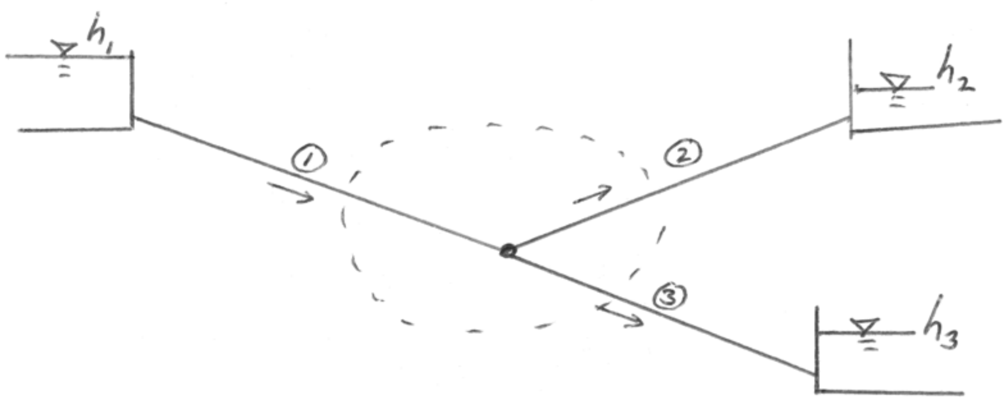
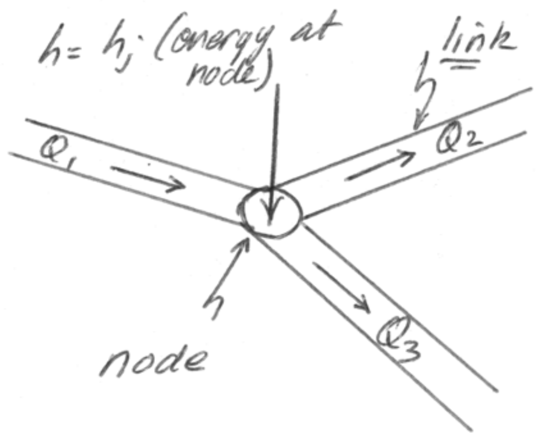
Branched & Looped Systems

Consider a system of multiple, interconnected pipes



Each pipe is called a link (arc)(pipe)
Each junction is called a node



Continuity at the node will require

$$0 = -\rho Q_1 + \rho Q_2 + \rho Q_3$$

$$\Rightarrow \quad \sum Q = 0$$

- Energy at the node is unique (only one value)

These two rules are used to analyze a branched system like the one shown

<u>Continuity</u>

$$Q_1 = Q_2 + Q_3$$

<u>Energy</u>

$$h_1 - f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g} = h_j$$

$$h_2 \pm f_2 \frac{L_2}{D_2} \frac{V_2^2}{2g} = h_j$$

$$h_3 \pm f_3 \frac{L_3}{D_3} \frac{V_3^2}{2g} = h_j$$

Sign from assumed flow directions

Solve each branch for head loss:

$$h_1 - h_j = f_1 \frac{L_1}{D_1} \frac{V_1^2}{2g} = f_1 K_1 Q_1^2$$

$$h_j - h_2 = f_2 \frac{L_2}{D_2} \frac{V_2^2}{2g} = f_2 K_2 Q_2^2$$

$$h_j - h_3 = f_3 \frac{L_3}{D_3} \frac{V_3^2}{2g} = f_3 K_3 Q_3^2$$

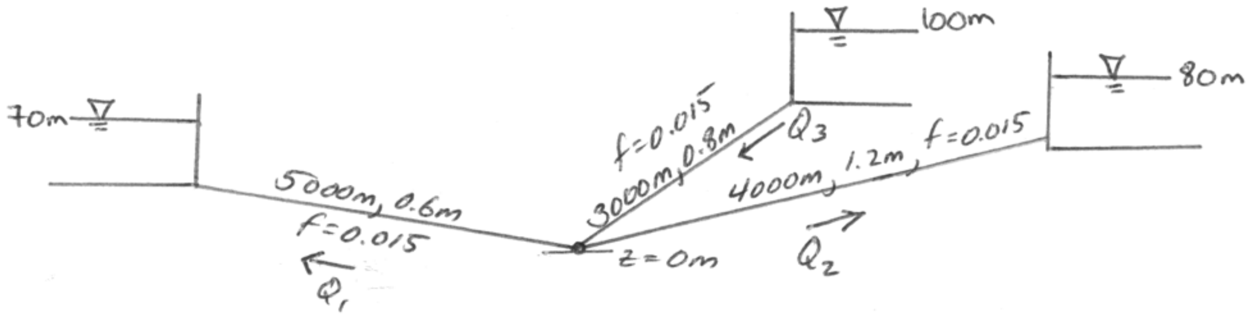
$$Q_1 = \sqrt{\frac{h_1 - h_j}{f_1 K_1}}$$

