CE 3354 Engineering Hydrology

Lecture 23: Transient (Time-Varying) Well Hydraulics; Superposition

Outline

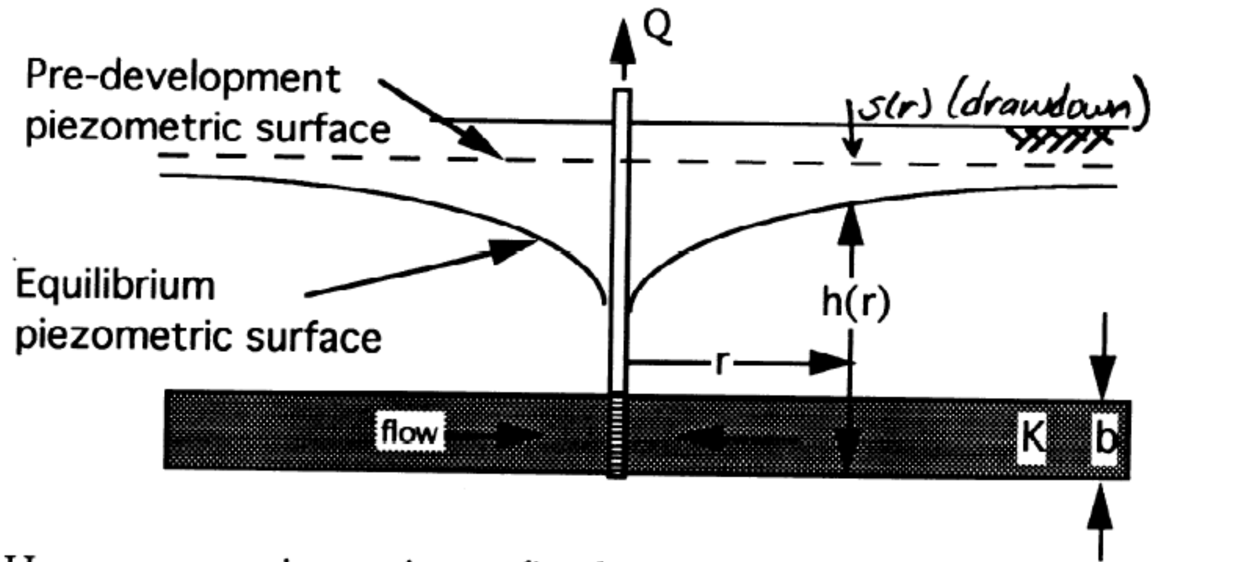
- Steady flow to well
  - Confined
  - Unconfined
- Superposition to represent
  - Multiple wells
  - Aquifer Boundaries

Outline

- Unsteady flow to a well
  - Confined (Theis Solution)
- Superposition
  - Multiple wells
  - Aquifer Boundaries
- Convolution
  - Time-varying pumping rates
- Leaky (Hantush Solution)
- Spreadsheets

Confined Aquifer

- Theim solution



Homogeneous, isotropic, confined aquifer. Flow in radial direction only. Steady state. No internal sources/sinks.

$$h(r) = h_0 + \frac{Q}{2\pi T} \ln\left(\frac{r}{R}\right)$$

$$s(r) = h_0 - h(r) = -\frac{Q}{2\pi T} \ln\left(\frac{r}{R}\right) = \frac{Q}{2\pi T} \ln\left(\frac{R}{r}\right)$$

Confined Aquifer

- Theim solution
  - Derivation in attached reading

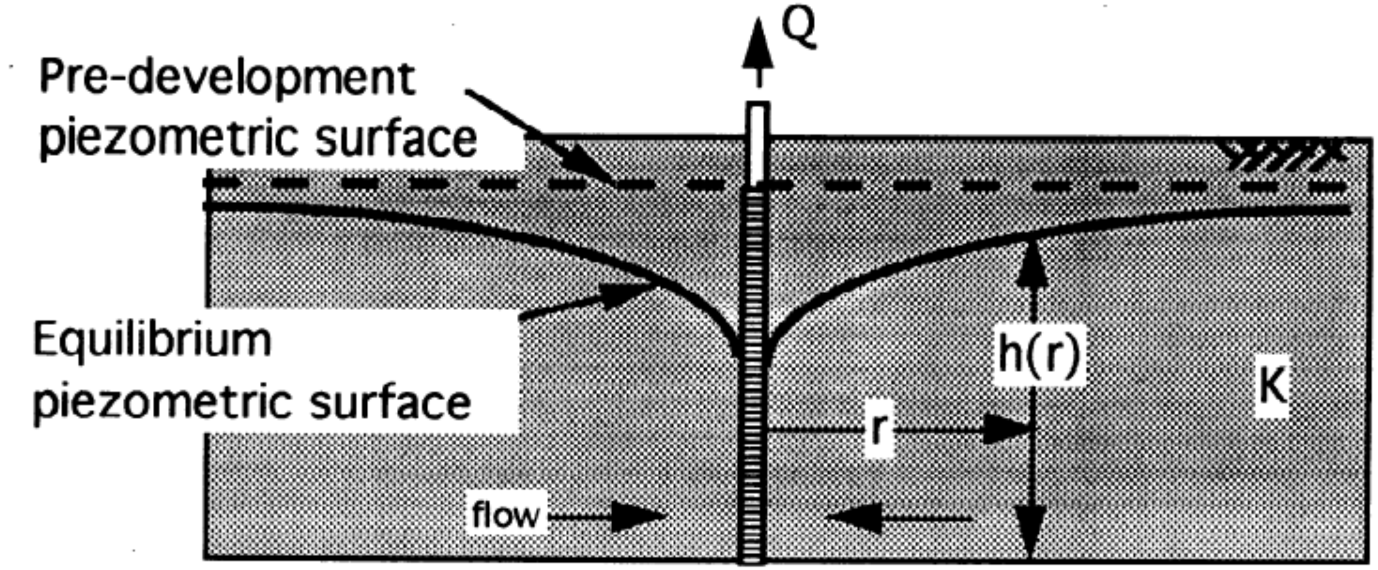
s(r) is a solution to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right)=0.$$

Theim equation $s_1 - s_2 = \frac{Q}{2\pi T}\ln\left(\frac{r_2}{r_1}\right)$ Used to Infer hydraulic properties.

Unconfined Aquifer

- Sketch



Homogeneous, isotropic, confined aquifer. Flow in radial direction only. Steady state. No internal sources/sinks.

$$h^2(r) = \frac{Q}{\pi K} \ln\left(\frac{r}{R}\right) + h_0^2$$

Unconfined Aquifer

- Several solutions:

Corrected drawdown form treats entire aquifer as if sat. thickness is $b = h_o$

$$s'(r) = \frac{Q}{2\pi K h_o} \ln\left(\frac{R}{r}\right)$$

and

$$s(r) = h_o - h(r) = h_o - \sqrt{h_o^2 + \frac{Q}{\pi K} \ln\left(\frac{r}{R}\right)} =$$

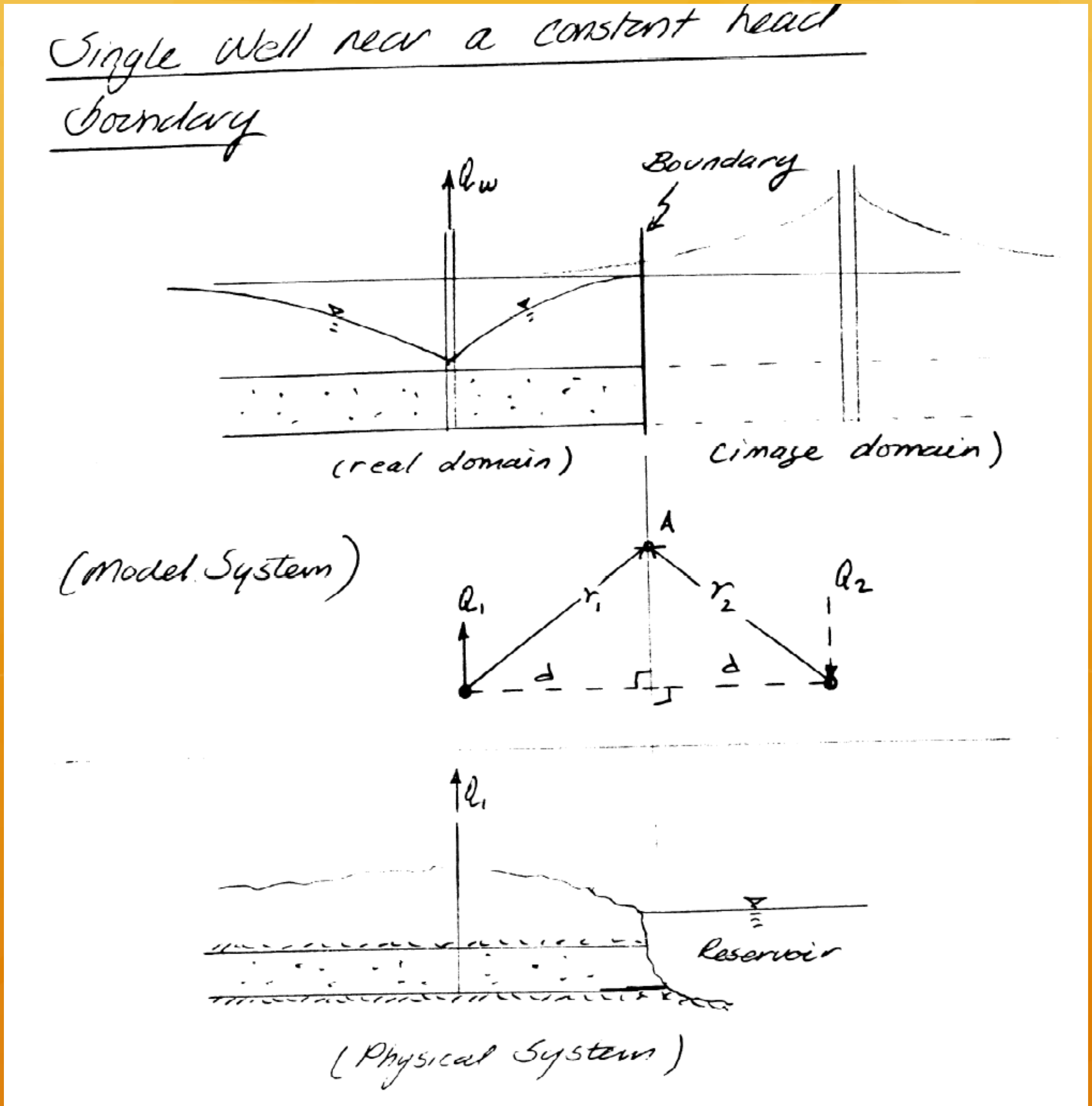
$$= h_o \left(1 - \sqrt{1 + \frac{2s'}{h_o}}\right)$$

