still hold at condition 2

$$\frac{h_2}{d_2} = F\left(\frac{\Upsilon_2}{\rho_2 g d_2^2}, \theta_2\right)$$

But

$$\frac{\Upsilon_2}{\rho_2 g d_2^2} = \frac{\frac{1}{2}\Upsilon_1}{2\rho_1 g(\frac{1}{2}d_1)^2} = \frac{\Upsilon_1}{\rho_1 g d_1^2}$$

Therefore, functionally,

$$\frac{h_2}{d_2} = F\left(\frac{\Upsilon_1}{\rho_1 g d_1^2}, \theta_1\right) = \frac{h_1}{d_1}$$

We are given a condition 2 which is exactly similar to condition 1, and therefore a scaling law holds

$$h_2 = h_1 \frac{d_2}{d_1} = (3 \text{ cm}) \frac{\frac{1}{2}d_1}{d_1} = 1.5 \text{ cm} \qquad \textit{Ans. (b)}$$

If the pi groups had not been exactly the same for both conditions, we would have to know more about the functional relation F to calculate h_2.

5.5 MODELING AND ITS PITFALLS

So far we have learned about dimensional homogeneity and two methods, the power product and the pi theorem, for converting a homogeneous physical relation into dimensionless form. This is straightforward mathematically, but there are certain engineering difficulties which need to be discussed.

First, we have more or less taken for granted that the variables which affect the process can be listed and analyzed. Actually, selection of the important variables requires considerable judgment and experience. The engineer must decide for example whether viscosity can be neglected. Are there significant temperature effects? Is surface tension important? What about wall roughness? Each pi group which is retained increases the expense and effort required. Judgment in selecting variables will come through practice and maturity; this book should provide some of the necessary experience.

Once the variables are selected and the dimensional analysis performed, the experimenter seeks to achieve *similarity* between the model tested and the prototype to be designed. With sufficient testing, the model data will reveal the desired dimensionless function between variables

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_k) \tag{5.49}$$

With Eq. (5.49) available in chart, graphical, or analytical form, we are then in a position to ensure complete similarity between model and prototype. A formal statement would be as follows:

> Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for model and prototype.

This follows mathematically from Eq. (5.49). If $\Pi_{2m} = \Pi_{2p}$, $\Pi_{3m} = \Pi_{3p}$, etc., Eq. (5.49) guarantees that the desired output Π_{1m} will equal Π_{1p}. But this is easier said than done, as we now discuss.

Instead of complete similarity, the engineering literature speaks of particular types of similarity, the most common being geometric, kinematic, dynamic, and thermal. Let us consider each separately.

Geometric Similarity

Geometric similarity concerns the length dimension $\{L\}$ and must be assured before any sensible model testing can proceed. A formal definition is as follows:

> A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio.

Note that *all* length scales must be the same. It is as if you took a photograph of the prototype and reduced it or enlarged it until it fitted the size of the model. If the model is to be made one-tenth the prototype size, its length, width, and height must each be one-tenth as large. Not only that, but its entire shape must be one-tenth as large, and technically we speak of *homologous* points, which are points which have the same relative location. For example the nose of the prototype is homologous to the nose of the model. The left wingtip of the prototype is homologous to the left wingtip of the model. Then geometric similarity requires that all homologous points be related by the same linear-scale ratio. This applies to the fluid geometry as well as the model geometry:

> All angles are preserved in geometric similarity. All flow directions are preserved. The orientation of model and prototype with respect to the surroundings must be identical.

Figure 5.4 illustrates a prototype wing and a one-tenth-scale model. The model lengths are all one-tenth as large, but its angle of attack with respect to the free stream is the same: 10° not 1°. All physical details on the model must be scaled, and some of them are rather subtle and sometimes overlooked:

