Polubarinova-Kochina, 1962
Theory of Groundwater Movement

that significant differences in the velocity are possible under influence of waviness. In Fig. 366 are given the lines of equal head and equal (reduced) flow rate.

§3. One-dimensional Vertical Flow for Constant Operating Head.

We examine the flow of water along a vertical line in the soil, considering the seepage coefficient k and the porosity m to be constant quantities.

We take the general equation of motion with the inertia terms (the y-axis is directed downward)

$$\frac{\partial v_y}{\partial t} + \frac{\partial v_y}{\partial y} v_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g - \frac{mg}{k} v_y . \tag{3.1}$$

We drop the rest of the equation, since we consider one-dimensional flow, parallel to the y-axis. Therefore, the equation of continuity takes on the form:

$$\frac{\partial v_y}{\partial y} = 0 .$$

This shows that v_y only depends upon time (which was noticed by N. N. Pavlovsky). Therefore (3.1) may be rewritten, introducing instead of v_y the seepage velocity v through

$$v_y = \frac{1}{m} v ,$$

in the following form

$$\frac{1}{m}\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g - \frac{g}{k} v . \tag{3.2}$$

But we have shown in Chapter I (and also in Chapter II), that the term $\frac{1}{m}\frac{\partial v}{\partial t}$ may be neglected in all practically interesting cases. If we introduce the quantity

$$h = \frac{p}{\rho g} - y , \tag{3.3}$$

we come back to Darcy's law

$$v = -k\frac{dh}{dy} , \tag{3.4}$$

valid, consequently, even in the case of unsteady flows. We assume now that water percolates in the soil under a constant head H (Fig. 368), and that at the moment t it has seeped to the depth y_0 from the boundary of the reservoir. The y-axis will be oriented vertically downward. Let $y_0 = 0$ at the origin of time $t = 0$.

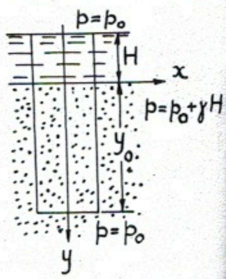

Fig. 368

Since $v = v(t)$ depends only upon time and not upon y, h is a linear function of y:

$$h = a(t)y + b(t) .$$

For $y = 0$, the head is equal to H.

$$h(0) = b(t) = H . \tag{3.5}$$

For $y = y_0$, considering the atmospheric pressure to be zero, we have after (3.3):

$$h(y_0) = -h_k - y_0 = ay_0 + b \,, \tag{3.6}$$

where by h_k the height of capillary rise is designated.

$$h_k = -\frac{p_k}{\rho g} \,.$$

Therefore for "a" we may write this expression [after (3.5) and (3.6)]:

$$a = \frac{dh}{dy} = -\frac{h_k + y_0 + H}{y_0} \,. \tag{3.7}$$

Seepage velocity v and derivative $\frac{dy_0}{dt}$ are related as

$$v = m\frac{dy_0}{dt} \,. \tag{3.8}$$

Comparison of (3.7) and (3.8) leads to the equation for y_0:

$$m\frac{dy_0}{dt} = k\frac{H + h_k + y_0}{y_0} \,. \tag{3.9}$$

We notice that according to the obtained equation, the capillary height is added to the acting head, as if instead of the head H we had the head $H + h_k$.

To integrate equation (3.9) it suffices to write it in the form

$$\frac{y_0 dy_0}{y_0 + H + h_k} = \frac{k}{m}dt$$

or

$$dy_0 - \frac{H + h_k}{y_0 + H + h_k}dy_0 = \frac{k}{m}dt \,,$$

after which we find, considering that $y_0 = 0$ for $t = 0$:

$$\frac{y_0}{H + h_k} - \ln\left(1 + \frac{y_0}{H + h_k}\right) = \frac{kt}{m(H + h_k)} \,. \tag{3.10}$$

Introducing the dimensionless quantities:

$$\left\{\begin{aligned} \frac{y_0}{H + h_k} &= \eta \,, \\ \frac{kt}{m(H + h_k)} &= \tau \,, \end{aligned}\right. \tag{3.11}$$

we rewrite (3.10) in the following form:

$$\tau = \eta - \ln(1 + \eta) \,. \tag{3.12}$$

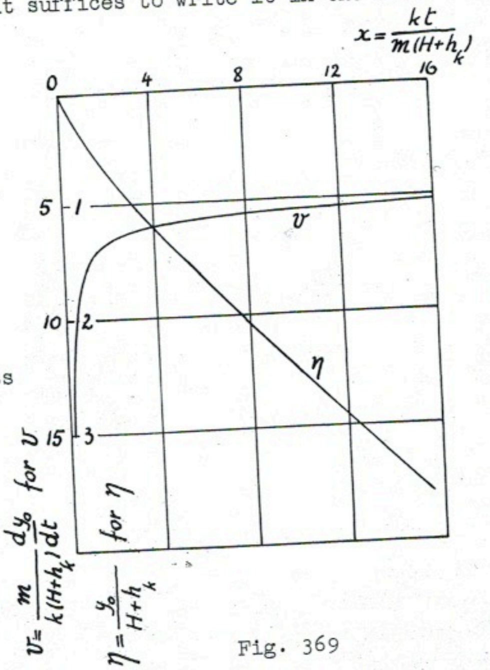

Fig. 369

The graph of the dependence of η on τ is given in Fig. 369. Also, the dependence of v upon τ is given there.