$$Q_2 = \sqrt{\frac{h_j - h_2}{f_2 K_2}}$$

$$Q_3 = \sqrt{\frac{h_j - h_3}{f_3 K_3}}$$

Example

Find flow in each pipe



① Assume flow directions.

$0 = +Q_1 + Q_2 - Q_3 \Rightarrow Q_3 = Q_1 + Q_2$ (continuity)

② Energy Equations from reservoir to junction

$$h_j - 70m = f_1 \frac{L_1}{D_1} \frac{Q_1^2}{A_1} \frac{1}{2g} = \frac{8 f_1 L_1}{\pi^2 g D_1^5} Q_1^2$$

$$h_j - 80m = \frac{8 f_2 L_2}{\pi^2 g D_2^5} Q_2^2$$

$$100m - h_j = \frac{8 f_3 L_3}{\pi^2 g D_3^5} Q_3^2$$

③ Evaluate constants

$$\frac{8 f_1 L_1}{\pi^2 g D_1^5} = \frac{(8)(0.015)(5000)}{\pi^2 (9.8)(0.6)^5} = 79.7$$

$$\frac{8 f_2 L_2}{\pi^2 g D_2^5} = \frac{(8)(0.015)(4000)}{\pi^2 (9.8)(1.2)^5} = 1.99$$

$$\frac{8 f_3 L_3}{\pi^2 g D^5} = \frac{(8)(0.015)(3000)}{\pi^2 (9.8)(0.8)^5} = 11.35$$

Energy Equations become:

$$h_j - 70 = 79.7\,Q_1^2 \qquad\qquad Q_1^2 = \frac{h_j - 70}{79.7}$$

$$h_j - 80 = 1.99\,Q_2^2 \quad \Rightarrow \quad Q_2^2 = \frac{h_j - 80}{1.99}$$

$$100 - h_j = 11.35\,Q_3^2 \qquad\qquad Q_3^2 = \frac{100 - h_j}{11.35}$$

Continuity: $Q_3 = Q_1 + Q_2$

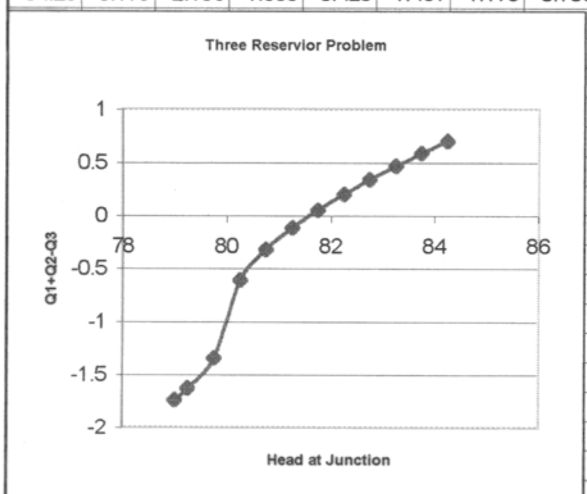
Construct a table

$$Q_1 \begin{cases} = \sqrt{\dfrac{h_j - 70}{79.7}} & \text{if } h_j > 70 \\ = -\sqrt{\dfrac{70 - h_j}{79.7}} & \text{if } h_j < 70 \end{cases}$$

$$Q_2 \begin{cases} = \sqrt{\dfrac{h_j - 80}{1.99}} & \text{if } h_j > 80 \\ = -\sqrt{\dfrac{80 - h_j}{1.99}} & \text{if } h_j < 80 \end{cases}$$

$$Q_3 \begin{cases} = \sqrt{\dfrac{100 - h_j}{11.35}} & \text{if } h_j < 100 \\ = \sqrt{\dfrac{h_j - 100}{11.35}} & \text{if } h_j > 100 \end{cases}$$

