A Statistical Mechanical Prediction of the Dimensionless Unit Hydrograph

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Abstract. That the form of the dimensionless unit hydrograph is very nearly independent of watershed properties implies that it should be predictable with a minimum use of such properties. Such a prediction is formulated in this study, using Boltzmann statistics. The prediction shows that a characteristic time is the only important watershed property and that the shape of the watershed has only a minor influence on the form of the hydrograph. The resulting theoretical hydrograph is compared with observations made in two watersheds and is found to represent them well.

Introduction. Much attention has been given to the use of the unit hydrograph during the past 30 years. This device has proved extremely helpful in the prediction of flood behavior and other waterway action. The use of hydrograph synthesis and the study of the related physical mechanisms have recently been reviewed by *Laurenson* [1963].

The present inquiry is concerned with the idea of the dimensionless unit hydrograph (see, e.g., the work of *Bender and Roberson* [1961] and *Bender* [1963]). This idea is that the usual hydrograph for storms of *short duration,*

$$\dot{Q} = f_1\,(t, \text{storm intensity, watershed properties}) \tag{1}$$

can be written in dimensionless form:

$$\dot{Q}/\dot{Q}_c = f_2(t/t_c) \tag{2}$$

and that the latter expression will apply to *any watershed.* $\dot{Q}$ is the discharge rate; $\dot{Q}_c$ is a characteristic of the storm intensity—usually the peak rate of discharge following the storm; and t_c—sometimes called the 'time lag'—is a characteristic of the particular watershed. Usually t_c measures the breadth of the hydrograph; it might, for example, be the time elapsed between the incidence of the storm and the maximum discharge.

That the observed function $f_2(t/t_c)$ is very nearly the same for any watershed strongly suggests that it might be derived by considering a minimum of physical detail of the watershed. Furthermore, the form of the observed function appears to closely approximate that of the Maxwell-Boltzmann molecular speed distribution. That is,

$$[\dot{Q}/\dot{Q}_c]_{\text{obs}} \approx C_1(t/t_c)^2 \exp\,[-C_2(t/t_c)^2] \tag{3}$$

This fact even more strongly suggests that the problem might be solved with a minimum knowledge of the watershed properties if Boltzmann statistics are used.

Analysis. Let us suppose that $(N + M)$ raindrops constitute a sudden storm over a watershed. By 'sudden' we mean that the storm is brief with respect to the characteristic time t_c. Of these raindrops, M disappear into the ground or evaporate into the air and N eventually find their way to the gaging station. The conventional hydrograph for this storm can be recast in the following way: the storm runoff past the gaging station can be interpreted in terms of the number of raindrops N_i in each of a series of time increments of duration Δt. The result is a histogram (Figure 1).

Each of the N raindrops will have the same a priori probability of reaching the gaging station during the ith time increment. Whether or not it does will depend upon where it lands and what obstacles it encounters. It will be assumed that each raindrop is distinguishable from and independent of the other raindrops.

Now let us consider the slug of water composed of the N_i raindrops that pass the gaging station during t_{i-1} to t_i. We must specify the g_i ways in which these drops could have found their way to the gaging station. It is reasonable

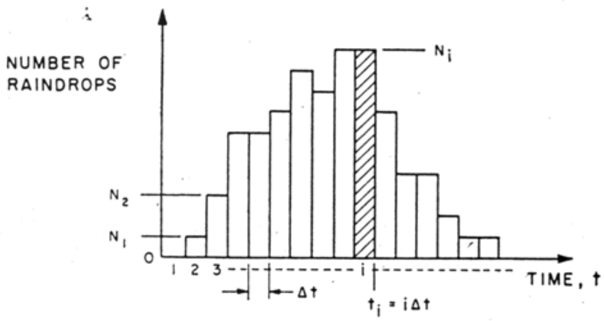

Fig. 1. The hydrograph interpreted as a raindrop histogram.

to assume that there are, on the average, a constant number of ways by which a raindrop must leave each unit of watershed area. Furthermore, the time required for a raindrop to arrive, after traveling any distance l_i to the gaging station, is approximately proportional to l_i.

