Since V is velocity, R is a radius, and n and S are dimensionless, the formula is not dimensionally homogeneous. This should be a warning that (1) the formula changes if the *units* of V and R change and (2) if valid, it represents a very special case. Equation (1.5) predates the dimensional-analysis technique and is valid only for water in rough channels at moderate velocities and large radii in English units.

Such dimensionally inhomogeneous formulas abound in the hydraulics literature. Another example is the Hazen-Williams formula [25] for volume flux of water through a straight smooth pipe

$$Q = 61.9D^{2.63}\left(\frac{dp}{dx}\right)^{0.54} \tag{5.25}$$

where D is diameter and dp/dx the pressure gradient. Some of these formulas arise because numbers have been inserted for fluid properties and other physical data into perfectly legitimate homogeneous formulas. We shall not give the units of Eq. (5.25) to avoid encouraging its use.

On the other hand, some formulas are "constructs" which cannot be made dimensionally homogeneous. The "variables" they relate cannot be analyzed by the dimensional-analysis technique. Most of these formulas are raw empiricisms convenient to a small group of specialists. Here are three examples:

$$B = \frac{25{,}000}{100 - R} \tag{5.26}$$

$$S = \frac{140}{130 + \mathrm{API}} \tag{5.27}$$

$$0.0147D_E - \frac{3.74}{D_E} = 0.26t_R - \frac{172}{t_R} \tag{5.28}$$

Equation (5.26) relates the Brinell hardness B of a metal to its Rockwell hardness R. Equation (5.27) relates the specific gravity S of an oil to its density in degrees API. Equation (5.28) relates the viscosity of a liquid in D_E, or degrees Engler, to its viscosity t_R in Saybolt seconds. Such formulas have a certain usefulness when communicated between fellow specialists, but we cannot handle them here. Variables like Brinell hardness and Saybolt viscosity are not suited to an $MLT\Theta$ dimensional system.

5.3 NONDIMENSIONALIZATION OF THE BASIC EQUATIONS

We could use the power-product method of the previous section to analyze problem after problem, finding the dimensionless parameters which govern in each case. Textbooks on dimensional analysis [e.g., 7] do this. An alternate and very powerful technique is to attack the basic equations of flow from Chap. 4. Even though these equations cannot be solved in general, they will reveal basic dimensionless parameters, e.g., Reynolds number, in their proper form and proper position, giving clues to when they are negligible. The boundary conditions must also be nondimensionalized.

Let us briefly apply this technique to the incompressible-flow continuity and momentum equations with constant viscosity:

Continuity: $$\nabla \cdot \mathbf{V} = 0 \tag{5.29a}$$

Momentum: $$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V} \tag{5.29b}$$

Typical boundary conditions for these two equations are

Fixed solid surface: $\mathbf{V} = 0$

Inlet or outlet: Known $\mathbf{V}$, p (5.30)

Free surface, $z = \eta$: $w = \dfrac{d\eta}{dt}$ $\quad p = p_a - \Upsilon(R_x^{-1} + R_y^{-1})$

We omit the energy equation (4.75) and assign its dimensionless form in the problems (Probs. 5.31 and 5.32).

Equations (5.29, 5.30) contain the three basic dimensions MLT. All variables p, $\mathbf{V}$, x, y, z, and t can be nondimensionalized using density and two reference constants which might be characteristic of the particular fluid flow:

$$\text{Reference velocity} = U \qquad \text{reference length} = L \tag{5.31}$$

For example, U may be the inlet or upstream velocity and L the diameter of a body immersed in the stream.

Now define all relevant dimensionless variables, denoting them by an asterisk:

$$\mathbf{V}^* = \frac{\mathbf{V}}{U}$$

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad z^* = \frac{z}{L} \tag{5.32}$$

$$t^* = \frac{tU}{L} \qquad p^* = \frac{p + \rho g z}{\rho U^2}$$

All these are fairly obvious except for p^*, where we have slyly introduced the gravity effect, assuming that z is "up." This is a hindsight idea suggested by Bernoulli's equation (3.63).