hj	Q1^2	Q2^2	Q3^2	Q1	Q2	Q3	Q1+Q2-Q3
79	0.113	-0.503	1.85	0.336	-0.709	1.36	-1.733
79.25	0.116	-0.377	1.828	0.341	-0.614	1.352	-1.625
79.75	0.122	-0.126	1.784	0.35	-0.354	1.336	-1.34
80.25	0.129	0.126	1.74	0.359	0.354	1.319	-0.606
80.75	0.135	0.377	1.696	0.367	0.614	1.302	-0.321
81.25	0.141	0.628	1.652	0.376	0.793	1.285	-0.117
81.75	0.147	0.879	1.608	0.384	0.938	1.268	0.054
82.25	0.154	1.131	1.564	0.392	1.063	1.251	0.205
82.75	0.16	1.382	1.52	0.4	1.176	1.233	0.343
83.25	0.166	1.633	1.476	0.408	1.278	1.215	0.471
83.75	0.173	1.884	1.432	0.415	1.373	1.197	0.592
84.25	0.179	2.136	1.388	0.423	1.461	1.178	0.706



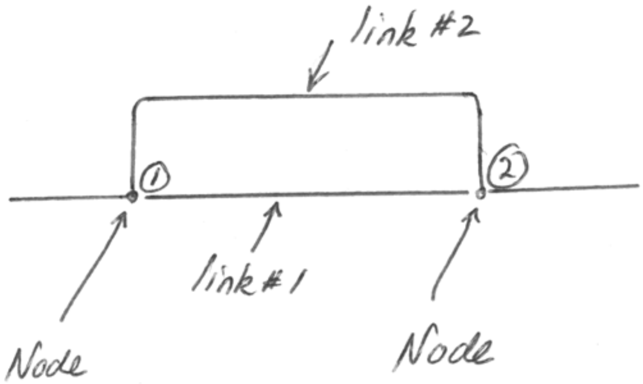
$\therefore Q_1 \approx 0.384\,m^3/s$

$Q_2 \approx 0.938\,m^3/s$

$Q_3 \approx 1.268\,m^3/s$

Looped Systems

Branched system where links define closed loops is called a pipe network.



Rules

1. $\Sigma Q = 0$ at each node.

2. head is unique at each node

3. From 2, the head loss around a closed loop is zero.

$$\therefore \quad h_{Loss\ link\#2} = h_{Loss\ link\#1}$$

or

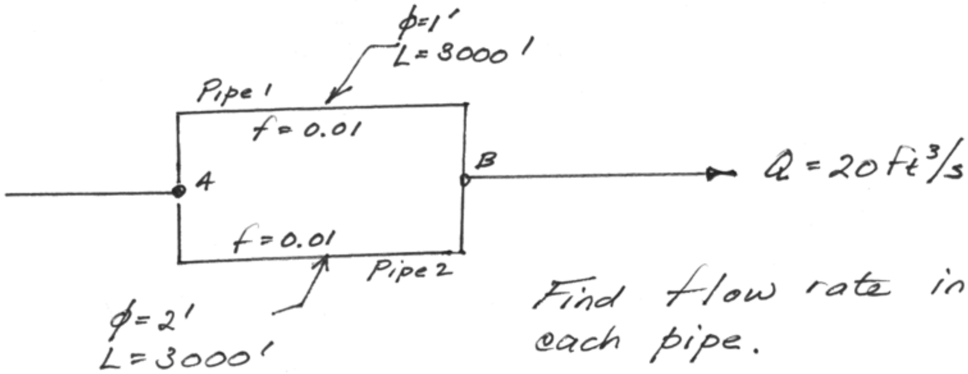
$$h_{1 \to 2} = \frac{f_1 L_1}{D_1} \frac{V_1^2}{2g} = \frac{f_2 L_2}{D_2} \frac{V_2^2}{2g}$$

Looped Systems

① $\Sigma Q = 0$ at each node

② h_L around a closed loop is zero

2 - Parallel Pipes



Find flow rate in each pipe.