Finally, in order to specify g_i in terms of t_i, it is necessary to specify the amount of area swept out by l_i. This area should increase roughly as the square of l_i (although for a particularly long, slender watershed it might be more nearly a linear function of l_i). Accordingly, we shall assume that

$$g_i \approx l_i^2$$

or, since $l_i \propto t_i$,

$$g_i = At_i^2 \quad (4)$$

where A is some constant.

We must now ask: 'How many ways can the set of distinguishable objects, $\langle N_1, N_2, \cdots, N_i, \cdots \rangle$, be selected from N distinguishable objects, if there are g_i distinguishable ways of placing N_i objects in the ith cell?' This is a well-known problem in Boltzmann statistics. The number of ways W (see, e.g., *Davidson* [1962]) is

$$W = N! \prod_{i=1}^{\infty} \frac{g_i^{N_i}}{N_i!} = N! \prod_{i=1}^{\infty} \frac{(At_i^2)^{N_i}}{N_i!} \quad (5)$$

The particular set of numbers, $\langle N_1, N_2, \cdots, N_i, \cdots \rangle$, for which W is maximum will be the *most probable* one. Since the numbers involved are very large, the most probable distribution will be *certain* for all practical purposes. It then remains to maximize W (or, since it is easier to do so, to maximize $\ln W$), subject to the appropriate constraints. The first of these is a simple conservation statement:

$$\sum_{i=1}^{\infty} N_i = N \quad (6)$$

The second constraint is one which characterizes the particular watershed with the root-mean-square time t_{rms} of the physical hydrograph. From the definition of t_{rms},

$$\sum_{i=1}^{\infty} t_i^2 N_i = t_{rms}^2 N \quad (6a)$$

The use of Stirling's approximation to the value of factorials of very large numbers in (5) gives

$$\ln W = N \ln N - N + \sum_{i=1}^{\infty} [N_i \ln (At_i^2) - N_i \ln N_i + N_i] \quad (5a)$$

Equation 5*a* is maximized with the aid of Lagrange's method of undetermined multipliers:

$$d \ln W = \sum_{i=1}^{\infty} [\ln (At_i^2) - \ln N_i] \, dN_i = 0 \quad (7)$$

and from the constraints:

$$-\sum_{i=1}^{\infty} \alpha \, dN_i = 0 \quad (8)$$

$$-\sum_{i=1}^{\infty} \beta t_i^2 \, dN_i = 0 \quad (9)$$

where $-\alpha$ and $-\beta$ are undetermined multipliers. Adding (7), (8), and (9) gives

$$\sum_{i=1}^{\infty} [\ln (N_i/At_i^2) + \alpha + \beta t_i^2] \, dN_i = 0 \quad (10)$$

Since the coefficients of the dN_i must vanish identically,

$$N_i = (Ae^{-\alpha})[t_i^2 \exp(-\beta t_i^2)] \quad (11)$$

or from (6),

$$\frac{N_i}{N} = \left[\sum_{i=1}^{\infty} t_i^2 \exp(-\beta t_i^2) \right]^{-1} t_i^2 \cdot \exp(-\beta t_i^2) \quad (11a)$$

Finally, the undetermined multiplier β should be evaluated. To do this, we first consider the sum

TABLE 1. Rainfall and Runoff in the Mill Creek Watershed (Illinois) on June 2 and 3, 1954

End of Hour of Observation	Rainfall, inches	Hourly Runoff, $ft^3 \times 10^6$	Per Cent of Q_{total}
2 P.M.	0		
3	0.01		
4	0.01		
5	0.02		
6	0.45		
7	0.14		
8	0.10	0	0
9	0	0.097	0.57
10		0.396	2.31
11		1.764	10.33
12		2.465	14.42
1 A.M.		2.66	15.61
2		2.555	15.00
3		2.068	12.12
4		1.726	10.21
5		1.284	7.53
6		0.900	5.27
7		0.594	3.47
8		0.360	2.10
9		0.180	1.06
10		0	0
Totals	0.73	17.06	100.00

Notes: The runoff, based on the 62.5-mi² watershed, is 0.1175 in. The values of t_{rms}, based on $t = 0$ at 6:30 P.M., 7:00 P.M., and 7:30 P.M., are 8.97 hr, 7.67 hr, and 7.20 hr, respectively.

