To



simplify the problem the Hazen-Williams equation will be used so that K and n in the exponential formula are constant. The values of K for the three pipes are: $K_{12} = 1.622$, $K_{23} = 0.667$, $K_{13} = 2.432$. The heat at junction 1 is known and equal to 100 ft. The heads H_2 and H_3 at junctions 2 and 3 are unknown and to be determined. To determine these two unknowns the H-equations will be written at junctions 2 and 3 (see Eq. 4-17 for the nature of these equations), giving

$$F_2 = -\left(\frac{H_1 - H_2}{K_{12}}\right)^{1/n_{12}} + \left(\frac{H_2 - H_3}{K_{23}}\right)^{1/n_{23}} + 1.5 = 0$$

$$F_3 = -\left(\frac{H_2 - H_3}{K_{23}}\right)^{1/n_{23}} - \left(\frac{H_1 - H_3}{K_{13}}\right)^{1/n_{13}} + 3.0 = 0$$

The equation at junction 1 is not written since $H_1 = 100$ ft is known, but might have been used in place of one of the above equations. Upon substituting known values for H_1 and the K's and n's those equations become:

$$F_2 = -\left(\frac{100 - H_2}{1.622}\right)^{0.54} + \left(\frac{H_2 - H_3}{0.667}\right)^{0.54} + 1.5 = 0$$

$$F_3 = -\left(\frac{H_2 - H_3}{0.667}\right)^{0.54} - \left(\frac{100 - H_3}{2.432}\right)^{0.54} + 3.0 = 0$$

$$\text{The Jacobian } D = \begin{vmatrix} \frac{\partial F_2}{\partial H_2} & \frac{\partial F_2}{\partial H_3} \\ \frac{\partial F_3}{\partial H_2} & \frac{\partial F_3}{\partial H_3} \end{vmatrix}$$

has the following elements

$$\frac{\partial F_2}{\partial H_2} = \frac{0.54}{K_{12}}\left(\frac{H_1 - H_2}{K_{12}}\right)^{\frac{1}{n_{12}}-1} + \frac{0.54}{K_{23}}\left(\frac{H_2 - H_3}{K_{23}}\right)^{\frac{1}{n_{23}}-1}$$

$$= 0.333\left(\frac{100 - H_2}{1.622}\right)^{-0.46} + 0.809\left(\frac{H_2 - H_3}{0.667}\right)^{-0.46}$$

$$\frac{\partial F_2}{\partial H_3} = -\frac{0.54}{K_{23}}\left(\frac{H_2 - H_3}{K_{23}}\right)^{\frac{1}{n_{23}}-1} = -0.809\left(\frac{H_2 - H_3}{0.667}\right)^{-0.46}$$

$$\frac{\partial F_3}{\partial H_2} = -\frac{0.54}{K_{23}}\left(\frac{H_2 - H_3}{K_{23}}\right)^{\frac{1}{n_{23}}-1} = -0.809\left(\frac{H_2 - H_3}{0.667}\right)^{-0.46}$$

$$\frac{\partial F_3}{\partial H_3} = \frac{0.54}{K_{23}}\left(\frac{H_2 - H_3}{K_{23}}\right)^{\frac{1}{n_{23}}-1} + \frac{0.54}{K_{13}}\left(\frac{H_1 - H_3}{K_{13}}\right)^{\frac{1}{n_{13}}-1}$$

$$= 0.809\left(\frac{H_2 - H_3}{0.667}\right)^{-0.46} + 0.222\left(\frac{100 - H_3}{2.432}\right)^{-0.46}$$

If the initialization $\vec{H} = \begin{vmatrix} H_2 \\ H_3 \end{vmatrix} = \begin{vmatrix} 95 \\ 85 \end{vmatrix}$ is used by the Newton-Raphson equation, Eq. 6-2, the solution to

$$\begin{vmatrix} 0.432 \\ -0.233 \end{vmatrix} \begin{vmatrix} -0.233 \\ 0.329 \end{vmatrix} \begin{vmatrix} z_2 \\ z_3 \end{vmatrix} = \begin{vmatrix} 3.98 \\ -3.98 \end{vmatrix}$$

is $z_2 = 4.34$ and $z_3 = -9.04$. When these are substracted from the initial guesses $H_2 = 90.66$, $H_3 = 94.04$. After completing six iterations the solution is: $H_2 = 91.45$ ft and $H_3 = 90.84$ ft. Using these heads the flow rates are computed as: $Q_{12} = 2.454$ cfs, $Q_{23} = 0.954$ cfs, and $Q_{13} = 2.046$ cfs.