① $Q_1 + Q_2 = Q_T = 20$

② $$h_A - \frac{8 f_1 L_1 Q_1^2}{\pi^2 g D_1^5} = h_B$$

$$h_A - \frac{8 f_2 L_2 Q_2^2}{\pi^2 g D_2^5} = h_B$$

$$\therefore \quad \frac{\cancel{8} f_1 L_1 Q_1^2}{\cancel{\pi^2 g} D_1^5} = \frac{\cancel{8} f_2 L_2 Q_2^2}{\cancel{\pi^2 g} D_2^5}$$

$$\frac{Q_1^2}{Q_2^2} = \frac{f_2 L_2 D_1^5}{f_1 L_1 D_2^5}$$

$$\frac{Q_1^2}{Q_2^2} = \frac{f_2 L_2 D_1^5}{f_1 L_1 D_2^5}$$

Because $f_1 = f_2$ & $L_1 = L_2$

$$\frac{Q_1^2}{Q_2^2} = \frac{D_1^5}{D_2^5} = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$\therefore$ $Q_1^2 = \frac{1}{32} Q_2^2$ or $Q_1 = \sqrt{\frac{1}{32}}\, Q_2$

Now substitute into continuity:

$$Q_1 + Q_2 = 20$$

or

$\sqrt{\frac{1}{32}}\, Q_2 + Q_2 = 20$ Solve for Q_2

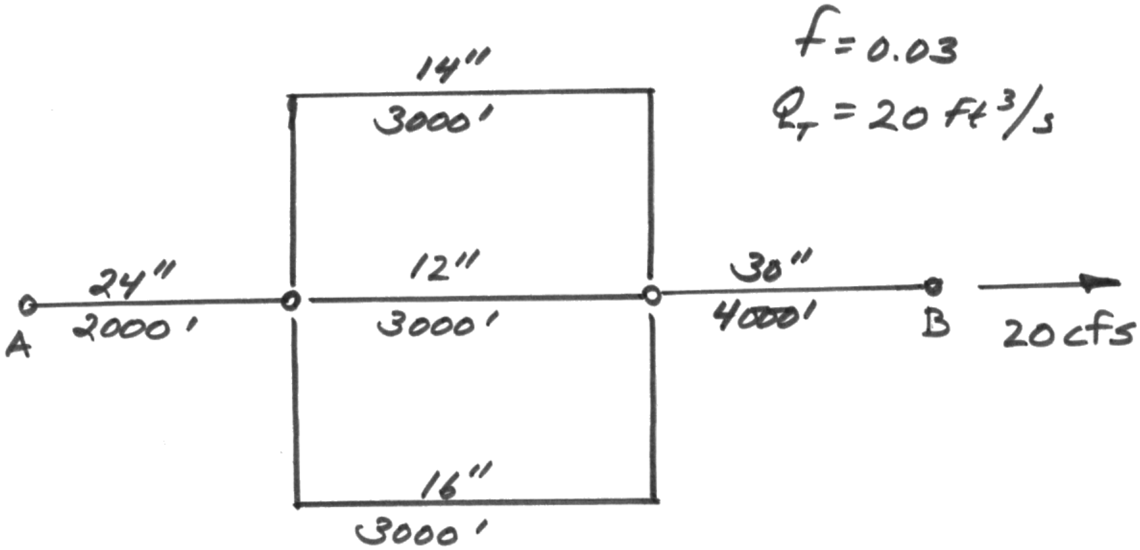
$$1.176\, Q_2 = 20$$

$$Q_2 = 20/1.176 = 17.006$$

$$Q_1 = 0.176(17.006) = 2.994$$

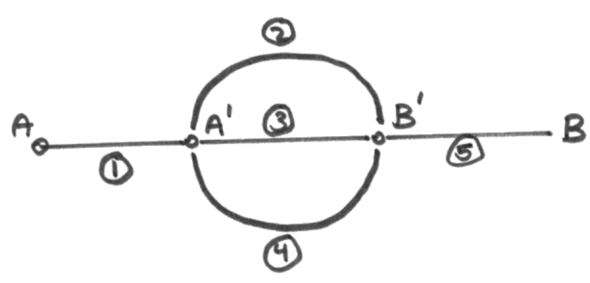
Several parallel pipes

Find head loss & flow distribution in system shown below



Solution

① label all pipes & nodes



② find losses in ① & ⑤; both have flow $Q = 20 cfs$

$$h_{L_1} = \frac{8 f L_1}{\pi^2 g D_1^5} Q^2 = \frac{(8)(0.03)(2000)(20)^2}{\pi^2 (32.2)(\frac{24}{12})^5} = 19.0 \text{ ft}$$

$$h_L = \frac{8 f L}{\pi^2 g D_5^5} Q^2 = \frac{(8)(0.03)(4000)(20)^2}{\pi^2 (32.2)(\frac{30}{12})^5} = 12.0 \text{ ft}$$