$$\sum_{i=1}^{\infty} t_i^2 \exp(-\beta t_i^2) = \Delta t^2 \sum_{i=1}^{\infty} i^2 \exp -(\beta\, \Delta t^2) i^2$$

This can be approximated with an integral,

$$\sum_{i=1}^{\infty} t_i^2 \exp(-\beta t_i^2) \approx \Delta t^2 \int_0^{\infty} i^2 \exp -(\beta\, \Delta t^2) i^2 \, di$$

or

$$\sum_{i=1}^{\infty} t_i^2 \exp(-\beta t_i^2) \approx \frac{\sqrt{\pi}}{4\, \Delta t} \beta^{-3/2} \qquad (12)$$

By the same token, the first and second constraints give

$$\sum_{i=1}^{\infty} N_i = Ae^{-\alpha} \sum_{i=1}^{\infty} t_i^2 \exp(-\beta t_i^2)$$

or

$$N \approx \frac{\sqrt{\pi}\, Ae^{-\alpha}}{4\, \Delta t} \beta^{-3/2} \qquad (13)$$

and

$$\sum_{i=1}^{\infty} t_i^2 N_i = Ae^{-\alpha} \sum_{i=1}^{\infty} t_i^4 \exp(-\beta t_i^2)$$

or

$$t_{rms}^2 N \approx \frac{3\sqrt{\pi}\, Ae^{-\alpha}}{8\, \Delta t} \beta^{-5/2} \qquad (14)$$

Eliminating $Ae^{-\alpha}$ from (13) and (14) gives

$$\beta = \frac{3}{2} \frac{1}{t_{rms}^2} \qquad (15)$$

and substitution of (15) and (12) into (11*a*) gives

$$\frac{N_i}{N} = \sqrt{54/\pi}\, (\Delta t / t_{rms})(t_i / t_{rms})^2 \cdot \exp\left[-\tfrac{3}{2}(t_i/t_{rms})^2\right] \qquad (16)$$

Equation 16 can be approximated as

$$f(t) = \sqrt{54/\pi}\, (1/t_{rms})(t/t_{rms})^2 \cdot \exp\left[-\tfrac{3}{2}(t/t_{rms})^2\right] \qquad (17)$$

TABLE 2. Rainfall and Runoff in the Mill Creek Watershed (Illinois) on May 24, 1944

End of Hour of Observation	Rainfall, inches	Hourly Runoff, $ft^3 \times 10^6$	Per Cent of Q_{total}
10 A.M.	0	0	0
11	0.40	0	0
12	0.01	0.072	0.82
1 P.M.	0	0.414	4.71
2		1.433	16.30
3		1.670	19.01
4		1.577	17.94
5		1.271	14.46
6		0.940	10.70
7		0.641	7.29
8		0.378	4.30
9		0.270	3.07
10		0.094	1.07
11		0.029	0.33
12		0	0
Totals	0.41	8.789	100.00

Notes: The runoff, based on the 62.5-mi² watershed, is 0.0605 in. The value of t_{rms}, based on $t = 0$ at 11:00 A.M., is 5.71 hr.

TABLE 3. Rainfall and Runoff in the Bay Creek Watershed (Illinois) on June 22, 1952

End of Hour of Observation	Rainfall, inches	Hourly Runoff, $ft^3 \times 10^6$	Per Cent of Q_{total}
12	0		
1 A.M.	0.30		
2	1.50	0	0
3	0.04	1.26	3.78
4	0	3.438	10.31
5		5.472	16.42
6		7.776	23.33
7		8.748	26.25
8		3.78	11.34
9		1.242	3.73
10		0.774	2.32
11		0.432	1.30
12		0.234	0.70
1 P.M.		0.1296	0.39
2		0.0432	0.13
3		0	0
Totals	1.84	33.33	100.00

Notes: The runoff, based on the 39.6-mi² watershed, is 0.362 in. The value of t_{rms}, based on $t = 0$ at 1:30 A.M., is 5.16 hr.

where the normalized distribution function $f(t)$ is

$$f(t) \equiv \frac{1}{N}\frac{dN(t)}{dt} = \frac{1}{Q_{total}}\frac{dQ(t)}{dt} \qquad (18)$$

and where, in turn, $N(t)$ is the limiting continuous distribution of the N_i's. The right-hand side of (18) is obtained when we note that N can be expressed as total discharge (ft^3 of rain) and $N(t)$ as the discharge at any time.