Use this one

Superposition

- Linear combination of solutions to model effect of
  - Multiple wells
  - Boundaries

Superposition



Superposition

$$s_A = s_{A\,(\text{from well \#1})} + s_{A\,(\text{from well \#2})}$$

But A is a constant head boundary $\therefore s_A = 0$

$$s_A = \frac{Q_1}{2\pi K b} \ln\left(\frac{R}{r_1}\right) - \frac{Q_2}{2\pi K b} \ln\left(\frac{R}{r_2}\right)$$

"$-$" because well #2 is "modeled" as injection to produce zero drawdown at A (anywhere along boundary)

$$|Q_1| = |Q_2| = |Q_w|$$

Superposition

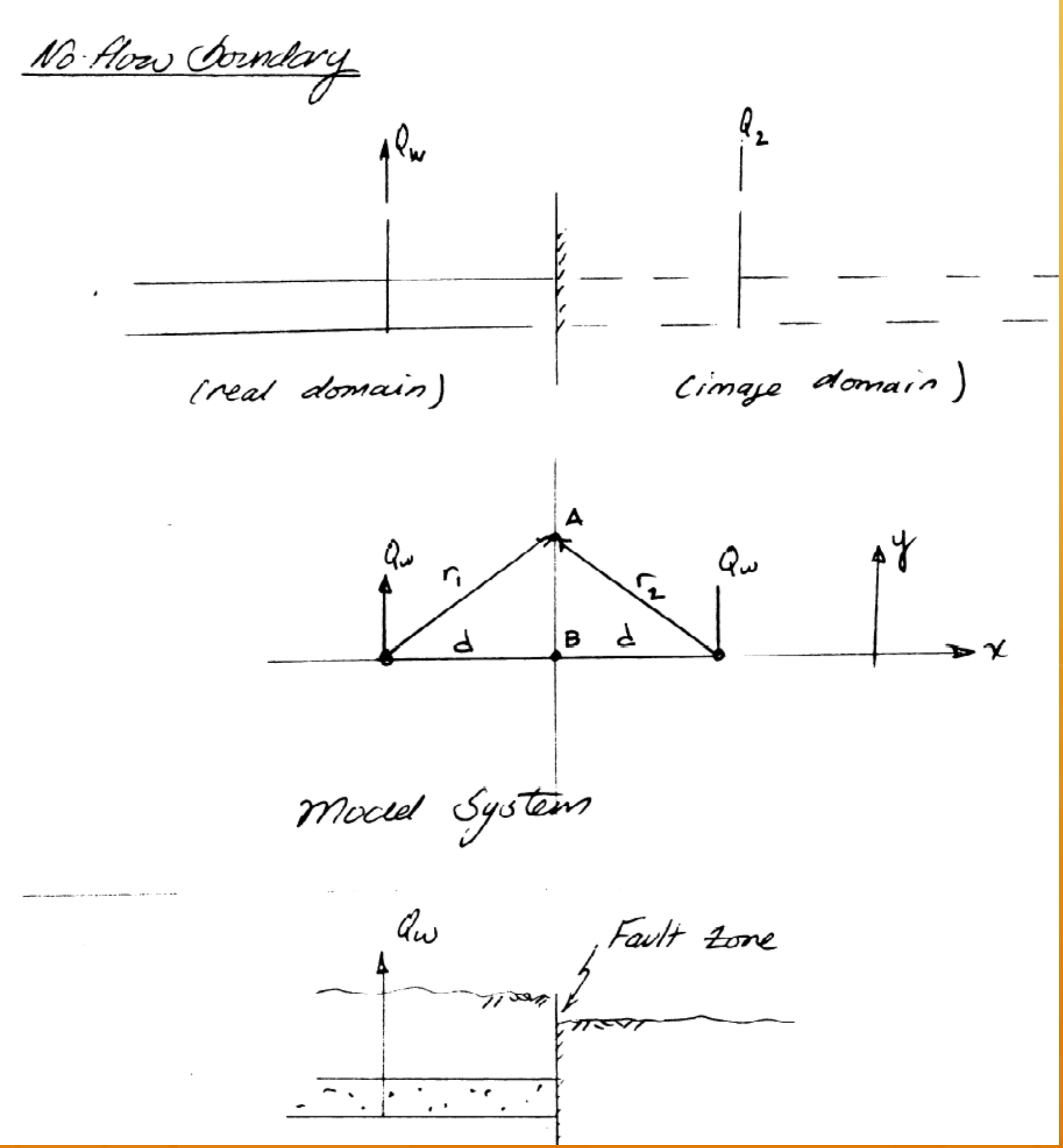
$$\therefore s_A = \frac{Q_w}{2\pi Kb} \ln\left|\frac{R}{r_1}\right| - \frac{Q_w}{2\pi Kb} \ln\left|\frac{R}{r_2}\right|$$

$$= \frac{Q_w}{2\pi Kb} \ln\left|\frac{r_2}{r_1}\right|$$

Now if Q_2 is located the same distance from the boundary as Q_1, $r_2 = r_1$

$\Rightarrow s_a = 0$ (as expected)

Superposition



Superposition

$$s_A = \frac{Q_w}{2\pi Kb}\ln\left(\frac{R}{r_1}\right) + \frac{Q_w}{2\pi Kb}\ln\left(\frac{R}{r_2}\right) = \frac{Q_w}{2\pi Kb}\ln\left(\frac{R^2}{r_1 r_2}\right)$$

$$s_B = \frac{Q_w}{2\pi Kb}\ln\left(\frac{R^2}{d^2}\right)$$

$$\left.\frac{\partial s_B}{\partial x}\right|_{\text{Well \#1}} = \frac{Q_w}{2\pi Kbd} \; ; \; \left.\frac{\partial s_B}{\partial x}\right|_{\text{Well \#2}} = \frac{-Q_w}{2\pi Kbd}$$

$$\left.\frac{\partial s_B}{\partial x}\right|_{\text{both wells}} = \frac{Q_w}{2\pi Kbd} - \frac{Q_w}{2\pi Kbd} = 0$$

(This will be the result for all points along the boundary)

Superposition

Summary

① drawdown in confined aquifer due to well (steady flow)

$$s(r) = \frac{Q_w}{2\pi K b} \ln\left|\frac{R}{r}\right|$$

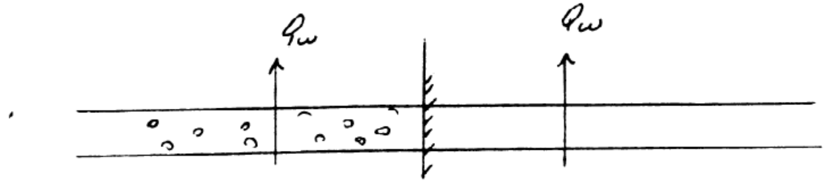
② Constant head boundary:

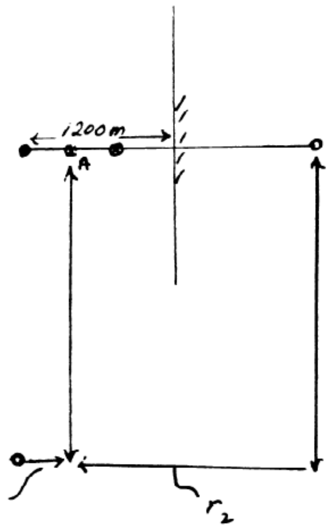
a) locate image well same distance from boundary as real well, opposite sense on Q_w

③ No flow boundary

b) locate image well same distance from boundary as real well, same sense on Q_w

Superposition





$Q_w = 100 \, m^3/d$

$Kb = 1 \, m^2/d$

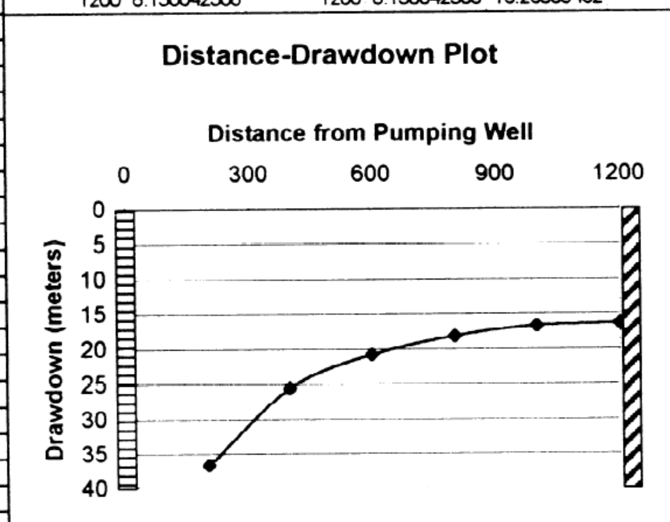
$R = 2000 \, m$

Plot distance-drawdown profile between well and boundary

$$s_A = \frac{Q_w}{2\pi Kb} \ln\left(\frac{R}{r_1}\right) + \frac{Q_w}{2\pi Kb} \ln\left(\frac{R}{r_2}\right)$$