(3) Interior pipes

<u>Continunity</u>: $Q_2 + Q_3 + Q_4 = 20$

<u>Energy</u> : $h_{L_2} = h_{L_3} = h_{L_4}$

$$h_{L_2} = \frac{8 f L_2}{\pi^2 g D_2^5} Q_2^2 = 1.05 Q_2^2$$

$$h_{L_3} = \frac{8 f L_3}{\pi^2 g D_3^5} Q_3^2 = 1.51 Q_3^2$$

$$h_{L_4} = \frac{8 f L_4}{\pi^2 g D_4^5} Q_4^2 = 0.54 Q_4^2$$

From energy: $h_{L_2} = h_{L_3} = h_{L_4}$

$\therefore \; 1.51 Q_3^2 = 1.05 Q_2^2 \Rightarrow Q_3^2 = \frac{1.05}{1.51} Q_2^2 \Rightarrow Q_3 = \sqrt{\frac{1.05}{1.51}}\, Q_2$

$0.54 Q_4^2 = 1.05 Q_2^2 \Rightarrow Q_4^2 = \frac{1.05}{0.54} Q_2^2 \Rightarrow Q_4 = \sqrt{\frac{1.05}{0.54}}\, Q_2$

So: $Q_3 = 0.834\, Q_2$

$\&$ $Q_4 = 1.394\, Q_2$

Now use continuity

$Q_2 + 0.834\, Q_2 + 1.394\, Q_2 = 20$

Solve for Q_2

$3.23\, Q_2 = 20 \Rightarrow Q_2 = 6.2\,cfs$

$Q_3 = (0.834)\, 6.2 = 5.2\,cfs$

$Q_4 = (1.394)\, 6.2 = 8.6\,cfs$

Now find head loss from A' to B'

$h_{L_2} = 1.05 Q_2^2 = 1.05(6.2)^2 = 40\ ft$

$\therefore$ Total system loss is

$h_{L_1} + h_{L_2} + h_{L_5} = 19 + 40 + 12 = 71\ ft.$

NETWORK ANALYSIS (NEWTON-RAPHSON)