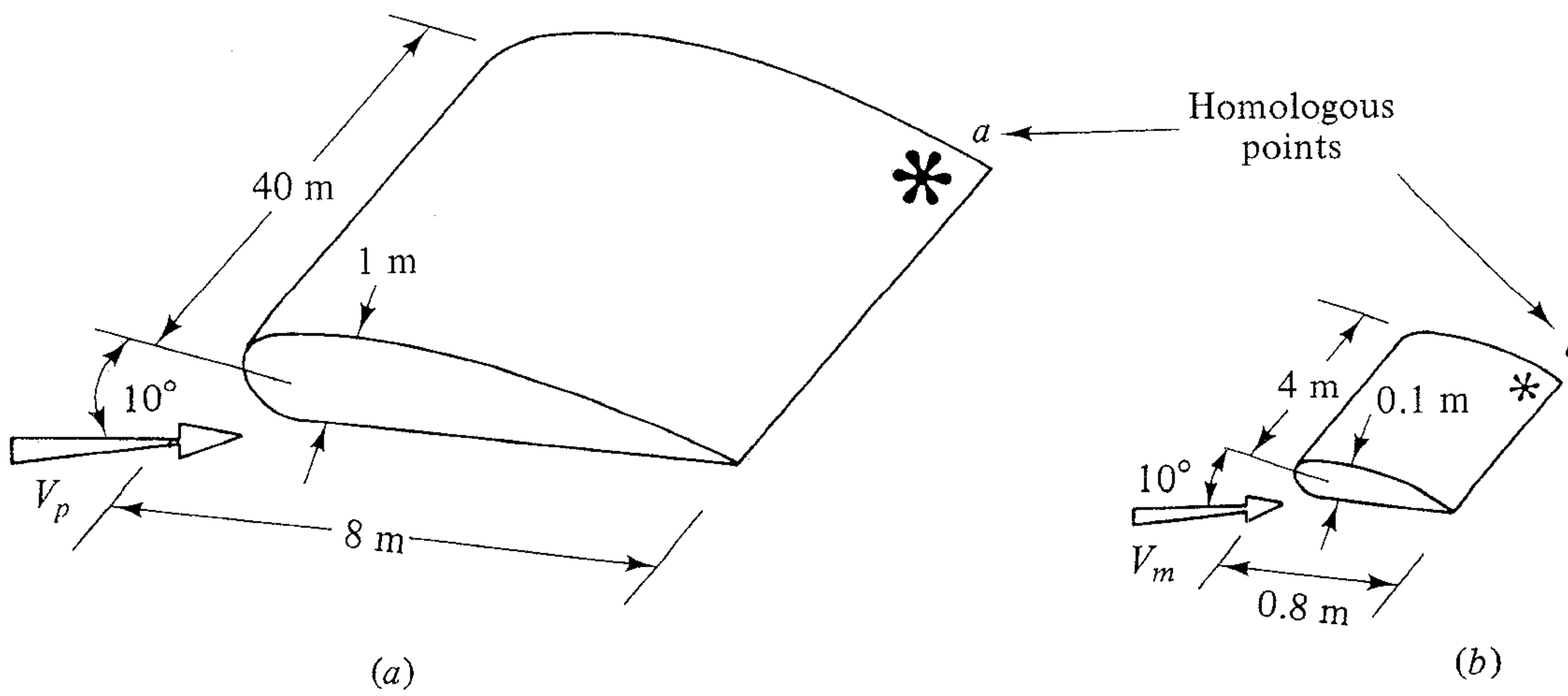

Fig. 5.4 Geometric similarity in model testing: (*a*) prototype; (*b*) a one-tenth scale model.