For small values of η it is possible to carry out calculation of equation (3.12), if we develop $\ell n(1+\eta)$ into a power series. We obtain:

$$\tau = \frac{\eta^2}{2} - \frac{\eta^3}{3} - \frac{\eta^4}{4} - \ldots = \frac{\eta^2}{2}\left(1 - \frac{2}{3}\eta + \frac{1}{2}\eta^2 + \ldots\right), \tag{3.13}$$

from where we find, extracting the quadratic root:

$$\sqrt{2\tau} = \eta\left(1 - \frac{2}{3}\eta + \frac{1}{2}\eta^2 + \ldots\right)^{\frac{1}{2}} = \eta - \frac{1}{3}\eta^2 + \frac{7}{36}\eta^8 + \ldots \tag{3.14}$$

We put

$$\sqrt{2\tau} = \mu \tag{3.15}$$

and express η in the form of a power series of μ:

$$\eta = \mu + A\mu^2 + B\mu^3 + \ldots, \tag{3.16}$$

where A, B, – unknown coefficients of the flow. Inserting the expression (3.16) into the series (3.14) and equating the coefficients for the same powers of μ, we find

$$\eta = \mu + \frac{1}{3}\mu^2 + \frac{1}{36}\mu^3 + \ldots = \sqrt{2\tau} + \frac{2}{3}\tau + \frac{\sqrt{2}}{18}\tau^{3/2} + \ldots \tag{3.17}$$

§4. Seepage for a Given Constant Flow Rate.

Let us assume that we took a certain quantity of water Q and quickly poured it in a pipe (of unit cross-section) and further added no more water. Then at the moment t, the quantity Q is used up in the head $H(t)$ and the moistening of the soil over a depth y_0 so that we will have (Fig. 370):

$$Q = H(t) + my_0 .$$

If we now replace H in equation (3.9) by its value

$$H = Q - my_0,$$

then we obtain the equation:

$$m\frac{dy_0}{dt} = k\frac{Q + h_k + (1-m)y_0}{y_0} =$$

$$= k(1-m)\frac{y_0 + \frac{Q + h_k}{1-m}}{y_0} . \tag{4.1}$$

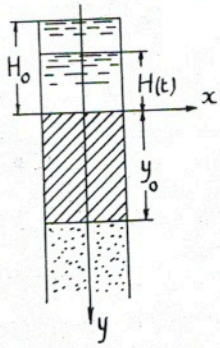

Fig. 370

The solution of this equation has the form (3.12) if we put

$$\eta = \frac{1-m}{Q + h_k}y_0, \quad \tau = \frac{k(1-m)^2}{m(Q + h_k)}t \tag{4.2}$$

(considering $y_0 = 0$ for $t = 0$).

A flow of such nature may take place in the following way. We drive a vertical pipe in the soil. In the upper part of this pipe, emerging above the surface of the soil, we quickly pour some water. If we now watch the change in time of the water level $H(t)$ in the upper part of the pipe, then we also will know the variation of y_0 in time. Comparing the curve $H(t)$ or $y_0(t)$, obtained in nature, with the theoretical one, it is possible to determine some parameters, characteristic of soil, for example the parameters

$$\frac{1 - m}{Q + h_k}, \quad \frac{k(1 - m)^2}{m(Q + h_k)} .$$

From these, knowing Q and h_k, we may find k and m.

Such kinds of experiments were carried out by a series of investigators, amongst whom M. M. Protodjakonov.

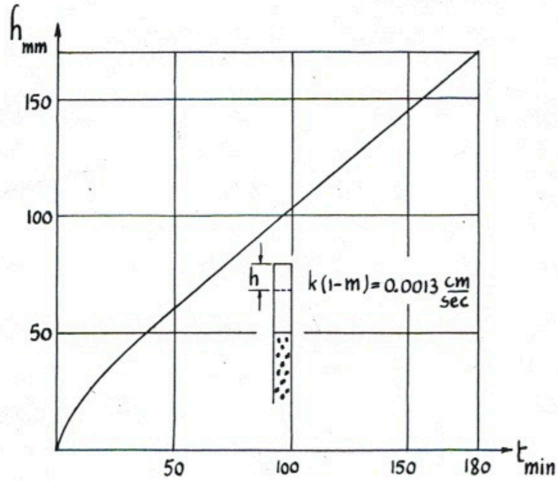

Fig. 371

However, he drove the cylinder only at a slight depth in the soil, so that the seepage that resulted was not one-dimensional, but three-dimensional. In order to obtain a movement much closer to the vertical, M. M. Protodjakonov took a second cylinder, concentric with the first one, of large diameter, and poured also water between the cylinders. In Fig. 371 is given one of the curves, resulting from such an experiment. Putting $h = H(0) - H(t)$, we find from (4.1) for large t: $\frac{dh}{dt} \approx k(1 - m)$. Therefore for large values of t the direction coefficient of the line of Fig. 371 gives the value of $k(1 - m)$.

§5. <u>Gradual Filling With Water</u>.

Actually it is impossible to carry out with precision the sudden filling of the upper part of the pipe up to a given height. And even if this were possible, then we would have a discontinuity in the pressure at the initial moment of time, when $y_0^* = 0$. Indeed, at that time, the water column of height H exerts on the horizontal area in the plane $y = 0$ a pressure, equal to the weight of this column per unit area. On the other side the pressure is zero (or corresponds to the capillary vacuum when