A simple version of a FORTRAN computer program for solving the H-equations by the Newton-Raphson method is listed below.

```
      INTEGER N1(50),N2(50),NN(45),JB(45,7),JC(45)
      REAL H(45),D(50),Q(50),CHW(50),QJ(45),K(45),F(45,46), V(2)
C     NP-NO. PIPES, NJ-NO. OF JUNCTIONS, KNOWN-
C     JUNCTION NO. OF KNOWN HEAD, MAX-MAX. NO. OF
C     JUNCTIONS ALLOWED, ERR-ERROR PARAMETER
   98 READ(5,100,END=99) NP,NJ,KNOWN,MAX,ERR
  100 FORMAT(4I5,4F10.5)
        DO 2 I=1,NP
C     N1(I) JUNCTION NO. FROM WHICH FLOW IN PIPE COMES
C     N2(I) JUNCTION NO. TO WHICH FLOW IN PIPE GOES
C     D(I)-DIAMETER OF PIPE IN INCHES
C     CHW(I)-HAZEN-WILLIAMS COEFFICIENT FOR PIPE
C     L-LENGTH OF PIPE IN FEET
        READ(5,101) N1(I),N2(I),D(I),CHW(I),L(I)
        D(I)=D(I)/12.
    2 K(I)=4.727328*L(I)/(CHW(I)**1.85185185*D(I)**4.87037)
  101 FORMAT(2I5,5F10.5)
        NJM=NJ-1
        DO 4 J=1,NJ
C     I-JUNCTION NO.
```