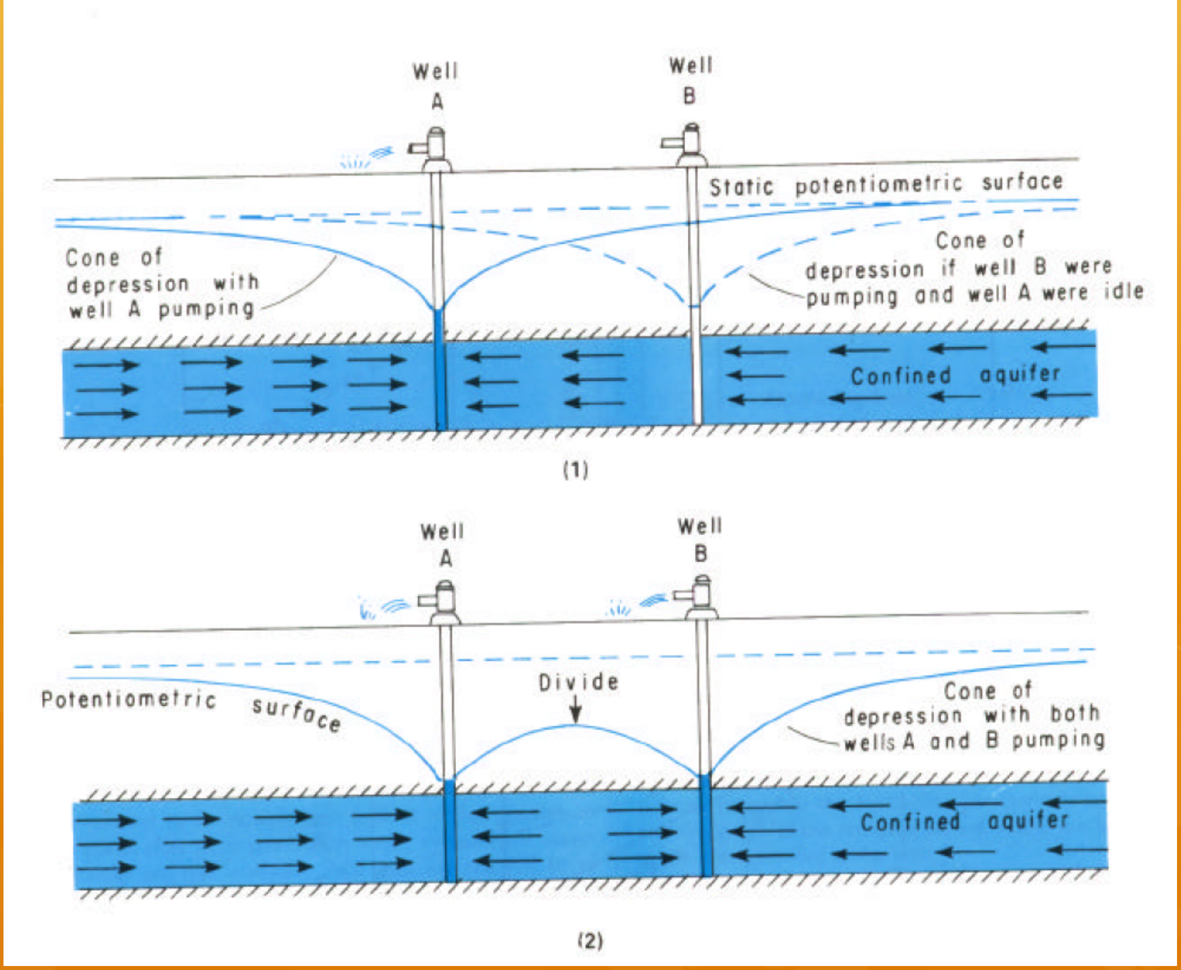
Superposition

	A	B	C	D	E	F
1	Qw	100	m3/d			
2	T	1	m2/d			
3	R	2000	m			
4		real well		image well		
5	distance from well to field point (m)	drawdown due to well	distance from well to field point (m)	drawdown due to well	drawdown at field point	
6	200	36.64677994	2200	0	36.64677994	
7	400	25.61499994	2000	0	25.61499994	
8	600	19.16182232	1800	1.676864687	20.838687	
9	800	14.58321993	1600	3.551439921	18.13465985	
10	1000	11.03178001	1400	5.67665804	16.70843805	
11	1200	8.130042308	1200	8.130042308	16.26008462	



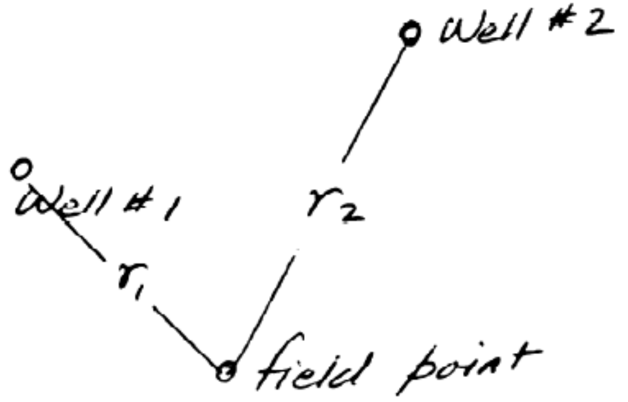
Well Interference

- Two (or more) wells operating near each other produce a combination drawdown that might affect operation of the wells
  - If the pump impellers are not deep enough, the well may not produce because of drawdown caused by nearby wells



Well Interference

- Superposition is used to model such situations



$$s_{\text{field point}} = \frac{Q_1}{2\pi K b} \ln\left|\frac{R}{r_1}\right| + \frac{Q_2}{2\pi K b} \ln\left|\frac{R}{r_2}\right|$$

Well Interference

- Superposition is used to model such situations

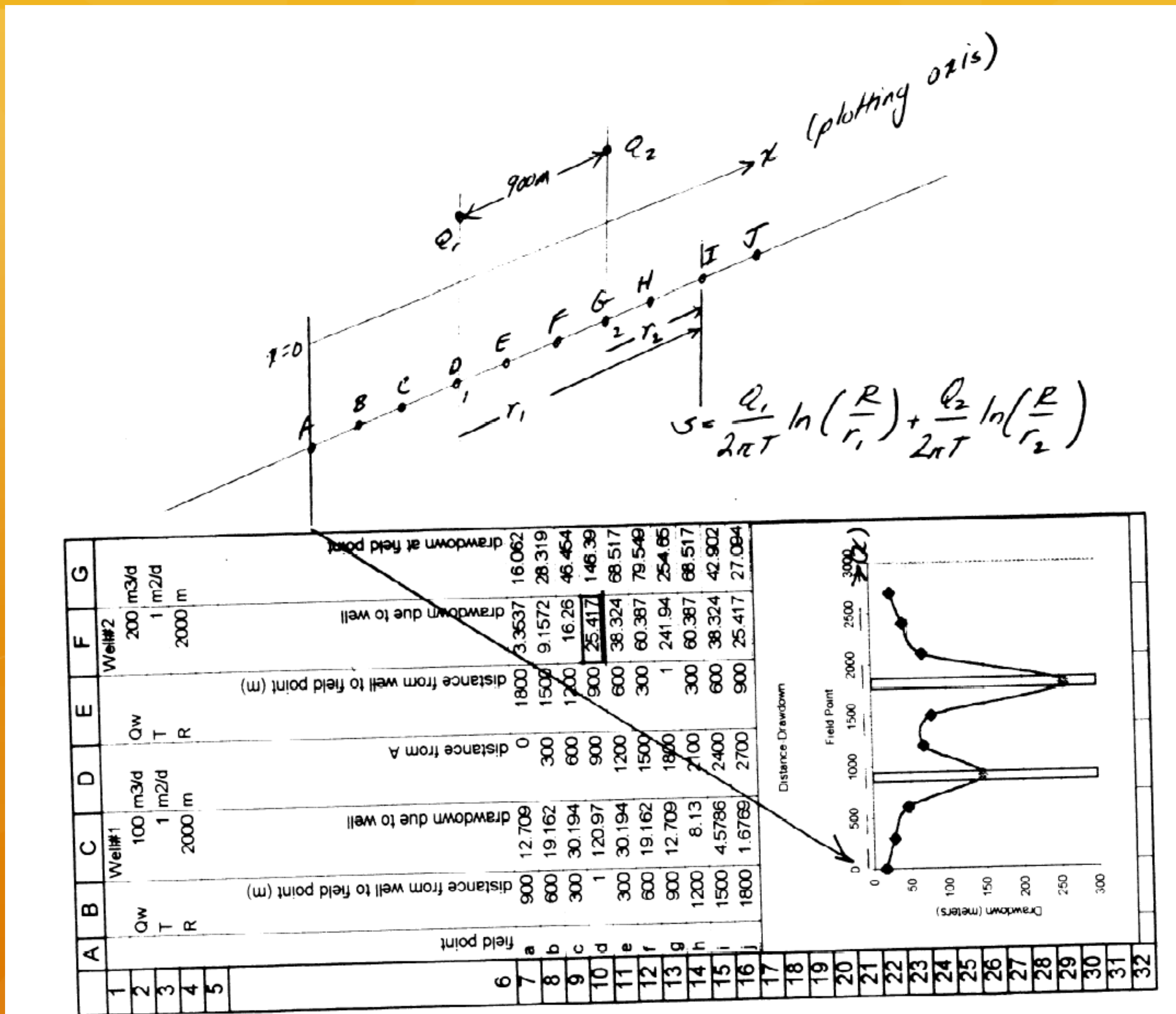
Example

$T = 1\ m^2/d$
$Q_1 = 100\ m^3/d$
$Q_2 = 200\ m^3/d$
$R = 2000\ m$

Wells located 900m apart. Show distance-drawdown profile.