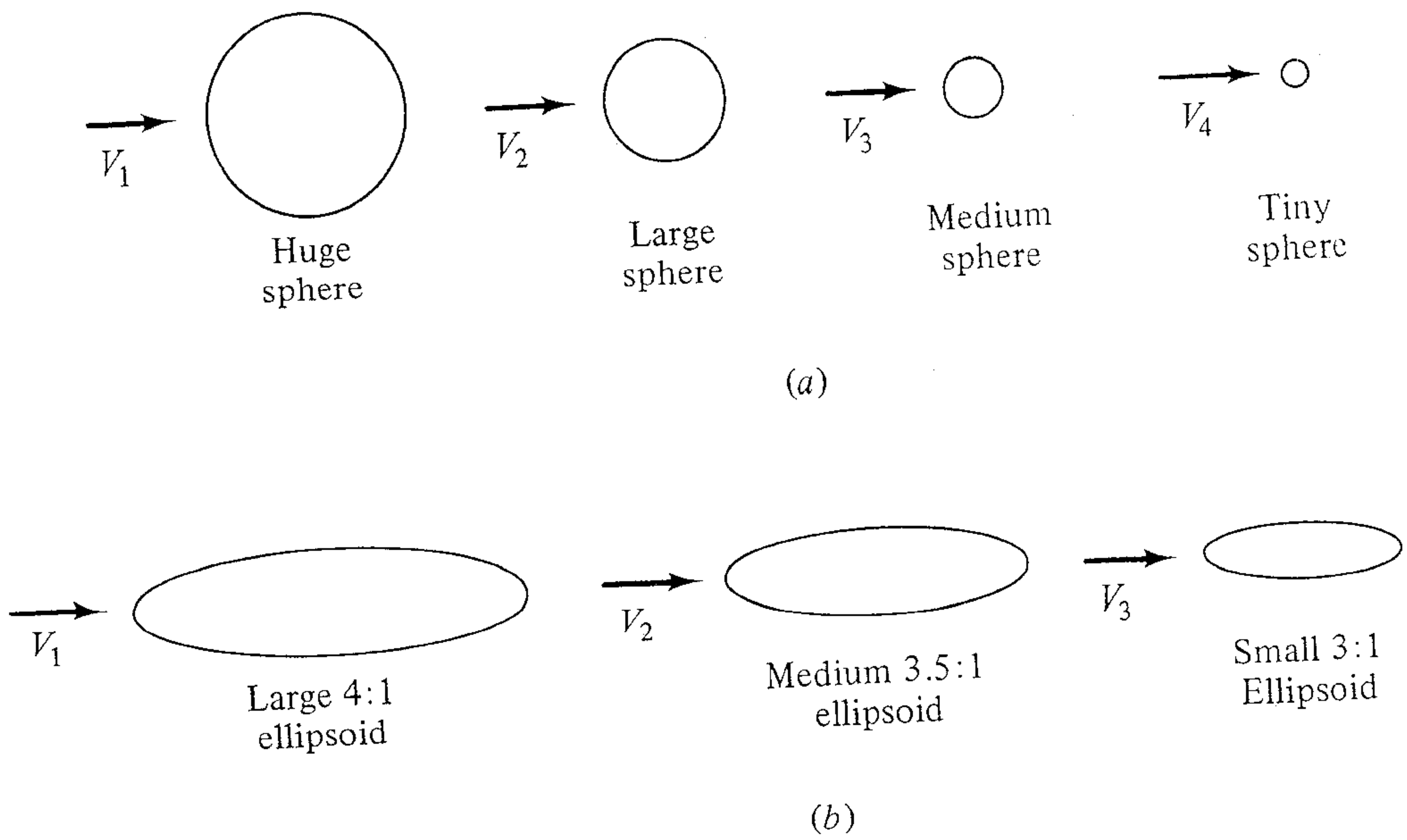

Fig. 5.5 Geometric similarity and dissimilarity of flows: (*a*) similar; (*b*) dissimilar.

1. The model nose radius must be one-tenth as large.
2. The model surface roughness must be one-tenth as large.
3. If the prototype has a 5-mm boundary-layer trip wire 1.5 m from the leading edge, the model should have a 0.5-mm trip wire 0.15 m from its leading edge.
4. If the prototype is constructed with protruding fasteners, the model should have homologous protruding fasteners one-tenth as large.

And so on. Any departure from these details is a violation of geometric similarity and must be justified by experimental comparison to show that the prototype behavior was not significantly affected by the discrepancy.

Models which appear similar in shape but which clearly violate geometric similarity should not be compared except at your own risk. Figure 5.5 illustrates this point. The spheres in Fig. 5.5*a* are all geometrically similar and can be tested with a high expectation of success if the Reynolds number or Froude number, etc., are matched. But the ellipsoids in Fig. 5.5*b* merely *look* similar. They actually have different linear-scale ratios and therefore cannot be compared in a rational manner, even though they may have identical Reynolds and Froude numbers, etc. The data will not be the same for these ellipsoids and any attempt to "compare" them is a matter of rough engineering judgment.

Kinematic Similarity

Kinematic similarity requires that the model and prototype have the same length-scale ratio and also the same time-scale ratio. The result is that the velocity-scale ratio will be the same for both. As Langhaar [8] states it:

> "The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times."

Length-scale equivalence simply implies geometric similarity, but time-scale equivalence may require additional dynamic considerations such as equivalence of the Reynolds and Mach numbers.

One special case is incompressible frictionless flow with no free surface, as sketched in Fig. 5.6*a*. These perfect-fluid flows are kinematically similar with independent length and time scales, and no additional parameters are necessary (see Chap. 8 for further details).

Frictionless flows with a free surface, as in Fig. 5.6*b*, are kinematically similar if their Froude numbers are equal

$$\mathrm{Fr}_m = \frac{V_m^2}{gL_m} = \frac{V_p^2}{gL_p} = \mathrm{Fr}_p \tag{5.50}$$

Note that Froude number contains only length and time dimensions and hence is a purely kinematic parameter which fixes the relation between length and time. From Eq. (5.50), if the length scale is

$$L_m = \alpha L_p \tag{5.51}$$

where α is a dimensionless ratio, the velocity scale is

$$\frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{1/2} = \sqrt{\alpha} \tag{5.52}$$

and the time scale is

$$\frac{T_m}{T_p} = \frac{L_m/V_m}{L_p/V_p} = \sqrt{\alpha} \tag{5.53}$$

These Froude-scaling kinematic relations are illustrated in Fig. 5.6*b* for wave-motion modeling. If the waves are related by the length scale α, the wave period, propagation speed, and particle velocities are related by $\sqrt{\alpha}$.

If viscosity, surface tension, or compressibility is important, kinematic similarity is dependent upon the achievement of dynamic similarity.

Dynamic Similarity

Dynamic similarity exists when model and prototype have the same length-scale ratio, time-scale ratio, and force-scale (or mass-scale) ratio. Again geometric similarity is a first requirement; otherwise proceed no further. Then dynamic similarity exists, simultaneous with kinematic similarity, if model and prototype forces are in a constant ratio. This is assured if:

1. Compressible flow: model and prototype Reynolds number and Mach number and specific-heat ratio are correspondingly equal.