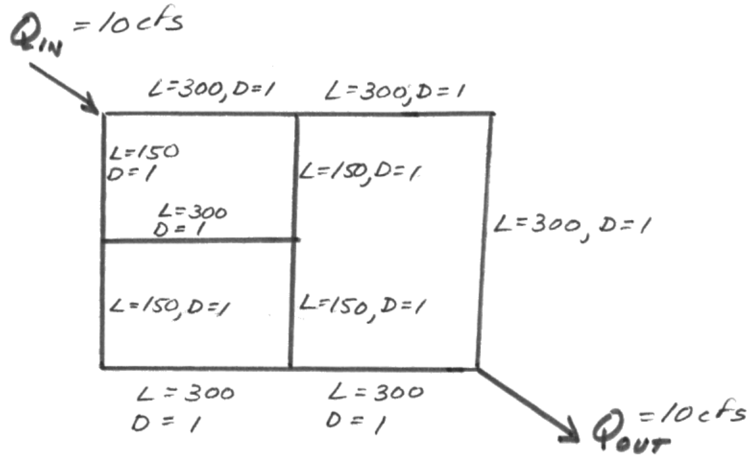
1) Sketch network

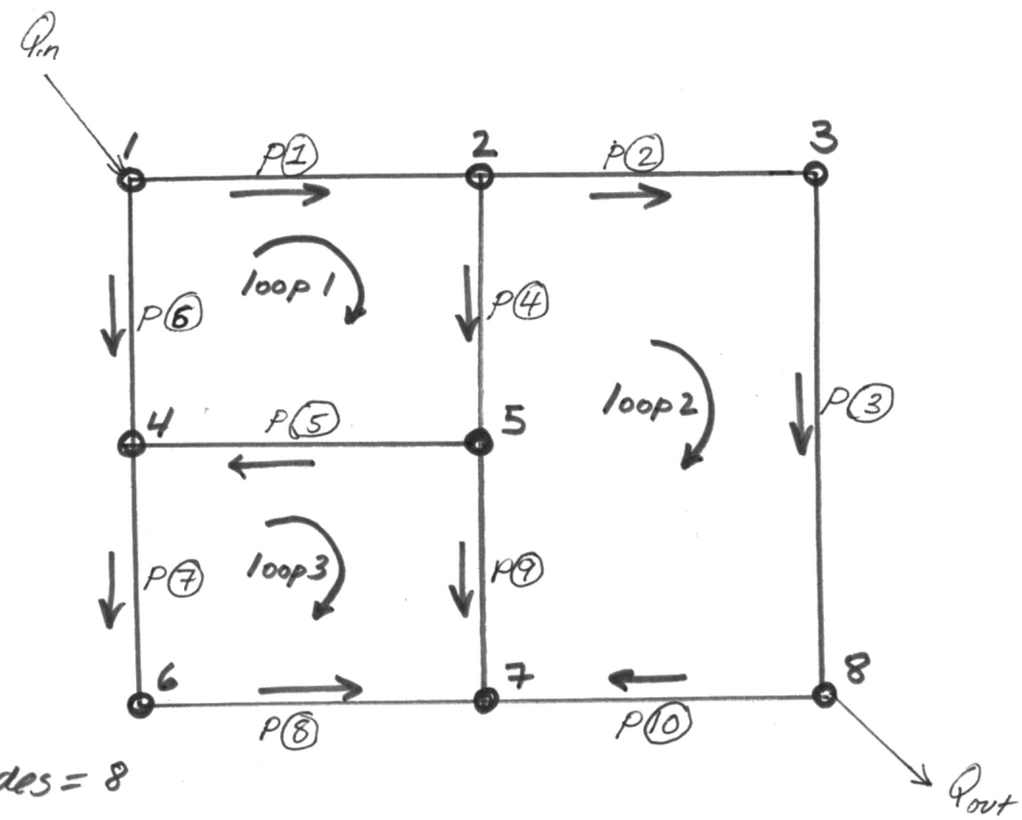
a) label pipes

b) label nodes

c) label loops

d) show assumed flow directions





nodes = 8

loops = 3

pipes = 10

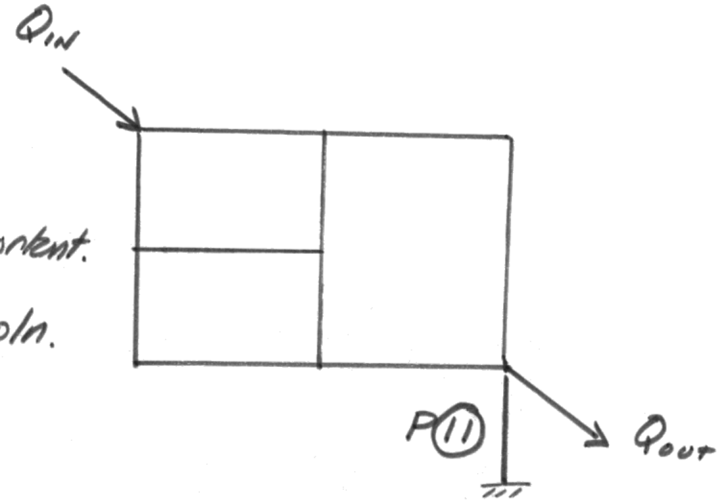
2) Check geometry:

The relationship: #nodes + #loops = #pipes must be satisfied to find a unique solution

Current example: $8 + 3 = 11$

but only show 10 pipes.

Add a pipe at a supply or demand node to satisfy geometry criterion



L&D unimportant.

At correct soln.

$Q_{11} = 0$.