Determine "interference" at well #1 due to operation of well #2

Well Interference



	A	B	C	D	E	F	G
1			Well#1			Well#2	
2		Qw	100	m3/d	Qw	200	m3/d
3		T	1	m2/d	T	1	m2/d
4		R	2000	m	R	2000	m
5							
6	field point	distance from well to field point (m)	drawdown due to well	distance from A	distance from well to field point (m)	drawdown due to well	drawdown at field point
7	a	900	12.709	0	1800	3.3537	16.062
8	b	600	19.162	300	1500	9.1572	28.319
9	c	300	30.194	600	1200	16.26	46.454
10	d	1	120.97	900	900	25.417	146.39
11	e	300	30.194	1200	600	38.324	68.517
12	f	600	19.162	1500	300	60.387	79.549
13	g	900	12.709	1800	1	241.94	254.65
14	h	1200	8.13	2100	300	60.387	68.517
15	i	1500	4.5786	2400	600	38.324	42.902
16	j	1800	1.6769	2700	900	25.417	27.094

Well Interference

	A	B	C	D	E	F	G
1			Well#1			Well#2	
2		Qw	100	m3/d	Qw	200	m3/d
3		T	1	m2/d	T	1	m2/d
4		R	2000	m	R	2000	m
5							
6	field point	distance from well to field point (m)	drawdown due to well	distance from A	distance from well to field point (m)	drawdown due to well	drawdown at field point
7	a	900	12.709	0	1800	3.3537	16.062
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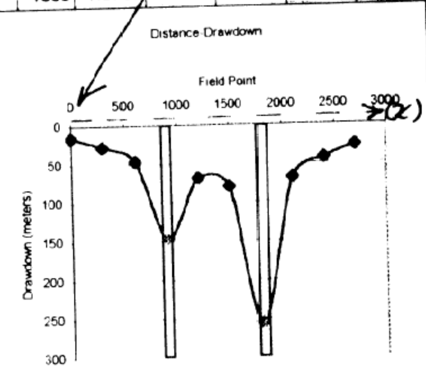
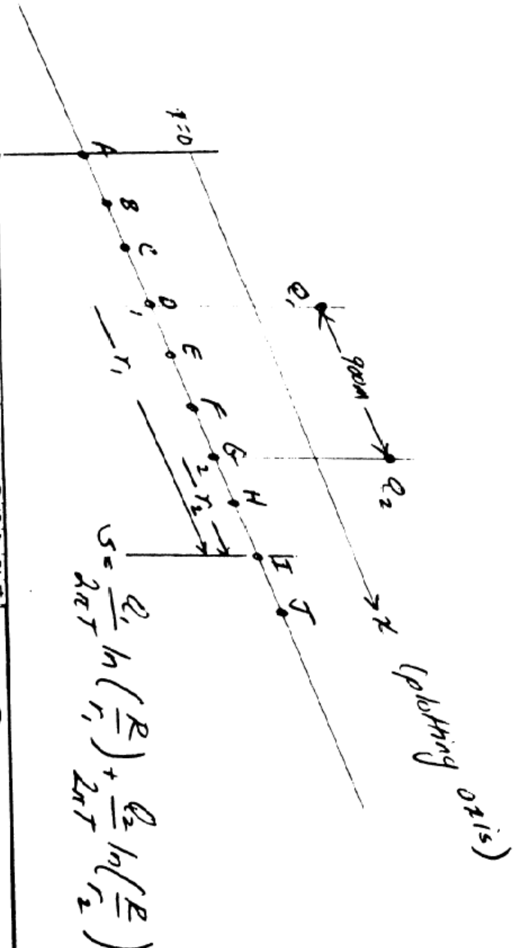
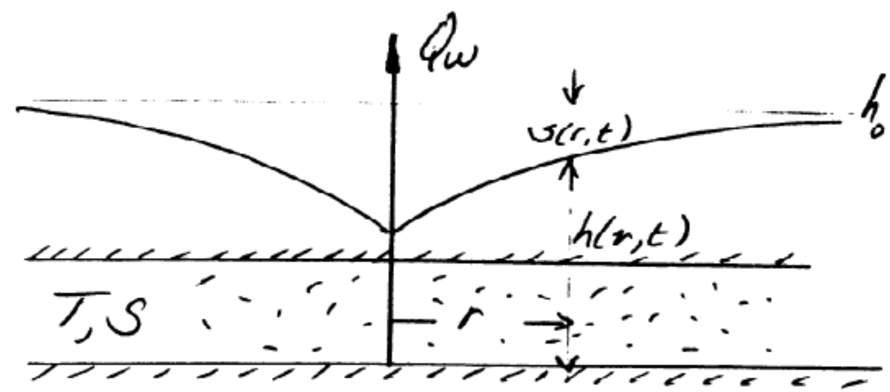
10	d	1	120.97	900	900	25.417	146.39
11	e	300	30.194	1200	600	38.324	68.517
12	f	600	19.162	1500	300	60.387	79.549
13	g	900	12.709	1800	1	241.94	254.65
14	h	1200	8.13	2100	300	60.387	68.517
15	i	1500	4.5786	2400	600	38.324	42.902
16	j	1800	1.6769	2700	900	25.417	27.094





Confined Aquifer

- Transient flow to a well
  (sketch of derivation – more in readings)



r = radial distance from well

s = drawdown $(h_0 - h(r,t))$

horizontal, radial flow

homogeneous, isotropic, aquifer

Confined Aquifer

- Governing PDE and BCs
  - Oddly enough Drawdown is lower case “s” and storativity is upper case “S”– need to be aware when reading.

$$\mathrm{div}\left(T\,\mathrm{grad}(h)\right) = S\frac{\partial h}{\partial t}$$

$$s = h_0 - h(r,t)$$

$$\frac{\partial s}{\partial t} = -\frac{\partial h}{\partial t} \qquad \frac{\partial^2 s}{\partial r^2} = -\frac{\partial^2 h}{\partial r^2} \qquad \frac{\partial s}{\partial r} = -\frac{\partial h}{\partial r}$$

Storage Coefficient

$$\therefore\quad T\frac{\partial^2 s}{\partial r^2} + \frac{T}{r}\frac{\partial s}{\partial r} = S\frac{\partial s}{\partial t} \qquad \text{or}$$

Drawdown

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t}$$

Confined Aquifer

- Governing PDE and BCs

Boundary Conditions

$r = \infty, \quad s = 0; \quad r \to 0, \quad \lim_{r \to 0}\left(2\pi T \frac{\partial s}{\partial r}\right) = -Q_w$

Initial Conditions

$t = 0, \quad s = 0$

Confined Aquifer

- Solving the PDE – apply a Boltzman Transformation

Obtaining a Solution

let $u = \dfrac{r^2 S}{4Tt}$

$t \to 0,\ s \to 0 \Rightarrow u \to \infty$

$r \to \infty,\ s \to 0 \Rightarrow u \to \infty$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q_w}{2\pi T}$$

$$\lim_{u \to 0} r\frac{\partial s}{\partial r} = \lim_{u \to 0} r\frac{\partial s}{\partial u}\frac{\partial u}{\partial r} = \lim_{u \to 0} \frac{2u\,\partial s}{\partial u} = -\frac{Q_w}{2\pi T}$$

$$\therefore \lim_{u \to 0} u\frac{\partial s}{\partial u} = -\frac{Q_w}{4\pi T}$$

Confined Aquifer

- Solving the PDE – apply a Boltzman Transformation
  - Convert PDE into an ODE

Now transform governing equation into an ODE

$$\frac{\partial s}{\partial t} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial t} = -\frac{u}{t}\frac{\partial s}{\partial u}$$

$$\frac{\partial s}{\partial r} = \frac{\partial s}{\partial u}\frac{\partial u}{\partial r} = \frac{2u}{r}\frac{\partial s}{\partial u}$$

$$\frac{\partial^2 s}{\partial r^2} = \frac{2u}{r^2}\frac{\partial s}{\partial u} + \frac{4u^2}{r^2}\frac{\partial^2 s}{\partial u^2}$$

Substitute into PDE

$$-\frac{u}{t}\frac{\partial s}{\partial u}\frac{S}{T} = \frac{4u^2}{r^2}\frac{\partial^2 s}{\partial u^2} + \frac{2u}{r^2}\frac{\partial s}{\partial u} + \frac{2u}{r}\frac{\partial s}{\partial u}$$