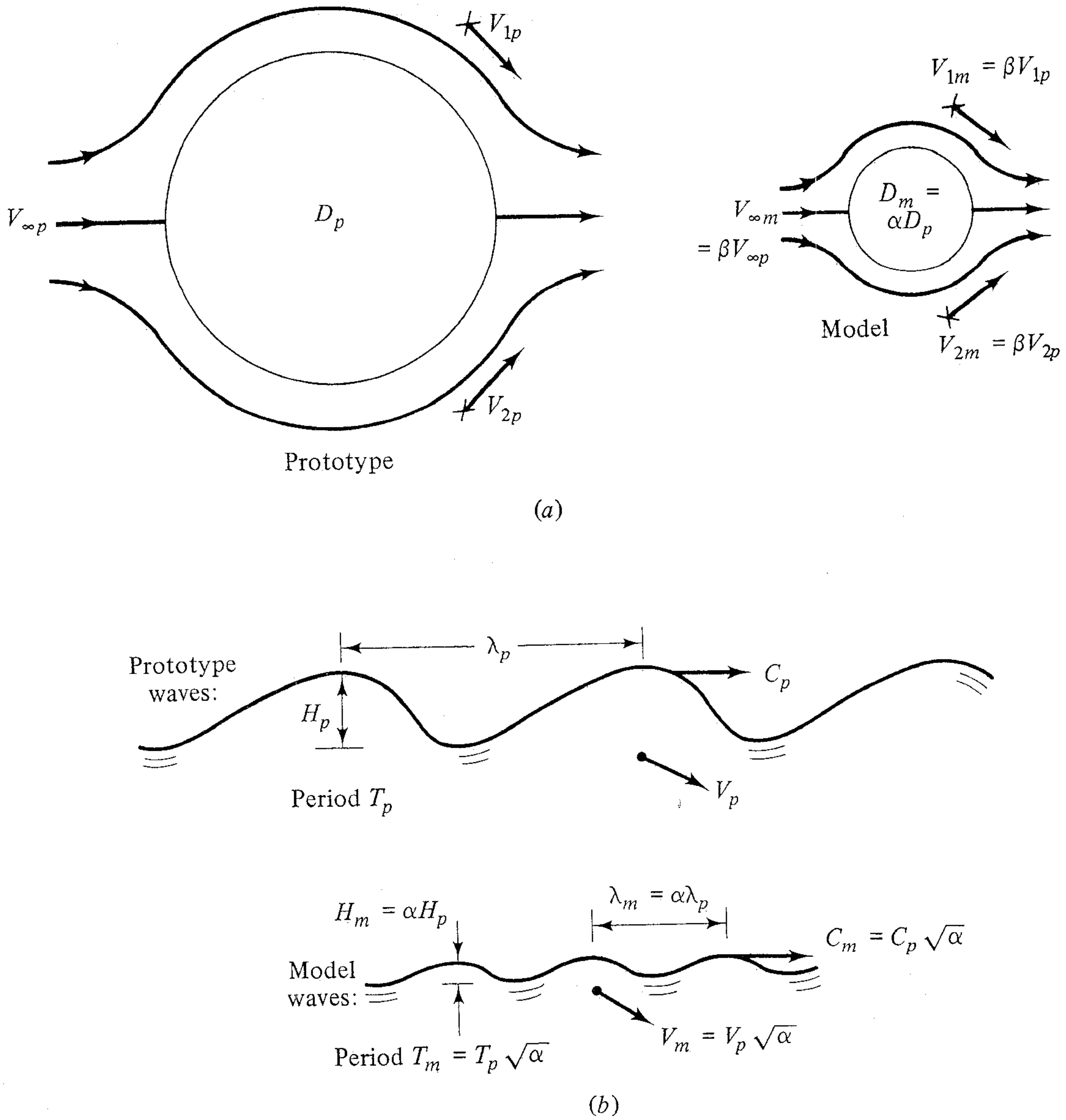

Fig. 5.6 Frictionless low-speed flows are kinematically similar: (*a*) flows with no free surface are kinematically similar with independent length- and time-scale ratios; (*b*) free-surface flows are kinematically similar with length and time scales related by the Froude number.

2. Incompressible flow

a. With no free surface: model and prototype Reynolds number are equal.

b. With a free surface: model and prototype Reynolds number, Froude number, and (if necessary) Weber number and cavitation number are correspondingly equal.

Mathematically, Newton's law for any fluid particle requires that the sum of the pressure force, gravity force, and friction force equal the acceleration term, or inertia force,