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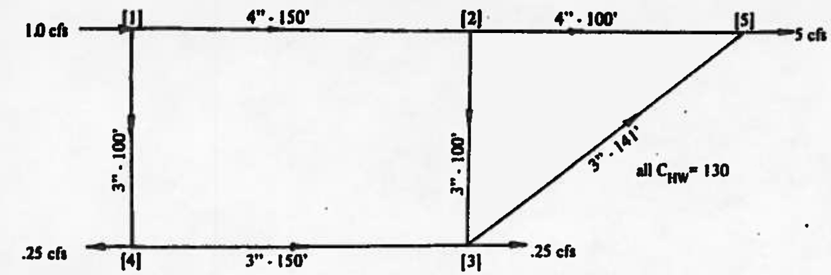
```
C  QJ(I)--EXTERNAL FLOW AT JUNCTION, MINUS IF OUT
C  FROM NETWORK
C  H(I)--ESTIMATE OF HEAD AT JUNCTION USED TO
C  INITIALIZE N-R SOLUTION
    4 READ(5,102) I,QJ(I),H(I)
  102 FORMAT(I5,5F10.5)
      DO 5 J=1,NJ
      NNP=0
      DO 6 I=1,NP
      IF(N1(I) .NE. J) GO TO 7
      NNP=NNP+1
      JB(J,NNP)=I
      GO TO 6
    7 IF(N2(I) .NE. J) GO TO 6
      NNP=NNP+1
      JB(J,NNP)=-I
    6 CONTINUE
    5 NN(J)=NNP
      NCT=0
   20 SUM=0.
      JE=0
      DO 10 J=1,NJ
      IF(J .EQ. KNOWN) GO TO 10
      JE=JE+1
      JJE=J-JE
      DO 15 JJ=1,NJ
   15 F(JE,JJ)=0.
      NNP=NN(J)
      DO 11 KK=1,NNP
      II=JB(J,KK)
      I=ABS(II)
      I1=N1(I)
      I2=N2(I)
      ARG=(H(I1)-H(I2))/K(I)
      FAC=II/I
      FAC5=.54*FAC
      ARGE=ARG**.54
   13 F(JE,NJ)=F(JE,NJ)+ARGE*FAC
      IF(I1 .EQ. KNOWN) GO TO 14
      IF(I1 .GT. KNOWN) I1=I1-1
      F(JE,I1)=F(JE,I1)+FAC5*ARGE/(K(I)*ARG)
   14 IF(I2 .EQ. KNOWN) GO TO 11
      IF(I2 .GT. KNOWN) I2=I2-1
      F(JE,I2)=F(JE,I2)-FAC5*ARGE/(K(I)*ARG)
   11 CONTINUE
      F(JE,NJ)=F(JE,NJ)-QJ(J)
   10 CONTINUE
      V(1)=4.
      CALL GJR(F,46,45,N,JM,NJ,$97,JC,V)
      JE=0
      DO 24 J=1,NJ
      IF(J .EQ. KNOWN) GO TO 24
      JE=JE+1
      DIF=F(JE,NJ)
      SUM=SUM+ABS(DIF)
      H(J)=H(J)-DIF
   24 CONTINUE
      DO 25 I=1,NP
      I1=N1(I)
      I2=N2(I)
      IF(H(I1) .GT. H(I2)) GO TO 25
      WRITE(6,225) I,I1,I2
  225 FORMAT(' FLOW HAS REVERSED IN PIPE',3I5)
      N1(I)=I2
      N2(I)=I1
      II=I1
   28 NNP=NN(II)
      DO 26 KK=1,NNP
      IF(IABS(JB(II,KK)) .NE. I) GO TO 26
      JB(II,KK)=-JB(II,KK)
      GO TO 27
   26 CONTINUE
   27 IF(II .EQ. I2) GO TO 25
      II=I2
      GO TO 28
   25 CONTINUE
      NCT=NCT+1
      WRITE(6,108) NCT,SUM
  108 FORMAT(' NCT=',I5,'  SUM=',E12.5)
      IF(NCT .LT. MAX .AND. SUM .GT. ERR) GO TO 20
      WRITE(6,103)(H(J),J=1,NJ)
  103 FORMAT('  HEADS AT JUNCTIONS',/,(1H  ,13F10.3))
      WRITE(6,104)
  104 FORMAT('  FROM       TO     DIAMETER LENGTH
     $CHW     FLOWRATE HEAD LOSS    HEADS AT
     $JUNCTIONS')
      DO 17 I=1,NP
      I1=N1(I)
      I2=N2(I)
      DH=H(I1)-H(I2)
      Q(I)=(DH/K(I))**.54
   19 WRITE(6,105) I1,I2,D(I),L(I),CHW(I),Q(I),DH,H(I1),H(I2)
  105 FORMAT(2I5,2F10.1,F10.0,4F10.3)
   17 CONTINUE
      GO TO 98
   97 WRITE(6,306) JC(1),V
  306 FORMAT('  OVERFLOW OCCURRED--CHECK SPEC.
     $FOR REDUNDANT EQ. RESULTING IN SINGULAR
     $MATRIX'  ,I5,2F10.2)
   99 STOP
      END
```

Example Problems Based on the H-Equations

1. Solve the network below for the heads at each junction. From these computed heads compute the flow rates in each pipe. The head at junction 3 is to be maintained at 100 ft.



Solution:

The system of equations for this network is:

$$F_1 = \left(\frac{H_1 - H_2}{18.19}\right)^{0.54} + \left(\frac{H_1 - H_4}{49.22}\right)^{0.54} - 1.0 = 0$$

$$F_2 = -\left(\frac{H_1 - H_2}{18.19}\right)^{0.54} + \left(\frac{H_2 - H_5}{49.22}\right)^{0.54} + \left(\frac{H_2 - H_3}{12.12}\right)^{0.54} = 0$$

$$F_4 = -\left(\frac{H_1 - H_4}{49.22}\right)^{0.54} + \left(\frac{H_4 - 100}{73.833}\right) + 0.25 = 0$$

$$F_5 = -\left(\frac{H_2 - H_5}{12.12}\right)^{0.54} - \left(\frac{100 - H_5}{69.40}\right)^{0.54} + 0.5 = 0$$