Confined Aquifer

- Solving the PDE – Algebra and Calculus

Multiply by $r^2/4$, divide by u^2, rearrange:

$$\frac{\partial^2 s}{\partial u^2} + \left(\frac{u+1}{u}\right)\frac{\partial s}{\partial u} = 0 \qquad \text{let } x = \frac{\partial s}{\partial u}$$

$$\frac{dx}{du} = -\left(\frac{u+1}{u}\right)x \quad \Rightarrow \quad -\int \frac{dx}{x} = \int \frac{u+1}{u}\,du$$

$$-\ln|x| = \ln|u| + u + \ln|c|$$

$$u = -\ln|x| - \ln|u| - \ln|c| = -\ln|xuc|$$

$$e^{-u} = xuc \qquad \text{but } x = \frac{ds}{du}$$

So:

$$\frac{ds}{du} = \frac{1}{c}\frac{e^{-u}}{u}$$

Confined Aquifer

- Apply IC and BCs

Apply B.C. $\quad u \frac{ds}{du} \rightarrow -\frac{Q_w}{4\pi T} \quad u \rightarrow 0$

$$\therefore \quad \frac{1}{C} = -\frac{Q_w}{4\pi T}$$

$$\frac{ds}{du} = -\frac{Q_w}{4\pi T}\frac{e^{-u}}{u} \Rightarrow \int_0^s ds = -\int_\infty^u \frac{Q_w}{4\pi T}\frac{e^{-u}}{u}\,du$$

$$s(u) = -\frac{Q_w}{4\pi T}\underbrace{\int_\infty^u \frac{e^{-u}}{u}\,du}_{-Ei(u)}$$

exponential integral

Confined Aquifer

- Apply IC and BCs

Apply B.C. $\quad u\dfrac{ds}{du} \to -\dfrac{Q_w}{4\pi T} \quad u \to 0$

$$\therefore \frac{1}{c} = -\frac{Q_w}{4\pi T}$$

$$\frac{ds}{du} = -\frac{Q_w}{4\pi T}\frac{e^{-u}}{u} \Rightarrow \int_0^s ds = -\int_\infty^u \frac{Q_w}{4\pi T}\frac{e^{-u}}{u}\,du$$

$$s(u) = -\frac{Q_w}{4\pi T}\underbrace{\int_\infty^u \frac{e^{-u}}{u}\,du}_{-Ei(u)}$$

exponential integral

Confined Aquifer

- Now we have a solution, but how to evaluate the integral?
  - Once upon a time you would look up values in a table (readings)
  - Alternatively, you can apply a series expansion of the integrand and find a series solution

Confined Aquifer

- Now we have a solution, but how to evaluate the integral?

EVALUATE $Ei(u)$ BY

- SERIES EXPANSION:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$Ei(u) = \int_u^\infty \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x}\,dx = \ln|x| - x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \cdots \Big|_u^\infty$$

NOTE: $\left[\ln|x| - x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \cdots\right]\Big|_{x=\infty} = \gamma \approx -0.5772$

$$\therefore\ Ei(u) = \gamma - \ln|u| + u - \frac{u^2}{2\cdot 2!} + \frac{u^3}{3\cdot 3!} \mp \cdots$$

Confined Aquifer

- Now we have a solution, but how to evaluate the integral?

- POLYNOMIAL APPROXIMATION

$$Ei(u) \simeq -\ln|u| + a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5$$

$a_0 = -0.577\,215\,66$ $\quad$ $a_3 = 0.055\,199\,68$

$a_1 = 0.999\,991\,93$ $\quad$ $a_4 = -0.009\,760\,04$

$a_2 = -0.249\,910\,55$ $\quad$ $a_5 = 0.001\,078\,57$

FOR $0 < u \leq 1$

$$Ei(u) \simeq \left(\frac{u^4 + a_1 u^3 + a_2 u^2 + a_3 u + a_4}{u^4 + b_1 u^3 + b_2 u^2 + b_3 u + b_4}\right)\left(\frac{1}{u\exp(u)}\right)$$

$a_1 = 8.573\,328\,740\,1$ $\quad$ $b_1 = 9.573\,322\,345\,4$

$a_2 = 18.059\,016\,973\,0$ $\quad$ $b_2 = 25.632\,956\,148\,6$

$a_3 = 8.634\,760\,892\,5$ $\quad$ $b_3 = 21.099\,653\,082\,7$

$a_4 = 0.267\,773\,734\,3$ $\quad$ $b_4 = 3.958\,496\,922\,8$

FOR $1 \leq u \leq \infty$

Confined Aquifer

- Now we have a solution, but how to evaluate the integral?

- POLYNOMIAL APPROXIMATION

$$Ei(u) \simeq -\ln|u| + a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5$$

$a_0 = -0.577\ 215\ 66$

$a_1 = 0.999\ 991\ 93$

$a_2 = -0.249\ 910\ 55$

$a_3 = 0.055\ 199\ 68$

$a_4 = -0.009\ 760\ 04$

$a_5 = 0.001\ 078\ 57$

FOR $0 < u \leq 1$

$$Ei(u) \simeq \left(\frac{u^4 + a_1 u^3 + a_2 u^2 + a_3 u + a_4}{u^4 + b_1 u^3 + b_2 u^2 + b_3 u + b_4}\right)\left(\frac{1}{u \exp(u)}\right)$$

$a_1 = 8.573\ 328\ 740\ 1$

$a_2 = 18.059\ 016\ 973\ 0$

$a_3 = 8.634\ 760\ 892\ 5$

$a_4 = 0.267\ 773\ 734\ 3$

$b_1 = 9.573\ 322\ 345\ 4$

$b_2 = 25.632\ 956\ 148\ 6$

$b_3 = 21.099\ 653\ 082\ 7$

$b_4 = 3.958\ 496\ 922\ 8$

FOR $1 \leq u \leq \infty$

Confined Aquifer

- VBA Code (to evaluate the well function)

TheisModel.xlsm - Module1 (Code)

(General) W

```
Function W(U) As Double
'Theis Well Function -- actually the exponential integral
If U <= 1 Then
A0 = -0.57721566
A1 = 0.99999193
A2 = -0.24991055
A3 = 0.05519968
A4 = -0.00976004
A5 = 0.00107857
W = (-Log(U) + A0 + A1 * U + A2 * U ^ 2 + A3 * U ^ 3 + A4 * U ^ 4 + A5 * U ^ 5)
Exit Function
Else
A1 = 8.5733287401
A2 = 18.059016973
A3 = 8.6347608925
A4 = 0.2677737343
B1 = 9.5733223454
B2 = 25.6329561486
B3 = 21.0996530827
B4 = 3.9584969228
W = ((U ^ 4 + A1 * U ^ 3 + A2 * U ^ 2 + A3 * U + A4) / (U ^ 4 + B1 * U ^ 3 + B2 * U ^ 2 + B3 * U + B4)) / (U * Exp(U))
Exit Function
End If
End Function
```

Confined Aquifer

TheisModel.xlsm

Model Name	1D_radial_flow_confined_aquifer_transient	
Model Type:	Hydraulic Model	
References:	Theis, C.V. 1935. The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage: American Geophysical Union Transactions, 16th Annual Meeting, v. 16, pt. 2, p. 519-524.	
	Walton, W. C. 1989. Analytical Groundwater Modeling, Flow and Contaminant Migration, Lewis Publishers, Chelsea, MI.	
Method:	Polynomial approximation ot exponential integral	
Author:	Dr. T.G. Cleveland for CIVE6361/7332 Students; Spring 1995	
	Rewrite in VBA Mac Excel 2011 Fall 2015	
Notes:	Infinite Confined Aquifer	
	Steady Pumping at Origin	
	Computes Drawdown at radial distance, r, for different time values t	
	Uses Polynomial approximation to well function (see references)	
Macros:	W(U)	Well Functions contained in Module1 as local to this worksheet macros

Conversion Calculator (Use GoalSeek to find U.S Customary values for SI Units)

Q	25	<=gpm	3.342246	<=ft^3/min	4812.8342	<=ft^3/day	0.094715	<=m^3/min	136.389	<=m^3/day
T	0.76	<=ft^2/min	1094.4	<=ft^2/day	5.6848	<=gpm/ft	101.7252	<=m^2/day	0.070642	<=m^2/min

MODEL INPUT VALUES

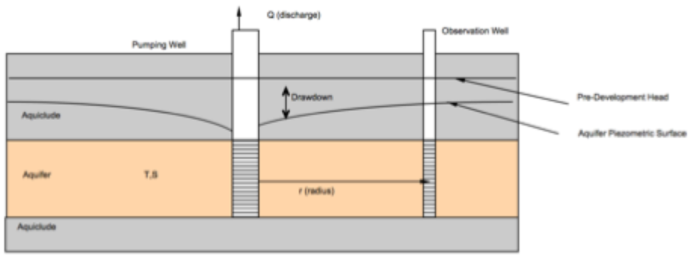
Input Data (must use consistent length and time units!)

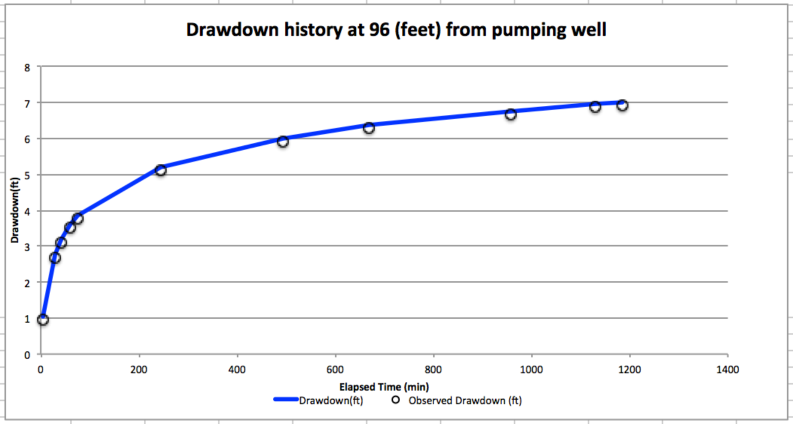
Item	Value	Units	Description
Q	11	ft^3/min	Pumping well discharge (L^3/t)
T	0.76	ft^2/min	Aquifer transmissivity (L^2/t)
S	0.0005		Aquifer storage coefficient
r	96	Radial distance of observation well from pumping well (L)	

Computed Constants

Q/(4πT)	1.15178
Chart Title	Drawdown history at 96 (feet) from pumping well
Time	Drawdown