$$\mathbf{F}_p + \mathbf{F}_g + \mathbf{F}_f = \mathbf{F}_i \tag{5.54}$$

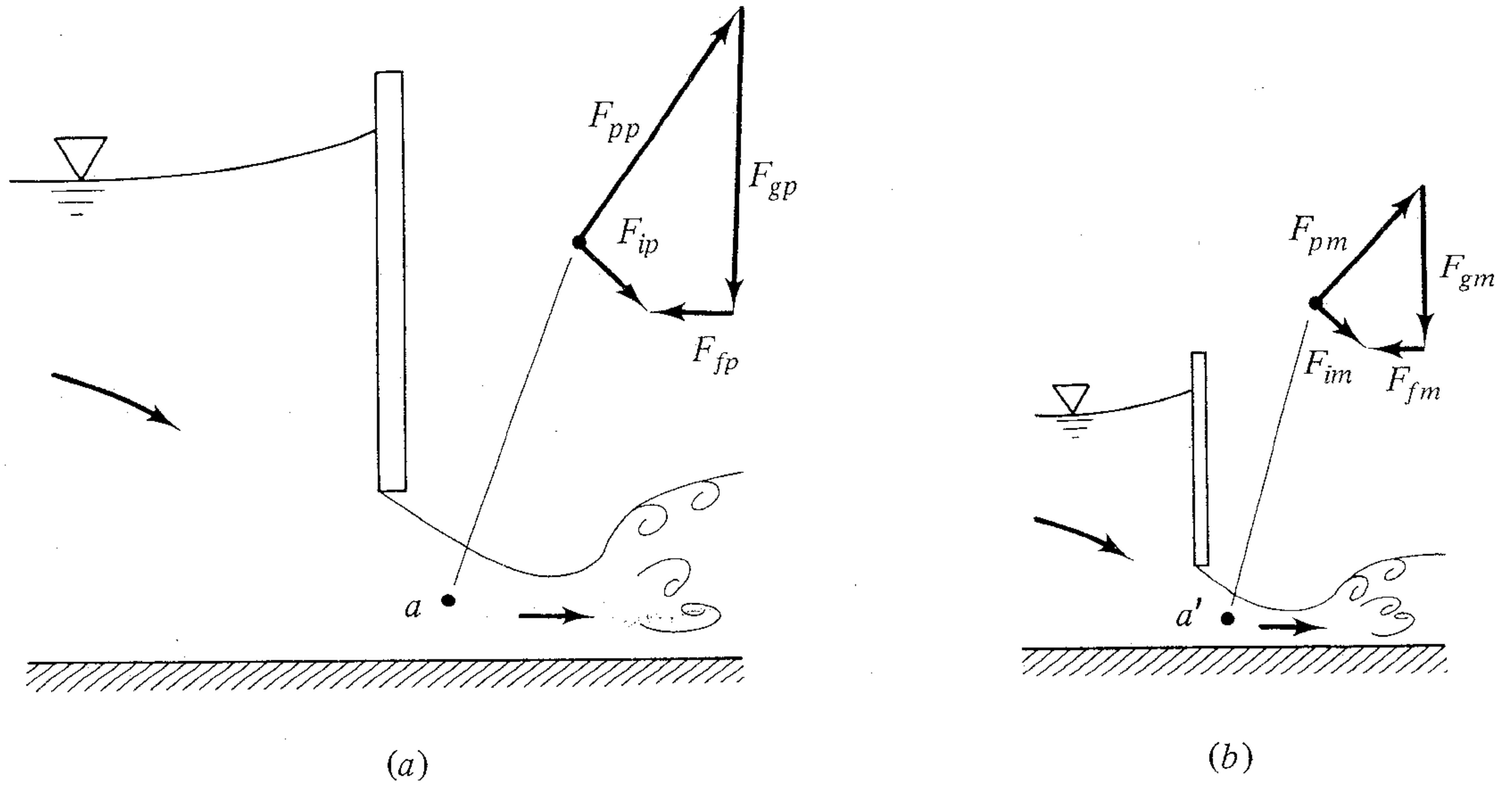

Fig. 5.7 Dynamic similarity in sluice-gate flow. Model and prototype yield identical homologous force polygons if the Reynolds and Froude numbers are the same corresponding values: (*a*) prototype; (*b*) model.

The dynamic-similarity laws listed above ensure that each of these forces will be in the same ratio and have equivalent directions between model and prototype. Figure 5.7 shows an example for flow through a sluice gate. The force polygons at homologous points have exactly the same shape if the Reynolds and Froude numbers are equal (neglecting surface tension and cavitation, of course). Kinematic similarity is also assured by these model laws.

Discrepancies in Water and Air Testing

The perfect dynamic similarity shown in Fig. 5.7 is more of a dream than a reality because true equivalence of Reynolds and Froude numbers can be achieved only by dramatic changes in fluid properties, whereas in fact most model testing is simply done with water or air, the cheapest fluids available.

First consider hydraulic model testing with a free surface. Dynamic similarity requires equivalent Froude numbers, Eq. (5.50), *and* equivalent Reynolds numbers

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \tag{5.55}$$

But both velocity and length are constrained by the Froude number, Eqs. (5.51) and (5.52). Therefore, for a given length-scale ratio α, Eq. (5.55) is true only if

$$\frac{\nu_m}{\nu_p} = \frac{L_m}{L_p}\frac{V_m}{V_p} = \alpha\sqrt{\alpha} = \alpha^{3/2} \tag{5.56}$$

For example, for a one-tenth-scale model, $\alpha = 0.1$, and $\alpha^{3/2} = 0.032$. Since ν_p is undoubtably water, we need a fluid with only 0.032 times the kinematic viscosity of water to achieve dynamic similarity. Referring back to Table 1.3, we see that this is impossible: even mercury has only one-ninth the kinematic viscosity of water, and a mercury hydraulic model would be expensive and bad for your health. In practice, water is used for both model and prototype, and the Reynolds-number similarity (5.55) is unavoidably violated. The Froude number is held constant since it is the dominant parameter in free-surface flows. Typically the Reynolds number of the model flow is too small by a factor of 10 to 1000. As shown in Fig. 5.8, the low-Reynolds-number model data are used to estimate by extrapolation the desired high-Reynolds-number prototype data. As the figure indicates, there is obviously considerable uncertainty in using such an extrapolation, but there is no other practical alternative in hydraulic model testing.