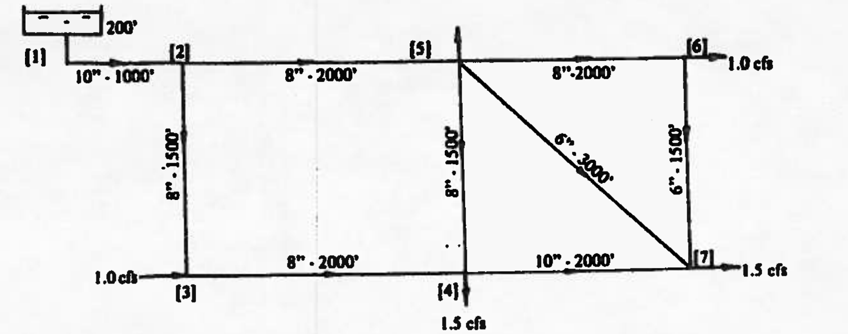
If as initialization, $H_1 = 115'$, $H_2 = 103'$, $H_4 = 103'$, and $H_5 = 99'$, then

$D\vec{z} = \vec{F}$ becomes

$$\begin{bmatrix} 0.057 & -0.036 & -0.021 & 0.0 \\ -0.036 & -0.150 & 0.0 & -0.074 \\ -0.021 & 0.0 & 0.53 & 0.0 \\ 0.0 & -0.074 & 0.0 & -0.129 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} 0.266 \\ -0.029 \\ -0.039 \\ -0.151 \end{bmatrix}$$

A solution of which gives: $z_1 = 5.78$, $z_2 = 0.86$, $z_4 = 1.55$, and $z_5 = -0.67$ producing $H_1 = 109.22$, $H_2 = 102.14$, $H_4 = 101.45$, $H_5 = 99.67$. After two additional iterations: $H_1 = 109.74'$, $H_2 = 102.19'$, $H_3 = 100'$, $H_4 = 101.63'$, $H_5 = 99.58'$, and $Q_{12} = 0.622$ cfs, $Q_{14} = 0.378$ cfs, $Q_{43} = 0.128$ cfs, $Q_{25} = 0.436$ cfs, $Q_{23} = 0.186$ cfs, $Q_{35} = 0.064$ cfs.

2. Obtain the elevation of the HGL at each junction and the flow in each pipe in the network below



Solution:

The system of equations is:

$$F_2 = -\left(\frac{200 - H_2}{1.622}\right)^{0.54} + \left(\frac{H_2 - H_5}{9.615}\right)^{0.54} + \left(\frac{H_2 - H_3}{7.211}\right)^{0.54} = 0$$

$$F_3 = -\left(\frac{H_2 - H_3}{7.211}\right)^{0.54} + \left(\frac{H_3 - H_4}{9.615}\right)^{0.54} - 1.0 = 0$$

$$F_4 = -\left(\frac{H_3 - H_4}{9.615}\right)^{0.54} - \left(\frac{H_5 - H_4}{7.211}\right)^{0.54} + \left(\frac{H_4 - H_7}{3.243}\right)^{0.54} + 0.5 = 0$$

$$F_5 = -\left(\frac{H_2 - H_5}{9.615}\right)^{0.54} + \left(\frac{H_5 - H_6}{9.615}\right)^{0.54} + \left(\frac{H_5 - H_4}{7.211}\right)^{0.54} + \left(\frac{H_5 - H_7}{58.55}\right)^{0.54} + 0.5 = 0$$

$$F_6 = -\left(\frac{H_5 - H_6}{9.615}\right)^{0.54} + \left(\frac{H_6 - H_7}{29.27}\right)^{0.54} + 1.0 = 0$$

$$F_7 = -\left(\frac{H_6 - H_7}{3.243}\right)^{0.54} - \left(\frac{H_5 - H_7}{58.55}\right)^{0.54} - \left(\frac{H_4 - H_7}{3.243}\right)^{0.54} + 1.5 = 0$$