Elapsed Time (min)	r (ft)	u = (r^2 S / 4 T t)	W(u)	Drawdown(ft)	Observed Drawdown (ft)	Sq. Error
5	96	0.30316	0.897932	1.034219	0.9615503	0.005281
28	96	0.05414	2.392464	2.75559	2.6659802	0.00803
41	96	0.03697	2.757051	3.175514	3.0855088	0.008101
60	96	0.02526	3.126297	3.600804	3.5109705	0.00807
75	96	0.02021	3.344445	3.852062	3.7625233	0.008017
244	96	0.00621	4.510219	5.194776	5.1081853	0.007498
493	96	0.00307	5.210429	6.001264	5.9170095	0.007099
669	96	0.00227	5.514896	6.351943	6.2687644	0.006919
958	96	0.00158	5.873277	6.764719	6.6828359	0.006705
1129	96	0.00134	6.037278	6.953611	6.872329	0.006607
1185	96	0.00128	6.085625	7.009296	6.9281921	0.006578
					SSE =>	0.078904





Superposition

ADDITIONAL SOLUTIONS BY SUPERPOSITION

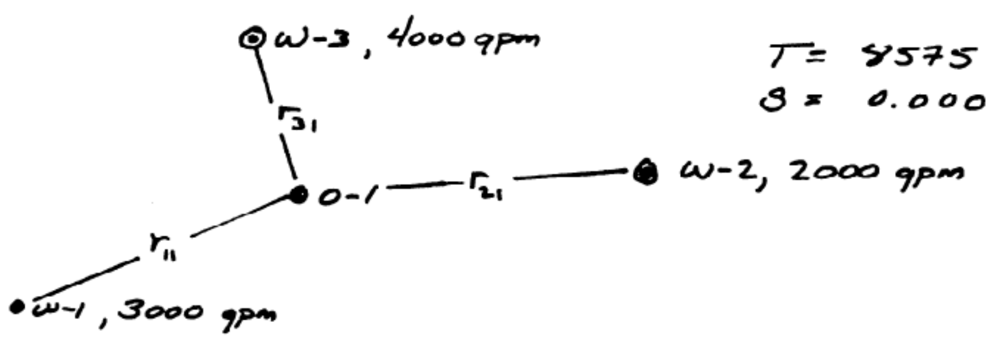
$$\frac{S}{T}\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}$$ IS LINEAR IN s & t

∴ CAN DEVELOP ADDITIONAL SOLUTIONS BY SUPERPOSITION

EXAMPLE

SUPPOSE WELLFIELD BELOW IS PLANNED TO OPERATE AS SHOWN.

WHAT IS THE DRAWDOWN AT O-1 AFTER 365 DAYS OF PUMPING?



$T = 8575\ ft^2/d$

$S = 0.0008$

Superposition

SOLUTION:

- FIND DRAWDOWN AT O-1 FROM EACH PUMPING WELL
- TOTAL DRAWDOWN IS SIMPLY SUM OF INDIVIDUAL DRAWDOWNS

SUPPOSE: $r_{11} = 1500 ft$ $\quad t = 365$ day

$r_{21} = 1470 ft$

$r_{31} = 1000 ft$

COMPUTE:

$$U_{11} = \frac{r_{11}^2 S}{4Tt} = 0.000144$$

$$U_{21} = \frac{r_{21}^2 S}{4Tt} = 0.000138$$

$$U_{31} = \frac{r_{31}^2 S}{4Tt} = 0.000064$$

Superposition

EVALUATE $Ei(u)$

			Q (gpm)	Q (ft³/day)
$Ei(u_{1,1}) =$	8.270182	;	3000	577540
$Ei(u_{2,1}) =$	8.310582	;	2000	385027
$Ei(u_{3,1}) =$	9.081032	;	4000	770053

Superposition

COMPUTE INDIVIDUAL DRAWDOWNS

$$s_{11} = \frac{Q_1}{4\pi T} E_i(u_{11}) = \frac{577540}{4\pi(8575)} 8.270182 = 44.325$$

$$s_{21} = \frac{Q_2}{4\pi T} E_i(u_{21}) = 29.694$$

$$s_{31} = \frac{Q_3}{4\pi T} E_i(u_{31}) = 64.895$$

TOTAL DRAWDOWN

$$s = \sum_{i=1}^{3} s_{i1} \qquad \Sigma = 139'$$

$\therefore$ TOTAL PREDICTED DRAWDOWN AT O-1 FROM THE PUMPING WELL ENSEMBLE IS 139' AFTER 365 DAYS OF PUMPING

Superposition

GENERAL FORM:

$$s_j = \sum_{i=1}^{NW} \frac{Q_i}{4\pi T} Ei\left(\frac{r_{ij}^2 S}{4Tt}\right)$$

r_{ij} IS RADIUS FROM i-th WELL TO FIELD POINT j

Q_i IS PUMPING RATE OF j-th WELL

Convolution

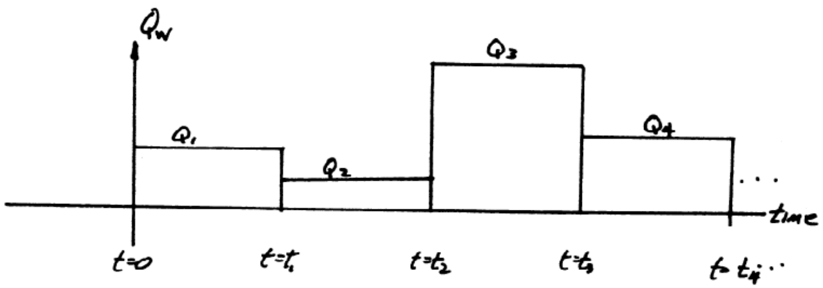
- Convolution == Superposition in Time

VARIABLE PUMPING RATES

USE CONVOLUTION IN TIME:

$$s(r,t) = \frac{Q}{4\pi T} Ei(u); \quad u = \frac{r^2 S}{4Tt}$$

ESTIMATE RESPONSE OVER SEVERAL PLANNING PERIODS (SEQUENTIAL) WITH DIFFERENT PUMP RATES:



FROM $t=0$ to $t=t_1$; $Q_w = Q_1$

$t=t_1$ to $t=t_2$; $Q_w = Q_2$

$\vdots$

Convolution

- Convolution == Superposition in Time

RESPONSE AT SOME ARBITRARY FIELD POINT:

$$0 \le t \le t_1 ; \quad s = \frac{Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4Tt}\right)$$

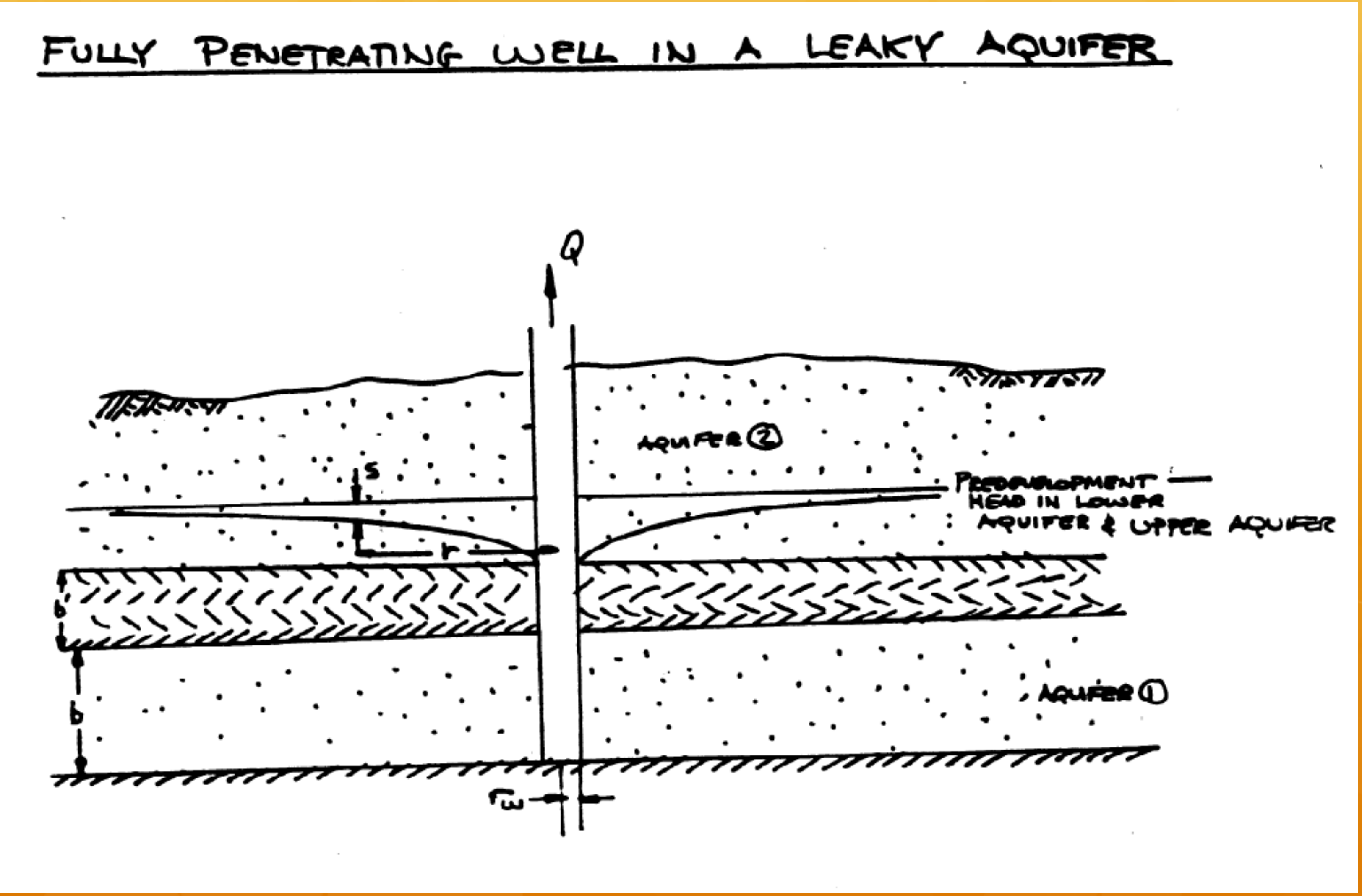
$$t_1 \le t \le t_2 ; \quad s = \frac{Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4Tt}\right) + \frac{Q_2 - Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_1)}\right)$$

$$t_2 \le t \le t_3 ; \quad s = \frac{Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4Tt}\right) + \frac{Q_2 - Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_1)}\right) + \frac{Q_3 - Q_2}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_2)}\right)$$

$$t_3 \le t \le t_4 ; \quad s = \frac{Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4Tt}\right) + \frac{Q_2 - Q_1}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_1)}\right) + \frac{Q_3 - Q_2}{4\pi T} Ei\left(\frac{r^2 S}{4T(t-t_2)}\right)$$