Second, consider aerodynamic model testing in air with no free surface. The important parameters are the Reynolds number and the Mach number. Equation (5.55) should be satisfied, plus the compressibility criterion

$$\frac{V_m}{a_m} = \frac{V_p}{a_p} \tag{5.57}$$

Elimination of V_m/V_p between (5.55) and (5.57) gives

$$\frac{\nu_m}{\nu_p} = \frac{L_m}{L_p}\frac{a_p}{a_m} \tag{5.58}$$

Since the prototype is no doubt an air operation, we need a wind-tunnel fluid of low viscosity and high speed of sound. Hydrogen is the only practical example, but clearly it is too expensive and dangerous. Therefore wind tunnels normally operate with air as the working fluid. Cooling and pressurizing the air will bring Eq. (5.58) into better agreement but not enough to satisfy a length-scale reduction of, say, one-tenth. Therefore Reynolds-number scaling is also commonly violated in aerodynamic testing, and an extrapolation like Fig. 5.8 is required here also.

In fact, as aerodynamic vehicle speeds and sizes increase, the Reynolds-number

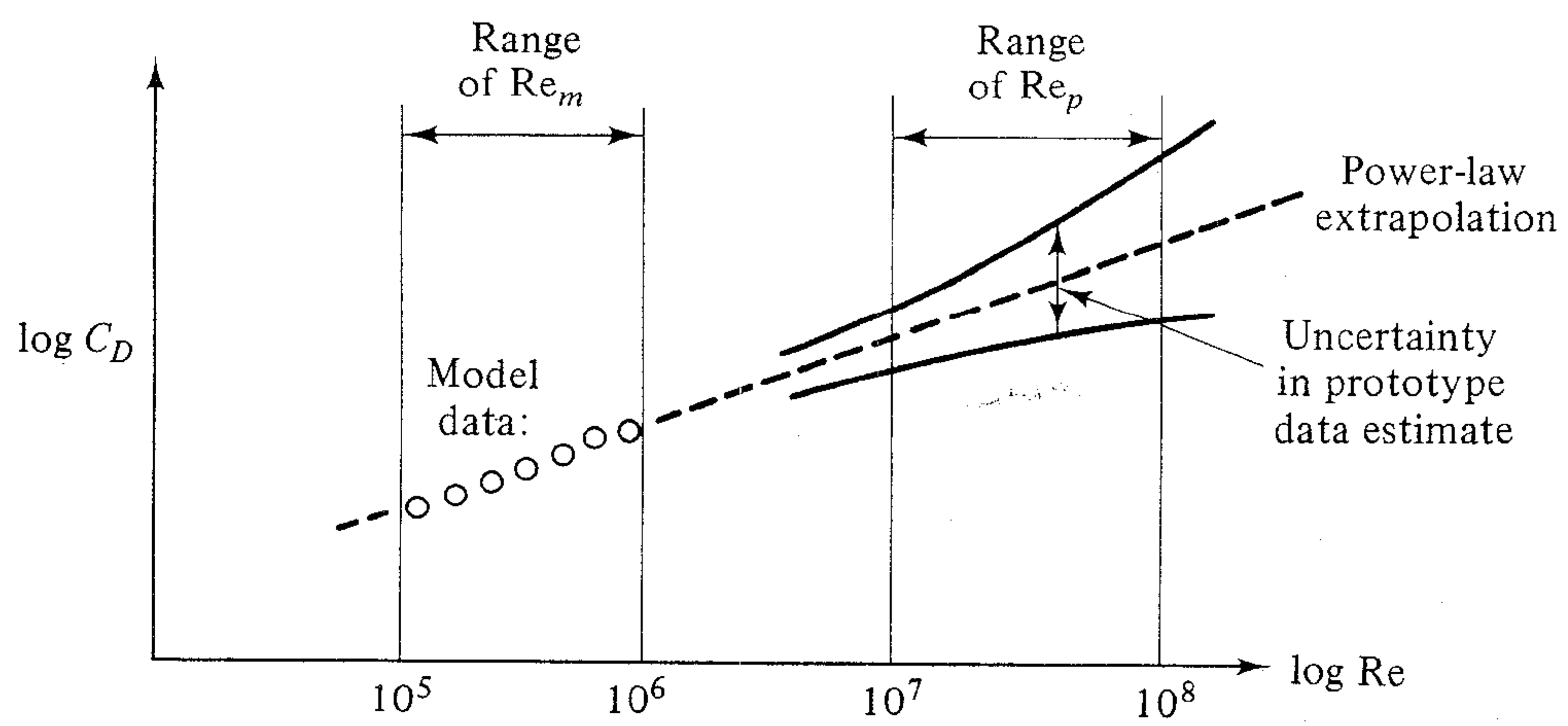

Fig. 5.8 Reynolds-number extrapolation, or scaling, of hydraulic data with equal Froude numbers.

gap between prototype and model is actually increasing, as shown in Fig. 5.9. Lukasiewicz [30] uses Fig. 5.9 to argue the need for new wind tunnels of higher-Reynolds-number capability.