Solving these equations by the Newton-Raphson method gives: $H_2 = 183.5'$, $H_3 = 174.1'$, $H_4 = 134.2'$, $H_5 = 137.0'$, $H_6 = 128.9'$, $H_7 = 129.2'$, and the flows are: $Q_{12} = 3.5$, $Q_{25} = 2.344$, $Q_{23} = 1.156$, $Q_{34} = 2.156$, $Q_{54} = 0.599$, $Q_{57} = 0.335$, $Q_{56} = 0.910$, $Q_{47} = 1.255$, $Q_{76} = 0.090$ all in cfs.

Corrective Flow Rate Equations

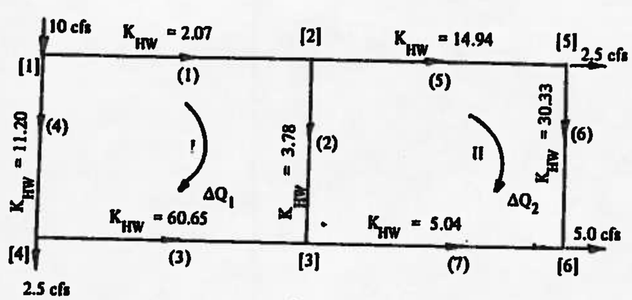
In applying the Newton-Raphson method to solve the system of equations which considers the corrective flow rates in each loop as the unknowns, the same procedure is followed except the unknown vector in Eq. 6-1 is ΔQ and the Jacobian is,

$$D = \begin{vmatrix} \frac{\partial F_1}{\partial \Delta Q_1} & \frac{\partial F_1}{\partial \Delta Q_2} & \cdots & \frac{\partial F_L}{\partial \Delta Q_L} \\ \frac{\partial F_2}{\partial \Delta Q_1} & \frac{\partial F_2}{\partial \Delta Q_2} & \cdots & \frac{\partial F_L}{\partial \Delta Q_L} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_L}{\partial \Delta Q_1} & \frac{\partial F_L}{\partial \Delta Q_2} & \cdots & \frac{\partial F_L}{\partial \Delta Q_L} \end{vmatrix}$$

With $\vec{z}$ defined as the solution to $D^{(m)}\vec{z}^{(m)} = \vec{F}^{(m)}$ as previous where F now becomes the equations evaluated from the m^{th} iterative values of $\Delta Q^{(m)}$, the Newton-Raphson method becomes

$$\Delta Q^{(m+1)} = \Delta Q^{(m)} - \vec{z} \quad \text{.................(6-3)}$$

The Newton-Raphson method will be illustrated in solving the ΔQ-equations by giving the details involved in solving the two loop network shown below. The table to the right of the sketch contains values of a possible initial flow rate in each pipe of the network which satisfy the junction continuity equations and which will be used to define the corrective loop flow rate equations. To simplify the illustration the Hazen-Williams formula will be used.



Pipe No.	Q_o (cfs)
1	5.0
2	2.0
3	2.5
4	5.0
5	3.0
6	0.5
7	4.5

Since there are two loops, there are two corrective flow rates, ΔQ_1 and ΔQ_2 which are unknown. Writing the energy equation around these two loops (with head losses in the clockwise direction as positive), gives the following two simultaneous equations to solve for these two unknowns.

$$F_1 = 2.07\,(5+\Delta Q_1)^{1.85} + 3.78\,(2+\Delta Q_1-\Delta Q_2)^{1.85} - 60.65\,(2.5-\Delta Q_1)^{1.85} - 11.20\,(5-\Delta Q_1)^{1.85} = 0$$

$$F_2 = 14.94\,(3+\Delta Q_2)^{1.85} + 30.33\,(.5+\Delta Q_2)^{1.85} - 5.04\,(4.5-\Delta Q_2)^{1.85} - 3.78\,(-2-\Delta Q_2+\Delta Q_1)^{1.85} = 0$$