3) Prepare K, f, Re tables for use in head loss equations

$$K = \frac{8L}{\pi^2 g D^5} \quad , \quad r = \frac{4\rho}{\mu \pi D}$$

$$h_L = f K Q^2 \quad ; \quad Re = r Q$$

$$= f K |Q| Q$$

$\leftarrow$ flow direction is carried by sign (Q)

4) Now write continuity for each node

$-Q_1$					$-Q_6$						=	-10	node 1
Q_1	$-Q_2$		$-Q_4$								=	0	node 2
	Q_2	$-Q_3$									=	0	node 3
				Q_5	$+Q_6$	$-Q_7$					=	0	node 4
			Q_4	$-Q_5$				$-Q_9$			=	0	node 5
					Q_6	$-Q_7$					=	0	node 6
							Q_8	Q_9	Q_{10}		=	0	node 7
		Q_3							$-Q_{10}$	Q_{11}	=	10	node 8

Flow into a node is $+1$

Flow out of a node is -1

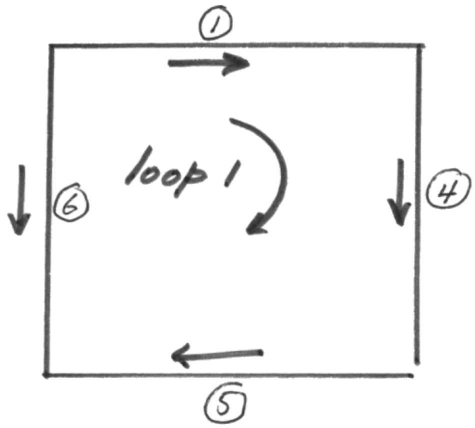
5) Loss equations for each loop

$f_1 K_1 |Q_1| Q_1$ $\quad$ $f_4 K_4 |Q_4| Q_4 + f_5 K_5 |Q_5| Q_5 - f_6 K_6 |Q_6| Q_6|$ $\quad = 0$

$f_2 K_2 |Q_2| Q_2 + f_3 K_3 |Q_3| Q_3 - f_4 K_4 |Q_4| Q_4$ $\quad$ $-f_9 K_9 |Q_9| Q_9 + f_{10} K_{10} |Q_{10}| Q_{10} = 0$

$f_5 K_5 |Q_5| Q_5$ $\quad$ $-f_7 K_7 |Q_7| Q_7 - f_8 K_8 |Q_8| Q_8 + f_9 K_9 |Q_9| Q_9| = 0$

Sign in loop equations is based on <u>assumed</u> flow directions



$$f_1 K_1 |Q_1| Q_1 + f_4 K_4 |Q_4| Q_4 + f_5 K_5 |Q_5| Q_5 - f_6 K_6 |Q_6| Q_6 = 0$$

6) Use some initial (guesses) values for $|Q_i|$ in the head loss equations.

When this is done the loop + node equations are a system of linear equations

At the correct solution:

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & \\ f_1K_1|\bar{Q}_1| & 0 & 0 & f_4K_4|\bar{Q}_4| & f_5K_5|\bar{Q}_5| & -f_6K_6|\bar{Q}_6| & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & \end{bmatrix}}_{\vec{A}_T} \cdot \underbrace{\begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{11} \end{bmatrix}}_{\vec{q}_T} = \underbrace{\begin{bmatrix} -10 \\ 0 \\ \vdots \\ 10 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{rhs}_T}$$

$\therefore$ $$\vec{A}_m \underbrace{\vec{q}_m}_{=\text{ current guess of } Q\text{'s}} = \vec{rhs}_m$$

Let $\vec{A}_m, \vec{q}_m$ be current guess of $\vec{A}_T$ & $\vec{q}_T$; $\vec{rhs}_m$ is current result of the matrix multiplication $\vec{A}_m \vec{q}_m$

$$g_1(\vec{q}) = f_1 K_1 |Q_1| Q_1 + \cdots$$

$$\frac{dg_1}{dq_1} = 2 f_1 K_1 |Q_1|$$

∴ Twice the value of the coefficients in the loop matrix is the jacobian of the head loss equations

Newton's method gives a formula to update the $\vec{q}$ vector to try to solve

$$\vec{g}(\vec{q}) = 0$$

$$\Delta \vec{q} = [\vec{J}(\vec{q})]^{-1} \vec{g}(\vec{q})$$

$$\vec{q}^{\,k+1} = \vec{q}^{\,k} - \Delta \vec{q}$$

Basis of all pipeline network models (including Hardy Cross method)