Finally, a serious discrepancy of another type occurs in hydraulic models of natural flow systems such as rivers, harbors, estuaries, and embayments. Such flows have large horizontal dimensions and small relative vertical dimensions. If we were to scale an estuary model by a uniform linear length ratio of, say, 1:1000, the resulting model would be only a few millimeters deep and dominated by entirely spurious surface-tension or Weber-number effects. Therefore such hydraulic models commonly violate *geometric* similarity by "distorting" the vertical scale by a factor of 10 or more. Figure 5.10 shows a hydraulic model of a barrier-beach inlet in South Carolina. The horizontal scale reduction is 1:300, but the vertical scale is only 1:60. Since a deeper channel flows more efficiently, the model channel bottom is deliberately roughened more than the natural channel to correct for the geometric discrepancy. Thus the friction effect of the discrepancy can be corrected, but its effect on say dispersion of heat and mass is less well known.

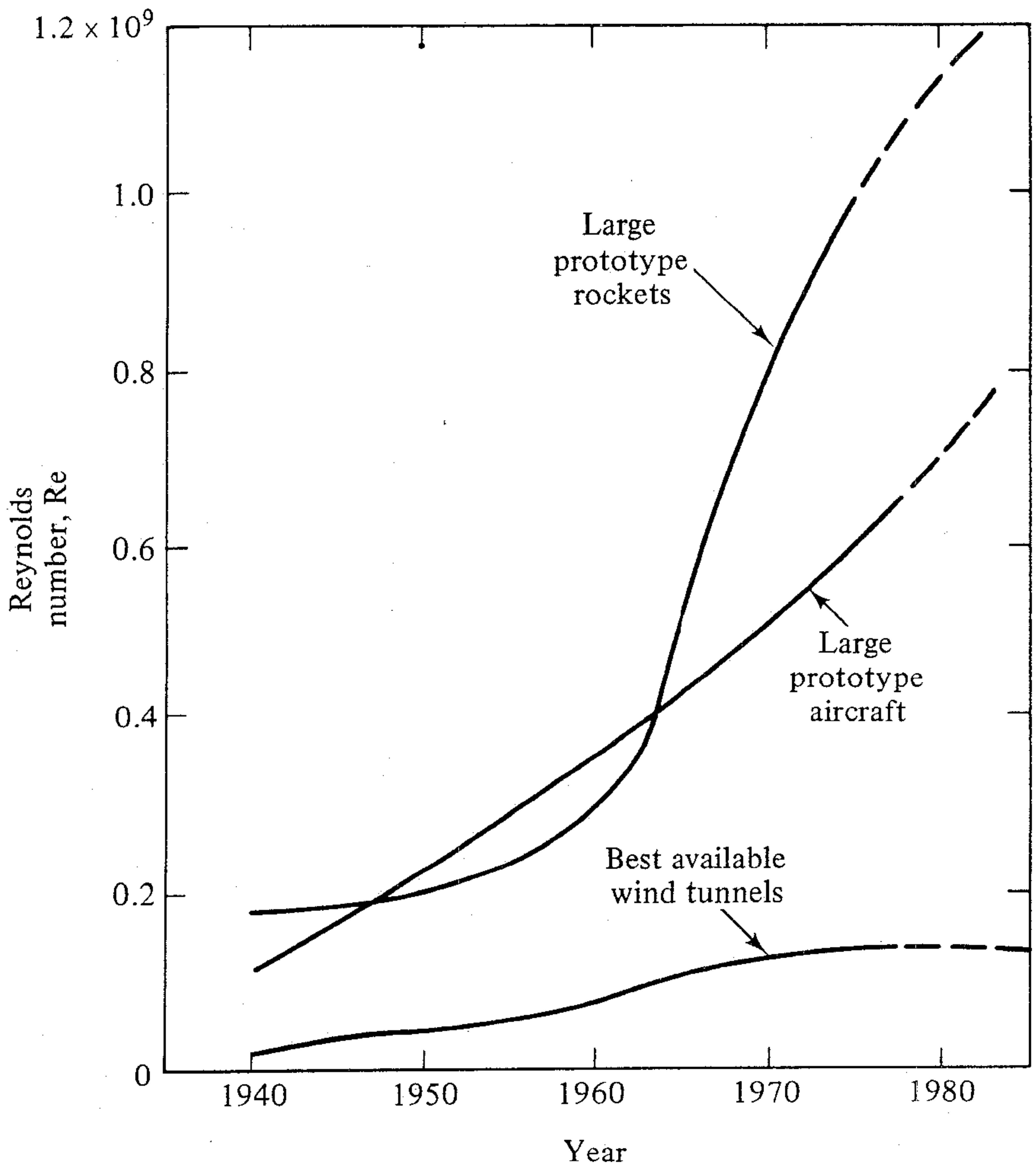

Fig. 5.9 The growing Reynolds-number gap in wind-tunnel testing. (*Adapted from Ref. 30, with additional data, by permission of the American Institute of Aeronautics and Astronautics.*)