Equations 17 and 18 will become the dimensionless hydrograph that we seek, when they are multiplied by t_{rms}.

Comparison of prediction with observation, and discussion. Rainfall and runoff data are presented for two relatively intense storms over each of two watersheds,[1] in Tables 1 through 4. Sketches of these watersheds are presented in Figure 2. The data have been expressed in terms of the distribution function $f(t)$, as suggested by (18), and plotted in Figures 3 and 4. The predicted distribution, equation 17, is included on these curves.

On each curve the assumed time of incidence of the storm and the corresponding value of t_{rms} are noted. Since none of the storms was absolutely sudden, some judgment had to be employed in naming the effective time of incidence. The proportion of rain that goes into the ground is probably highest at the beginning of the storm, so that the time of incidence is shifted to later times insofar as runoff is concerned.

A second difficulty in the specification of zero time stems from our ignorance of the motion of the storm in relation to the precipitation station and the watershed. When, for example, the storm passes the rainfall station before it arrives over the watershed, an erroneous time decrement appears between the precipitation and runoff data.

The first three curves (Figures 3*a*, 3*b*, and 3*c*) therefore compare results for three assumed zero times over the span of 1 hour. The best correspondence between observation and prediction occurs when $t = 0$ at 7:30 P.M. The

TABLE 4. Rainfall and Runoff in the Bay Creek Watershed (Illinois) on June 18, 1942

End of Hour of Observation	Rainfall, inches	Hourly Runoff, $ft^3 \times 10^6$	Per Cent of Q_{total}
2 A.M.	0		
3	0.06		
4	0.91		
5	0.03	0	0
6	0	0.03	0.10
7		2.23	6.72
8		5.4	16.22
9		8.06	24.23
10		7.11	21.36
11		4.86	14.60
12		2.93	8.81
1 P.M.		1.29	3.87
2		0.61	1.84
3		0.40	1.21
4		0.24	0.72
5		0.11	0.32
6		0	0
Totals	1.00	33.27	100.00

Notes: The runoff, based on the 39.6-mi² watershed, is 0.362 in. The value of t_{rms}, based on $t = 0$ at 5:00 A.M., is 5.20 hr.

[1] I am indebted to Dr. D. L. Bender for providing these data. They were, in turn, obtained from the U. S. Geological Survey, Surface Water Branch at Champagne, Ill., and the U. S. Department of Commerce, Weather Bureau.