Leaky Aquifer

- Hantush Model



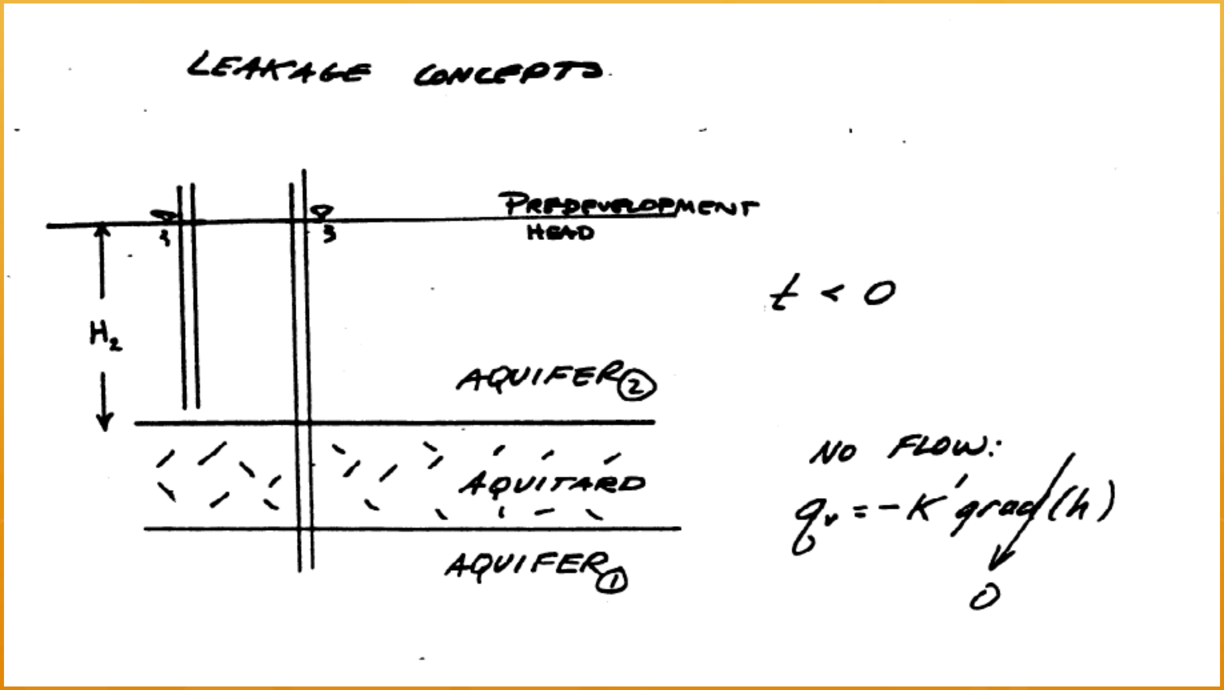
Leaky Aquifer

- Hantush Model

- WELL DISCHARGES AT CONSTANT RATE Q
- INFINITESMAL WELL DIAMETER
- AQUIFER ① OVERLAIN BY CONFINING BED (AQUITARD) OF THICKNESS b', HYDRAULIC CONDUCTIVITY K'
- AQUIFER ② OVERLIES AQUITARD AND HAS CONSTANT HEAD
- HYDRAULIC GRADIENT ACROSS CONFINING BED CHANGES INSTANTLY — NO STORAGE IN AQUITARD
- AQUIFER FLOW IS 2D HORIZONTAL, AQUITARD FLOW IS VERTICAL

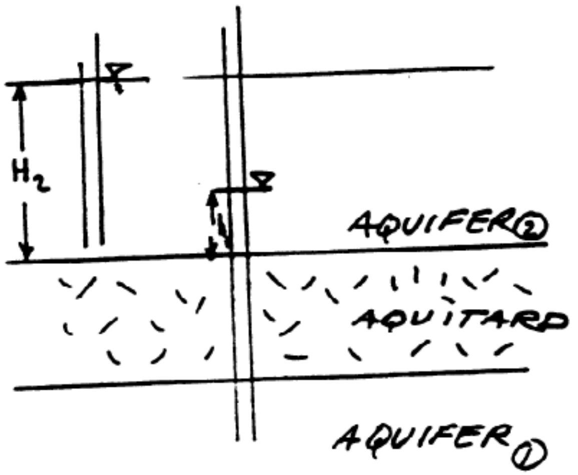
Leaky Aquifer

- Leakage prior to pumping



Leaky Aquifer

- Leakage after pumping begins



$t > 0$

FLOW: $\neq 0$

$$q_v = -K' \cancel{\text{grad}}(h)$$

$$q_v = -K' \cdot \frac{h_1 - H_2}{b'}$$

$$q_v = K' \cdot \frac{H_2 - h_1}{b'}$$

Leaky Aquifer

- Leakage – drawdown relationship

BUT: $H_2 - h_1 = s$ (DRAWDOWN)

$$\therefore \; q_v = \frac{K' s}{b'}$$

- NO STORAGE IN AQUITARD ⇒ CHANGE IN HEAD CAUSES INSTANTANEOUS CHANGE IN FLOW
- H_2 = CONSTANT (ASSUMPTION 4)

Leaky Aquifer

- Governing PDE and BCs

BASIC EQUATIONS

$$S\frac{\partial h}{\partial t} = \mathrm{div}\left(\mathbf{T}\,\mathrm{grad}(h)\right) \pm \text{sources}$$

OR

$$S\frac{\partial h}{\partial t} = T\frac{\partial^2 h}{\partial r^2} + \frac{T}{r}\frac{\partial h}{\partial r} - \frac{(H_2-h)K'}{b'}$$

Leaky Aquifer

- Governing PDE and BCs

IN TERMS OF DRAWDOWN:

$$\frac{S}{T}\frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s K'}{T b'}$$

SUBJECT TO:

$$s(r,0) = 0$$

$$s(\infty,t) = 0$$

$$\lim_{r=r_w \to 0} \frac{\partial s}{\partial r} = -\frac{Q}{2\pi T}$$

Leaky Aquifer

- Solution(s)

SOLUTION BY

LÁPLACE TRANSFORM:

$$s = \frac{Q_w}{4\pi T} \int_u^\infty \frac{e^{-u - \frac{v^2}{u}}}{u} \, du$$

WHERE

$$u = \frac{r^2 S}{4Tt}$$

$$v^2 = \frac{r^2 K'}{4Tb'}$$

THE EXPONENTIAL INTEGRAL:

$$\int_u^\infty \frac{e^{-u - \frac{v^2}{u}}}{u} \, du = L(u,v)$$

Leaky Aquifer

- Solution(s)

L(u,v) IS SOMETIMES DENOTED BY

$W(u, r/B)$.

THE TERM $B = \sqrt{\dfrac{Tb'}{K'}}$ IS CALLED

THE LEAKAGE FACTOR

OBSERVE: $W(u, r/B) = L(u, 2v)$

Leaky Aquifer

- Solution(s)

L(u,v) IS SOMETIMES DENOTED BY

$W(u, r/B)$.

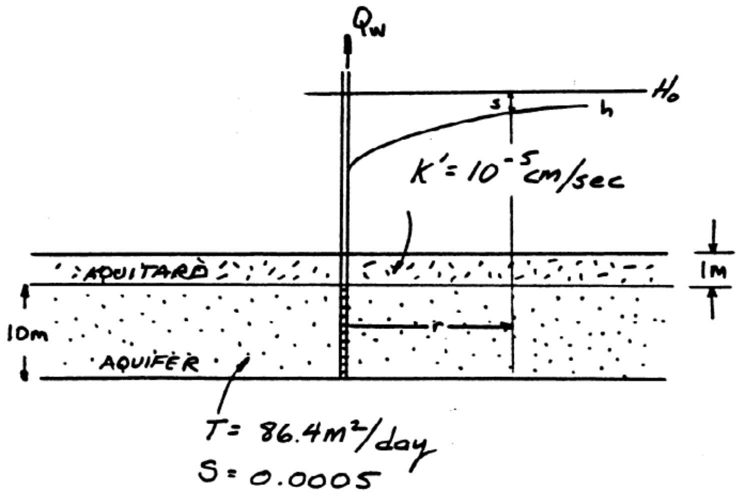
THE TERM $B = \sqrt{\frac{Tb'}{K'}}$ IS CALLED

THE LEAKAGE FACTOR

OBSERVE: $W(u, r/B) = L(u, 2v)$

Leaky Aquifer

- Example



AQUIFER 10m THICK IS OVERLAIN BY A 1m THICK AQUITARD. STORAGE IN AQUITARD IS ASSUMED NEGLIGIBLE. THE WELL PUMPS AT 500 m³/DAY. WHAT IS THE DRAWDOWN AT 1, 5, 10, 50, 100, 500, AND 1000 METERS AFTER ONE DAY OF PUMPING?