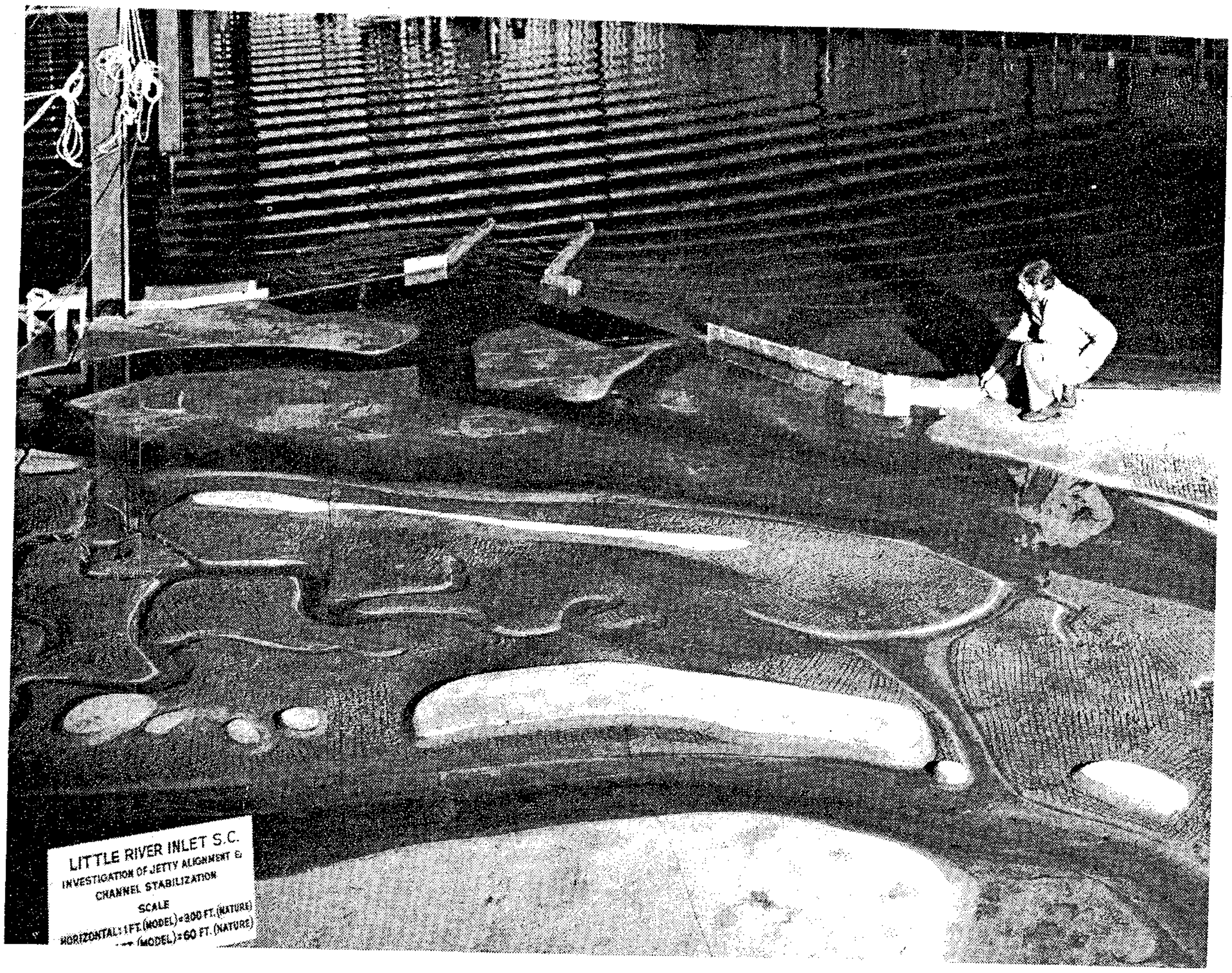

Fig. 5.10 Hydraulic model of a barrier-beach inlet at Little River, South Carolina. Such models of necessity violate geometric similarity and do not model the Reynolds number of the prototype inlet. (*Courtesy of U.S. Army Engineer Waterways Experiment Station.*)

5.6 INVENTIVE USE OF THE DATA

The methods of dimensional analysis discussed here allow one to organize both theory and experiment efficiently. The parameters arrived at are customary and traditional: Reynolds number, Froude number, drag coefficient, etc. They are not necessarily the best parameters for a given task, and sometimes they do not give a clear indication of what is happening physically in an experiment. The remedy for this is to regroup the parameters until the particular problem under investigation is most clearly revealed.

As an example of a regrouping procedure, consider Fig. 5.3*a* for the drag coefficient of a sphere in a uniform stream. This figure is a classic and is reproduced in nearly every textbook on fluid mechanics, but it is a drag-oriented figure. One is supposed to be given the fluid, the diameter, and the velocity, and hence compute the Reynolds number, read the drag coefficient, and compute the sphere drag. Suppose instead that the drag is known but the fluid velocity is not. Then, since V is contained in both C_D and Re, one must iterate back and forth on the chart in Fig. 5.3*a* until the proper velocity is found. With luck the iteration procedure converges. Consider the following numerical example.