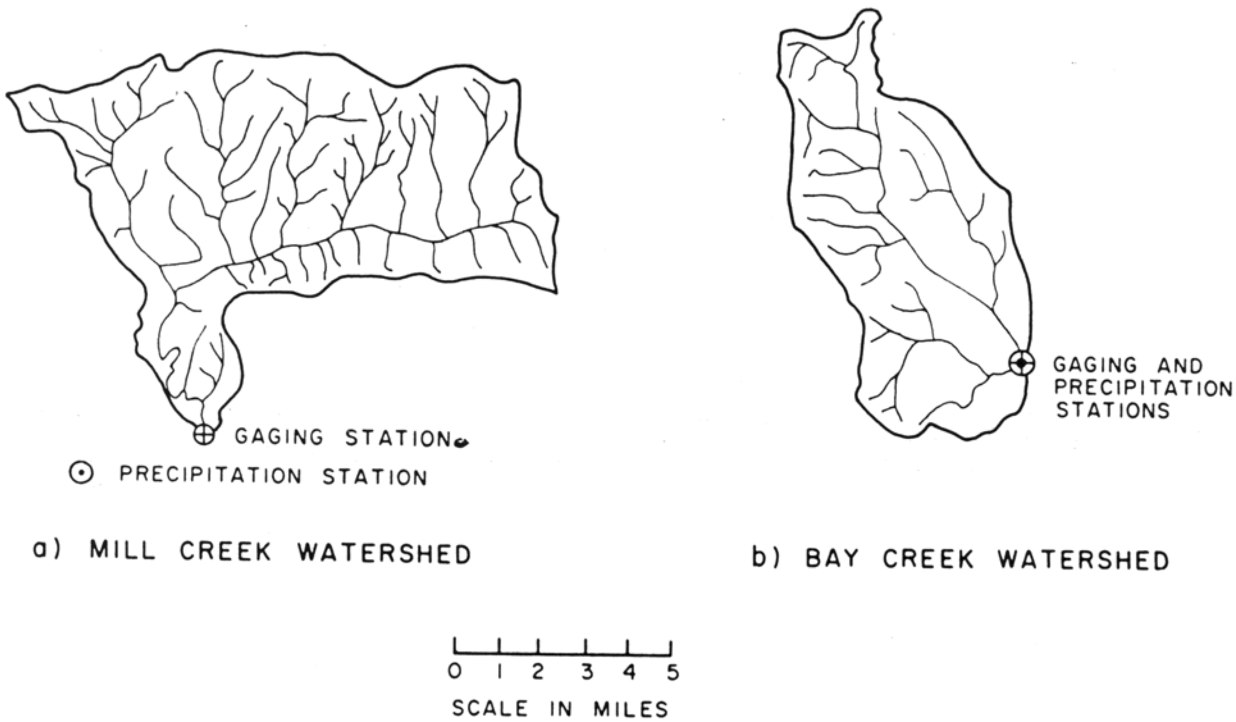

Fig. 2. Physical arrangement of observed watersheds.

three remaining curves (Figures 3*d*, 4*a*, and 4*b*) represent single estimates of zero time. These comparisons could doubtless be improved slightly by making minor changes in zero time.

Such questions of adjustment are not of first importance, however. The comparisons are remarkably good in all cases. The extent to which (17) fails or succeeds must be credited to the analytical model. The model incorporated two physical assumptions in fixing g_i, namely that t_i was proportional to l_i and that it subtended an area proportional to l_i^2.

The latter restriction can be relaxed somewhat by assuming instead that t_i subtends an area proportional to l_i^n. If the analysis is then repeated, with the general exponent n substituted for 2, the resulting distribution function is

$$f(t) = \left\{ \frac{2}{\Gamma\left(\frac{n+1}{2}\right)} \left(\frac{n+1}{2}\right)^{(n+1)/2} \right\} \frac{1}{t_{\mathrm{rms}}} \left(\frac{t}{t_{\mathrm{rms}}}\right)^n \cdot \exp\left[-\frac{n+1}{2} \left(\frac{t}{t_{\mathrm{rms}}}\right)^2 \right] \qquad (19)$$

which reduces to (17) when n is 2. When n is only unity, (19) becomes

$$f(t) = (2/t_{\mathrm{rms}})(t/t_{\mathrm{rms}}) \exp\left[-(t/t_{\mathrm{rms}})^2\right] \qquad (20)$$

It is fortunate, however, that (17) and (20), while representing vastly different physical assumptions, do not differ greatly from one another. The comparison between these cases is shown graphically in Figure 5. The two curves generally differ by less than 20%, except at very early times.

A still more accurate representation of watershed shape could be obtained by making the exponent n dependent upon t_i. This could improve the prediction slightly; however, it would result in an equation of the form

$$\dot{Q}/\dot{Q}_c = t_{\mathrm{rms}} f(t) = f_2(t_{\mathrm{rms}},\, t/t_{\mathrm{rms}}) \qquad (21)$$

This implies that the form of the hydrograph should, in general, change with the watershed, but Figure 5 implies that any such effect will be small.