Leaky Aquifer

- Example

① MODEL: $s(r,t) = \frac{Q_w}{4\pi T} L(u,v)$

$u = \frac{r^2 S}{4Tt}$

$v^2 = \frac{r^2 K'}{4Tb'}$

Leaky Aquifer

- Example

② REDUCE DATA

$t = 1$ day

$Q_w = 500 \, m^3/d$

$K' = 8.64 \cdot 10^{-3} \, m/d$

$T = 86.4 \, m^2/d$

$S = 0.0005$

$b' = 1m$

$$u = \frac{r^2 (0.0005)}{4(86.4)(1)} = 1.45 \cdot 10^{-6} r^2$$

$$v^2 = \frac{r^2 (8.64 \cdot 10^{-3})}{4(86.4)(1)} = 2.5 \cdot 10^{-5} r^2$$

$$\frac{Q_w}{4\pi T} = \frac{500}{4(\pi)(86.4)} = 0.46$$

Leaky Aquifer

- Example

③ MAKE A TABLE

r	u	v^2	v	$2v\ (= \frac{r}{B})$	$W(u, \frac{r}{B})$
1 m	$1.45 \cdot 10^{-6}$	$2.5 \cdot 10^{-5}$	$5.0 \cdot 10^{-3}$	0.01	9.44
5 m	$3.63 \cdot 10^{-5}$	$6.25 \cdot 10^{-4}$	$2.5 \cdot 10^{-2}$	0.05	6.23
10 m	$1.45 \cdot 10^{-4}$	$2.5 \cdot 10^{-3}$	$5.0 \cdot 10^{-2}$	0.1	4.85
50 m	$3.63 \cdot 10^{-3}$	$6.25 \cdot 10^{-2}$	$2.5 \cdot 10^{-1}$	0.5	1.85
100 m	$1.45 \cdot 10^{-2}$	$2.5 \cdot 10^{-1}$	$5.0 \cdot 10^{-1}$	1	0.842
500 m	$3.63 \cdot 10^{-1}$	6.25	2.5	5	0.007
1000 m	1.45	25	5.0	10	0.0001

Leaky Aquifer

- Table-look up

TABLE 4.1.—*Selected values of* W(u,r/B)

[From Hantush (1961e)]

u	r/B: 0.001	0.003	0.01	0.03	0.1	0.3	1	3
1×10^{-6}	13.0031	11.8153	9.4425	7.2471	4.8541	2.7449	0.8420	0.0695
2	12.4240	11.6716						
3	12.0581	11.5098	9.4425					
5	11.5795	11.2248	9.4413					
7	11.2570	10.9951	9.4361					
1×10^{-5}	10.9109	10.7228	9.4176					
2	10.2301	10.1332	9.2961	7.2471				
3	9.8288	9.7635	9.1499	7.2470				
5	9.3213	9.2818	8.8827	7.2450				
7	8.9863	8.9580	8.6625	7.2371				
1×10^{-4}	8.6308	8.6109	8.3983	7.2122				
2	7.9390	7.9290	7.8192	7.0685				
3	7.5340	7.5274	7.4534	6.9068	4.8541			
5	7.0237	7.0197	6.9750	6.6219	4.8530			
7	6.6876	6.6848	6.6527	6.3923	4.8478			
1×10^{-3}	6.3313	6.3293	6.3069	6.1202	4.8292			
2	5.6393	5.6383	5.6271	5.5314	4.7079	2.7449		
3	5.2348	5.2342	5.2267	5.1627	4.5622	2.7448		
5	4.7260	4.7256	4.7212	4.6829	4.2960	2.7428		
7	4.3916	4.3913	4.3882	4.3609	4.0771	2.7350		
1×10^{-2}	4.0379	4.0377	4.0356	4.0167	3.8150	2.7104		
2	3.3547	3.3546	3.3536	3.3444	3.2442	2.5688		
3	2.9591	2.9590	2.9584	2.9523	2.8873	2.4110	.8420	
5	2.4679	2.4679	2.4675	2.4642	2.4271	2.1371	.8409	
7	2.1508	2.1508	2.1506	2.1483	2.1232	1.9206	.8360	
1×10^{-1}	1.8229	1.8229	1.8227	1.8213	1.8050	1.6704	.8190	
2	1.2226	1.2226	1.2226	1.2220	1.2155	1.1602	.7148	.0695
3	.9057	.9057	.9056	.9053	.9018	.8713	.6010	.0694
5	.5598	.5598	.5598	.5596	.5581	.5453	.4210	.0681
7	.3738	.3738	.3738	.3737	.3729	.3663	.2996	.0639
1×10^{0}	.2194	.2194	.2194	.2193	.2190	.2161	.1855	.0534
2	.0489	.0489	.0489	.0489	.0488	.0485	.0444	.0210
3	.0130	.0130	.0130	.0130	.0130	.0130	.0122	.0071
5	.0011	.0011	.0011	.0011	.0011	.0011	.0011	.0008
7	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001

all same (6.23) ② (4.35) ③ all same all same all same (1.45) ④ etc

LOG SCALE

USE LINEAR INTERPOLATION FOR MISSING VALUES

Leaky Aquifer

- Table-look up

④ APPLY: $s = \frac{Q_w}{4\pi T} W\left(u, \frac{r}{B}\right)$

(5) Solution

r (meters)	s (meters)
1	4.34
5	2.87
10	2.23
50	0.85
100	0.39
500	0.003
1000	0.000046

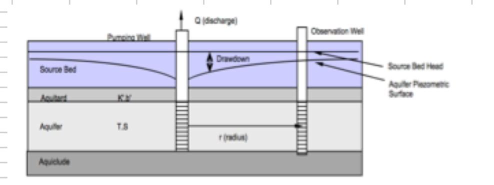
Leaky Aquifer

- More Modern Approach:
- VBA Script in Excel to evaluate W(u,r/B)
  - Complex uses:
  - ERFC (complimentary error function)
  - BESSELI (Bessel Function Type I)
  - BESSELJ (Bessel Function Type J)
- Script too complex to display – but ultimately it is just a function that can be evaluated just like SQRT(Z).

Leaky Aquifer

HantushLeakyModel.xlsm

Model Name	1D_radial_flow_leaky_aquifer_transient_hantush	
Model Type:	Hydraulic Model	
References:	Hantush	
Method:		
Author:	Dr. T.G. Cleveland for CIVE6361/7332 Students; Spring 1995	
	Rewrite in VBA Mac Excel 2011 Fall 2015	
Notes:	Infinite Confined Aquifer	
	Steady Pumping at Origin	
	Drawdown at radial distance, r, for different time values t in leaky aquifer system	
	Uses Polynomial approximation to well function (see references)	
Macros:	W(U),LEAKY(U,V)	Well Functions contained in Module1 as local to this worksheet macros



Conversion Calculator (Use GoalSeek to find U.S Customary values for SI Units)

Q	25	<=gpm	3.34225	<=ft^3/min	4812.83	<=ft^3/day	0.09736	<=m^3/min	140.201	<=m^3/day
T	0.1722	<=ft^2/min	247.968	<=ft^2/day	1.28806	<=gpm/ft	23.0488	<=m^2/day	0.01601	<=m^2/min

Input Data (must use consistent length and time units!)

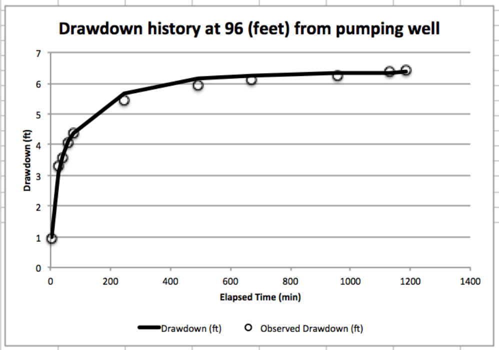
Item	Value	Units	Description
Q	3.34	ft^3/min	Pumping well discharge (L^3/t)
T	0.17	ft^2/min	Aquifer transmissivity (L^2/t)
S	0.00017		Aquifer storage coefficient
K'	5.7E-06	ft/min	Aquitard vertical hydraulic conductivity (L/t)
b'	14	ft	Thickness of aquitard (L)
r	96		Radial distance of observation well from pumping well (L)

Computed Constants

B	648.55
r/B	0.14802
Q/4πT	1.56346
Chart Title	Drawdown history at 96 (feet) from pumping well

Time

Elapsed Time (min)	Radial Distance (ft)	u	r/B	W(u,r/B)	Drawdown (ft)	Observed Drawdown (ft)	Error^2
5	96	0.44898	0.14802	0.62239	0.97309	0.96	0.00017
28	96	0.08017	0.14802	1.97392	3.08616	3.3	0.04573
41	96	0.05475	0.14802	2.30223	3.59945	3.59	8.9E-05
60	96	0.03741	0.14802	2.62436	4.10309	4.08	0.00053
75	96	0.02993	0.14802	2.80751	4.38944	4.39	3.2E-07
244	96	0.0092	0.14802	3.62976	5.67499	5.47	0.04202
493	96	0.00455	0.14802	3.92868	6.14234	5.96	0.03325
669	96	0.00336	0.14802	4.00373	6.25968	6.11	0.0224
958	96	0.00234	0.14802	4.055	6.33984	6.27	0.00488
1129	96	0.00199	0.14802	4.06801	6.36019	6.4	0.00159
1185	96	0.00189	0.14802	4.07089	6.36468	6.42	0.00306
						SSE	0.15372



1DRFLAH-HYDM

Next Time

- Contaminant Transport Concepts
  - Advection, Dispersion, Retardation, Decay
- Aquifer Numerical Modeling
  - Flow Nets