The four elements of the Jacobian are:

$$\frac{\partial F_1}{\partial \Delta Q_1} = 3.83\,(5+\Delta Q_1)^{0.85} + 6.99\,(2+\Delta Q_1-\Delta Q_2)^{0.85} + 112.20\,(2.5+\Delta Q_1)^{0.85} + 20.72\,(5+\Delta Q_1)^{0.85}$$

$$\frac{\partial F_1}{\partial \Delta Q_2} = -6.99\,(2+\Delta Q_1-\Delta Q_2)^{0.85}$$

$$\frac{\partial F_2}{\partial \Delta Q_1} = -6.99\,(2+\Delta Q_2-\Delta Q_1)^{0.85}$$

$$\frac{\partial F_2}{\partial \Delta Q_2} = 27.64\,(3+\Delta Q_2)^{0.85} + 56.11\,(.5+\Delta Q_2)^{0.85} + 9.32\,(4.5-\Delta Q_2)^{0.85} + 6.99\,(2-\Delta Q_2+\Delta Q_1)^{0.85}$$

Note that the Jacobian is a symmetric matrix as was the Jacobian from the H-equations.

Starting the Newton iteration with $\Delta Q_1 = \Delta Q_2 = 0$, then

$$\begin{bmatrix} 353.5 & -12.6 \\ -12.6 & 147.5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -497. \\ 27.4 \end{bmatrix}$$

results upon evaluating the functions F_1 and F_2 and the elements in the Jacobian. Solution of this system produces $z_1 = -\Delta Q_1^{(1)} = -1.399$, $z_2 = -\Delta Q_2^{(0)} = 0.0654$. For the second Newton-Raphson iteration,

$$\begin{bmatrix} 222.5 & -20.2 \\ -20.2 & 151.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 90.3 \\ -5.46 \end{bmatrix}$$

which gives $z_1 = -0.414$, $z_2 = -0.0914$, and $\Delta Q_1^{(2)} = 1.813$ and $\Delta Q_2^{(2)} = 0.026$. After two additional iterations changes in corrective flow rates are insignificant and the solution is accepted as $\Delta Q_1 = 1.866$ and $\Delta Q_2 = 0.0331$. The flow rates in each pipe can now be computed by adding these corrective flow rates to the initially assumed values which satisfy the junction continuity equations. From these flow rates, the frictional head losses can be computed. These results are:

Pipe No.	1	2	3	4	5	6	7
Q (cfs)	6.87	3.83	0.634	3.134	3.03	0.533	4.47
h_f (ft)	73.5	45.5	26.1	92.9	116.6	9.46	80.6

A listing is given below of a simplifed FORTRAN computer program for carrying out the computations described above for solving the ΔQ-equations. Below this listing is a listing of the input data required to solve example problem 1 below by this program.