The present derivation shows that if the form of the hydrograph depends upon the shape of the particular watershed, it does so only in a weak way. The statistical mechanical method has clearly provided a 'first-order' explanation

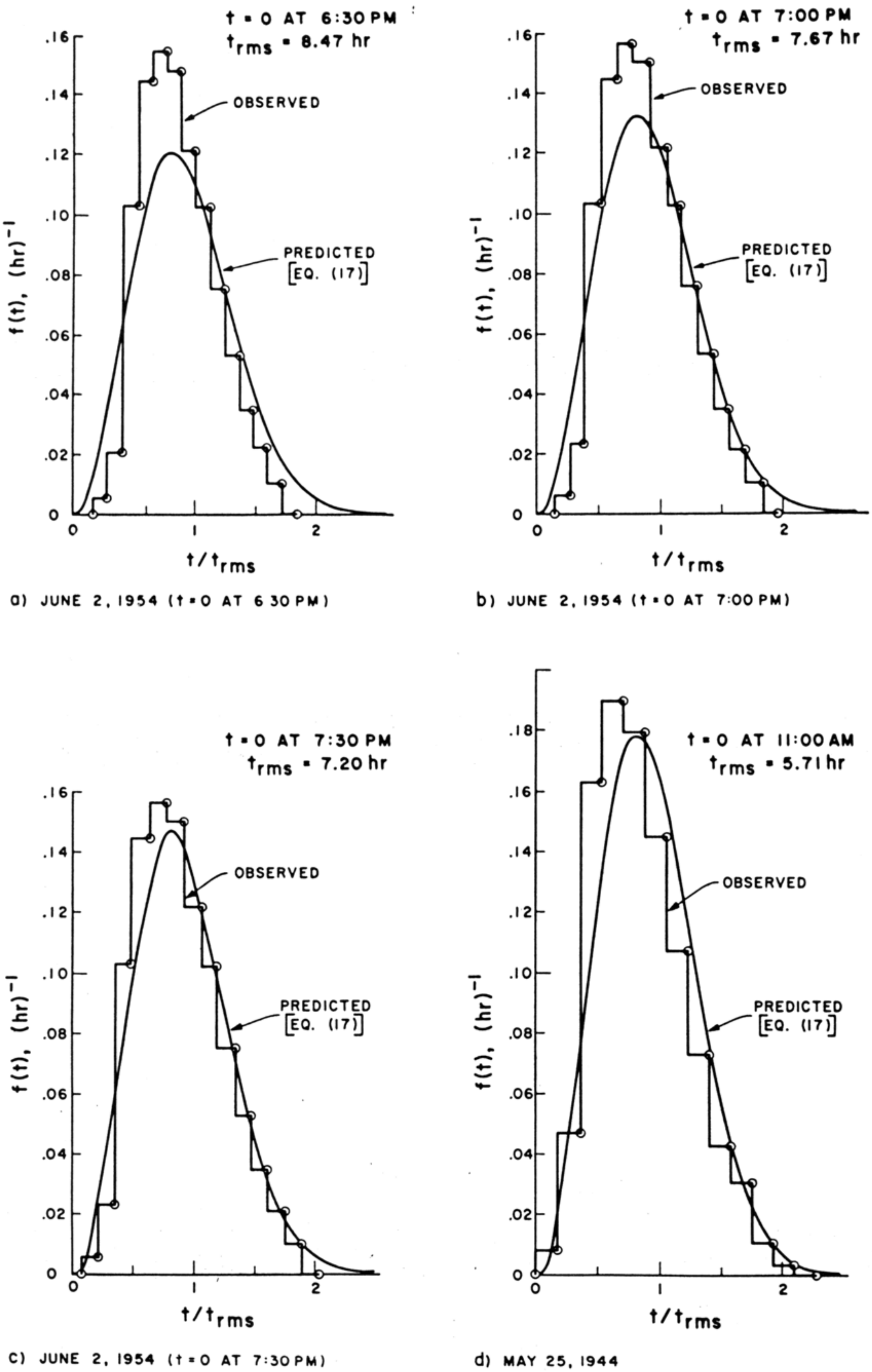

Fig. 3. Comparison of predicted and observed hydrographs at Mill Creek, Illinois.