4

$r f_i K_i |Q_i|$

	Pipe Network Model - Iterative Method														
Reset	1														
Iterations	100														
	Pipes	1	2	3	4	5	6	7	8	9	10	11			
	Diameters	1	1	1	1	1	1	1	1	1	1	1			
	Length	300	300	300	150	300	150	150	300	150	300	1			
	K-factors	7.55188946	7.55188946	7.55188946	3.77594473	7.55188946	3.77594473	3.77594473	7.55188946	3.77594473	7.55188946	0.02517296			
	f-factors	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.015	0.015			
	r-factor	117674.634	117674.634	117674.634	117674.634	117674.634	117674.634	117674.634	117674.634	117674.634	117674.634	117674.634			
	Q-initial	1	1	1	1	1	1	1	1	1	1	1			
	Q-k	4.62079511	4.29767158	4.29767158	0.32312353	-2.6336342	5.37920489	2.74557067	2.74557067	2.95675775	-5.7023284	-1.128E-15	← CURRENT GUESS		
	Re-guess	5.4E+5	5.1E+5	5.1E+5	3.8E+4	3.1E+5	6.3E+5	3.2E+5	3.2E+5	3.5E+5	6.7E+5	1.3E-10			
	fK|Q|	0.52343601	0.48683311	0.48683311	0.01830145	0.29833372	0.30467371	0.15550685	0.31101369	0.16746831	0.64595031	4.2588E-19			
		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	RHS(mod)	RHS(true)	G(Q)
Nodes	FUNC(Q)														
	1	-1	0	0	0	0	-1	0	0	0	0	0	-10	-10	0
	2	1	-1	0	-1	0	0	0	0	0	0	0	-1.11E-16	0	-1.11E-16
	3	0	1	-1	0	0	0	0	0	0	0	0	0	0	0
	4	0	0	0	0	1	1	-1	0	0	0	0	0	0	0
	5	0	0	0	1	-1	0	0	0	-1	0	0	0	0	0
	6	0	0	0	0	0	0	1	-1	0	0	0	0	0	0
	7	0	0	0	0	0	0	0	1	1	1	0	0	0	0
	8	0	0	1	0	0	0	0	0	0	-1	-1	10	10	1.7764E-15
Loops	1	0.52343601	0	0	0.01830145	0.29833372	-0.3046737	0	0	0	0	0	1.7764E-15	0	1.7764E-15
	2	0	0.48683311	0.48683311	-0.0183014	0	0	0	0	-0.1674683	0.64595031	0	1.3323E-15	0	1.3323E-15
	3	0	0	0	0	-0.2983337	0	-0.1555068	-0.3110137	0.16746831	0	0	-4.996E-16	0	-4.996E-16
		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	DQ	Q-k	Q-k+1
Nodes	JACOB(Q)														
	1	-1	0	0	0	0	-1	0	0	0	0	0	7.9229E-16	4.62079511	4.62079511
	2	1	-1	0	-1	0	0	0	0	0	0	0	4.3216E-16	4.29767158	4.29767158
	3	0	1	-1	0	0	0	0	0	0	0	0	4.3216E-16	4.29767158	4.29767158
	4	0	0	0	0	1	1	-1	0	0	0	0	4.7115E-16	0.32312353	0.32312353
	5	0	0	0	1	-1	0	0	0	-1	0	0	7.4901E-16	-2.6336342	-2.6336342
	6	0	0	0	0	0	0	1	-1	0	0	0	-7.923E-16	5.37920489	5.37920489
	7	0	0	0	0	0	0	0	1	1	1	0	-4.328E-17	2.74557067	2.74557067
	8	0	0	1	0	0	0	0	0	0	-1	-1	-4.328E-17	2.74557067	2.74557067
Loops	1	1.04687202	0	0	0.0366029	0.59666744	-0.6093474	0	0	0	0	0	-2.779E-16	2.95675775	2.95675775
	2	0	0.97366622	0.97366622	-0.0366029	0	0	0	0	-0.3349366	1.29190062	0	3.2114E-16	-5.7023284	-5.7023284
	3	0	0	0	0	-0.5966674	0	-0.3110137	-0.6220274	0.33493662	0	0	-1.665E-15	-1.128E-15	5.3746E-16
	Q-k+1	4.62079511	4.29767158	4.29767158	0.32312353	-2.6336342	5.37920489	2.74557067	2.74557067	2.95675775	-5.7023284	5.3746E-16			

$\vec{RHS}_M$ = MMULT([ ], TRANSPOSE (CURRENT GUESS))

□ = 2 * [ ]