FORTRAN program for solving the corrective loop flow rate equation by the Newton-Raphson method for analyzing pipe networks.

```
      REAL D(50),L(50),K(50),CHW(50),QI(50),DQ(50),DR(50,
     $51),V(2),A(3),B(3),HO(3),DELEV(3)
      INTEGER LP(42,7),NN(42),LO(3),LLP(3),LOP(50,4),
     $NLOP(50)
C     NP-NO. OF PIPES, NL-NO. OF LOOPS, MAX-MAX. NO.
C     OF ITERATIONS ALLOWED, NPUMP-NO. OF PUMPS,
C     NSL-NO. OF PSEUDO LOOPS.
   98 READ(5,100,END=99) NP,NL,MAX,NPUMP,NSL
      NLP=NL+1
  100 FORMAT(10I5)
C     II-PIPE NO., D(II)-DIAMETER OF PIPE IN INCHES, L(II)-
C     LENGTH OF PIPE IN FT, CHW(II)-HAZEN WILLIAMS COEF.,
C     QI(II)-INITIAL FLOW SATISFYING CONTINUITY EQS.
      DO 1 I=1,NP
      READ(5,101) II,D(II),L(II),CHW(II),QI(II)
      D(II)=D(II)/12.
    1 K(II)=4.77*L(II)/(D(II)**4.87*CHW(II)**1.852)
  101 FORMAT(I5,7F10.5).
      DO 2 I=1,NL
      DQ(I)=0.
C     NNP IS THE NUMBER OF PIPES AROUND THE LOOP
C     LP(I,J) ARE THE PIPE NO. AROUND THE LOOP. IF
C     COUNTERCLOCKWISE THIS NO. IS -
      READ(5,100) NNP,(LP(I,J),J=1,NNP)
    2 NN(I)=NNP
C     LLP(I)-LINE NO. CONTAINING PUMP (MINUS IF
C     COUNTERCLOCKWISE, A, B, HO-PUMP CHAR
      IF(NPUMP .EQ. 0) GO TO 30
      DO 31 I=1,NPUMP
   31 READ(5,101)  LLP(I),A(I),B(I),HO(I)
C     LO(I)-NO. OF PSEUDO LOOP, DELEV(I)-ELEV. DIFF. ON
C     RIGHT OF = IN ENERGY EQ.
      DO 32 I=1,NSL
   32 READ(5,101) LO(I),DELEV(I)
   30 DO 50 I=1,NP
      NLO=0
      DO 51 L1=1,NL
      NNP=NN(L1)
      DO 51 KK=1,NNP
      IF (IABS(LP(L1,KK)) .NE. I) GO TO 51
      NLO=NLO+1
      LOP(I,NLO)=L1*LP(L1,KK)/I
   51 CONTINUE
   50 NLOP(I)=NLO
      NCT=0
   10 SUM=0.
      DO 3 I=1,NL
      DO 12 J=1,NLP
   12 DR(I,J)=0.
      NNP=NN(I)
      DO 3 J=1,NNP
      IJ=LP(I,J)
      IIJ=IABS(IJ)
      Q=QI(IIJ)
      NLO=NLOP(IIJ)
      DO 52 KK=1,NLO
      L1=LOP(IIJ,KK)
      LL=IABS(L1)
   52 Q=Q+FLOAT(L1/LL)*DQ(LL)
      QE=ABS(Q)**.852
      FAC=IJ/IIJ
      DR(I,NLP)=DR(I,NLP)+FAC*K(IIJ)*Q*QE
      DO 53 KK=1,NLO
      L1=LOP(IIJ,KK)
      LL=IABS(L1)
   53 DR(I,LL)=DR(I,LL)+FAC*FLOAT(L1/LL)*1.852*K(IIJ)*QE
    3 CONTINUE
      IF(NSL .EQ. 0) GO TO 40
      DO 33 I=1,NSL
      II=LO(I)
      DR(II,NLP)=DR(II,NLP)-DELEV(I)
      NNP=NN(II)
      DO 33 IK=1,NPUMP
      IL=IABS(LLP(IK))
      DO 33 KK=1,NNP
      IF(IL .NE. IABS(LP(II,KK))) GO TO 33
      Q=ABS(FLOAT(LLP(IK)/IL)*QI(IL)+DQ(II)
      HP=(A(IK)*Q+B(IK))*Q+HO(IK)
      IF(LLP(IK) .LT. 0) GO TO 35
      DR(II,NLP)=DR(II,NLP)-HP
      DR(II,II )=DR(II,II )+2.*A(IK)*Q+B(IK)
      GO TO 33
   35 DR(II,NLP)=DR(II,NLP)+HP
      DR(II,II )=DR(II,II )-2.*A(IK)*Q+B(IK)
   33 CONTINUE
   40 V(1)=4.
      CALL GJR(DR,51,50,NL,NLP,$98,D,V)
      DO 7 I=1,NL
      SUM=SUM+ABS(DR(I,NLP))
    7 DQ(I)=DQ(I)-DR(I,NLP)
      NCT=NCT+1
      WRITE(6,202) NCT,SUM,(DQ(I),I=1,NL)
```

```
202 FORMAT(' NCT='I2,(12F10.3))
    IF(SUM .GT. .001 .AND. NCT .LT. MAX) GO TO 10
    IF(NCT .EQ. MAX) WRITE(6,102) NCT,SUM
102 FORMAT(' DID NOT CONVERGE --NCT=',I5,' SUM=',
   $E12.5)
    DO 8 I=1,NL
    NNP=NN(I)
    DO 8 J=1,NNP
    IJ=LP(I,J)
    IIJ=IABS(IJ)
  8 QI(IIJ)=QI(IIJ)+FLOAT(IJ/IIJ)*DQ(I)
    WRITE(6,105) (QI(I),I=1,NP)
105 FORMAT(' FLOWRATES IN PIPES',/,(1H ,13F10.3))
    DO 9 I=1,NP
  9 D(I)=K(I)*ABS(QI(I))**1.852
    WRITE(6,106) (D(I),I=1,NP)
106 FORMAT(' HEAD LOSSES IN PIPES',/,(1H ,13F10.3))
    GO TO 98
 99 STOP
    END
```