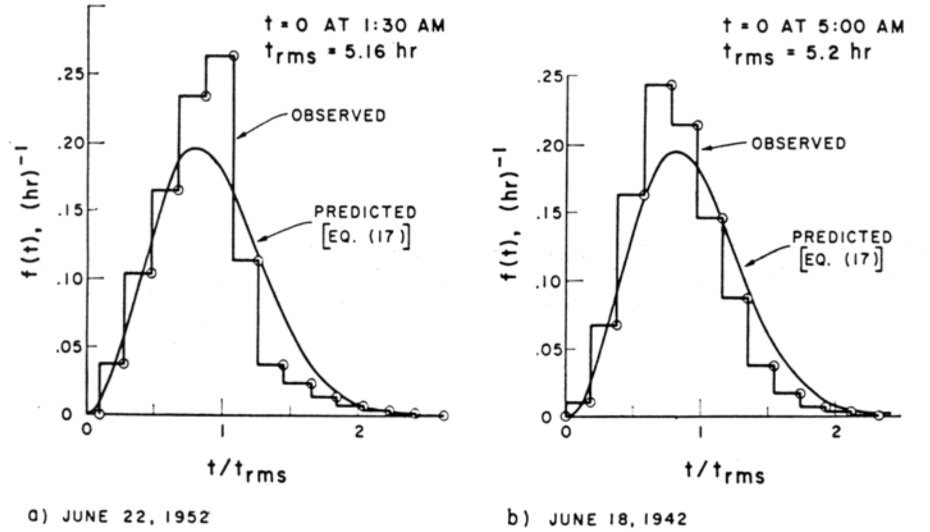

Fig. 4. Comparison of predicted and observed hydrographs at Bay Creek, Illinois.

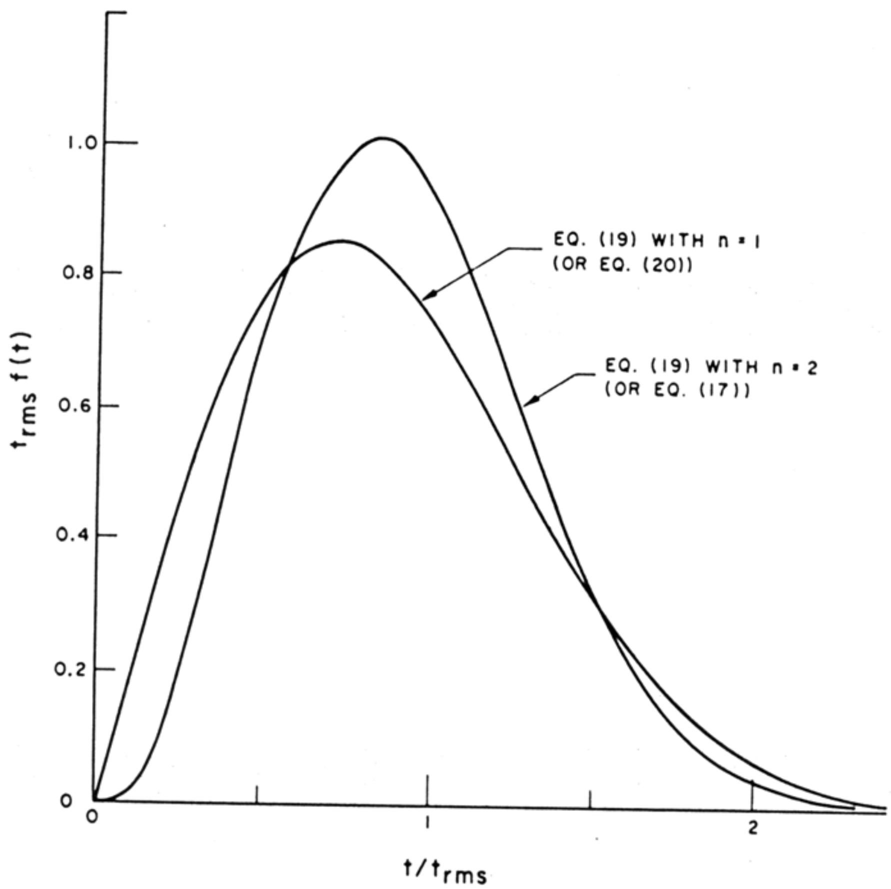

Fig. 5. The effect on the hydrograph of varying n, where $t_i = l_i^n$.

for the dimensionless unit hydrograph. Furthermore, it is a tool that is susceptible to subsequent refinement.

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