The input data cards needed in order to use the previously listed program follow:

```
16    7    15    1    1
 1   12.   1000.   130.   2.5  }
 2    8.   1000.   130.   1.0  }
 3   10.   1200.   130.    .75 }
 4   10.    900.   130.   2.0  }
 5   10.   1000.   130.    .5  }  Pipe diameters,
 6   10.   2500.   130.   1.0  }  lengths, and coef.
 7    8.   1000.   130.    .25 }  C_HW and initial
 8   10.   1400.   130.    .75 }  flow rate Q_o
 9   12.   2000.   130.    .5  }
10    8.   1200.   130.   2.   }
11    8.   1300.   130.   1.   }
12   10.   1600.   130.    .5  }
13   10.   1000.   130.   1.   }
14   10.   1300.   130.   1.   }
15   12.    900.   130.   2.25 }
16   12.    500.   130.   2.5  }
 4    2    3   -4   -5  }
 4   -7    8  -15   -3  }
 4    2   -7    8   -6  }
 3   11   12   -9       }  Pipe Nos. in Loops
 3    9  -10   -8       }
 3   10   13  -14       }
 4   16    7   -2   -1  }
-1   -2.505   16.707   155.286   Pump characteristics
 7   150.          Pseudo Loop No. & Δ Elev.
```

Input data needed for example problem 1 which follows:

```
8    3   10
1   12.   2000.   95.   5.   }
2   10.   1500.   95.   2.   }
3    6.   2000.   95.   2.5  }
4    8.   1500.   95.   5.   }  Pipe Information
5    8.   2000.   95.   3.   }
6    6.   1000.   95.    .5  }
7   10.   2000.   95.   4.5  }
8   10.   3000.   95.   2.   }
3    1    2   -8        }
4    5    6   -7   -2   }  Pipes in Loop
3    8   -3   -4        }
```

The input data needed for a more extensive computer program using this method is given in Appendix C. This latter computer program permits pumps, reservoirs, etc., to exist in the network does not require information about pipe numbers in each loop, generates its own initialization and for large networks is efficient in computer time and storage since it orders loops in such a manner so that all nonzero elements of the Jacobian are concentrated in a band near the diagonal elements.

Including Pressure Reducing Valves in Analyses Based on the ΔQ Equations

An important fact which allows pipe networks to be analyzed by the system of ΔQ equations is that junction continuity is satisfied regardless of what values each corrective loop flow rate ΔQ takes on provided the initializing flows, Q_{oi}, satisfy all junction continuity equations. Thus after establishing initial flows which satisfy continuity at each junction no additional equations, or provisions, need be incorporated in the solution procedure to insure that the solution satisfies the conservation of mass principle. Special procedures must be followed as described below, however, if PRV's are present in the network and the analysis is to be based on solving a system of equations in which corrective flow rates (the ΔQ's) are the only unknowns.

For networks without PRV's each equation in the system of ΔQ equations is obtained by summing the head losses around the loop through which that particular corrective flow rate ΔQ is thought of as circulating around. That is the m^{th} equation will contain the m^{th} ΔQ in each term of the equation. This procedure always automatically produces exactly as many independent equations as there are unknown ΔQ's for which a solution is sought. When PRV's are present this