Since ρ, U, and L are all constants, the derivatives in Eqs. (5.29) can all be handled in dimensionless form with dimensional coefficients. For example,

$$\frac{\partial u}{\partial x} = \frac{\partial (Uu^*)}{\partial (Lx^*)} = \frac{U}{L} \frac{\partial u^*}{\partial x^*} \tag{5.33}$$

Substitute the variables from Eqs. (5.32) into Eqs. (5.29) and (5.30) and divide through by the leading dimensional coefficient, in the same way we handled Eq. (5.12). The resulting dimensionless equations of motion are

Continuity: $$\nabla^* \cdot \mathbf{V}^* = 0 \tag{5.34a}$$

Momentum: $$\frac{d\mathbf{V}^*}{dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L} \nabla^{*2}(\mathbf{V}^*) \tag{5.34b}$$

The dimensionless boundary conditions are

Fixed solid surface: $\mathbf{V}^* = 0$

Inlet or outlet: Known $\mathbf{V}^*$, p^*

Free surface, $z^* = \eta^*$: $$w^* = \frac{d\eta^*}{dt^*} \tag{5.35}$$

$$p^* = \frac{p_a}{\rho U^2} + \frac{gL}{U^2} z^* + \frac{\Upsilon}{\rho U^2 L}(R_x^{*-1} + R_y^{*-1})$$

These equations reveal a total of four dimensionless parameters, one in the momentum equation and three in the free-surface-pressure boundary condition.

Dimensionless Parameters

In the continuity equation there are no parameters. The momentum equation contains one, generally accepted as the most important parameter in fluid mechanics:

$$\text{Reynolds number Re} = \frac{\rho U L}{\mu} \tag{5.36}$$

It is named after Osborne Reynolds (1842–1912), a British engineer who first proposed it in 1883 (Ref. 4 of Chap. 6). The Reynolds number is always important, with or without a free surface, and can be neglected only in flow regions away from high velocity gradients, e.g., away from solid surfaces, jets, or wakes.

The no-slip and inlet-exit boundary conditions contain no parameters. The free-surface-pressure condition contains three:

$$\text{Euler number (pressure coefficient) Eu} = \frac{p_a}{\rho U^2} \tag{5.37}$$

This is named after Leonhard Euler (1707–1783) and is rarely important unless the pressure drops low enough to cause vapor formation (cavitation) in a liquid. The Euler number is often written in terms of pressure differences, $\text{Eu} = \Delta p/\rho U^2$. If Δp involves vapor pressure p_v, it is called the cavitation number $\text{Ca} = (p_a - p_v)/(\rho U^2)$.

The second pressure parameter is much more important:

$$\text{Froude number Fr} = \frac{U^2}{gL} \tag{5.38}$$

It is named after William Froude (1810–1879), a British naval architect who, with his son Robert, developed the ship-model towing-tank concept and proposed similarity rules for free-surface flows (ship resistance, surface waves, open channels). The Froude number is the dominant effect in free-surface flows and is totally unimportant if there is no free surface. Chapter 10 investigates Froude-number effects in detail.

The final free-surface parameter is

$$\text{Weber number We} = \frac{\rho U^2 L}{\Upsilon} \tag{5.39}$$

It is named after Moritz Weber (1871–1951) of the Polytechnic Institute of Berlin, who developed the laws of similitude in their modern form. It was Weber who named Re and Fr after Reynolds and Froude. The Weber number is important only if it is of order unity or less, which typically occurs when the surface curvature is comparable in size to the liquid depth, e.g., in droplets, capillary flows, ripple waves, and very small hydraulic models. If We is large, its effect may be neglected.

If there is no free surface, Fr, Eu, and We drop out entirely, except for the possibility of cavitation of a liquid at very small Eu. Thus, in low-speed viscous flows with no free surface, the Reynolds number is the only important dimensionless parameter.

Compressibility Parameters

In high-speed flow of a gas there are significant changes in pressure, density, and temperature which must be related by an equation of state such as the perfect-gas law, Eq. (1.29). These thermodynamic changes introduce two additional dimensionless parameters mentioned briefly in earlier chapters:

$$\text{Mach number Ma} = \frac{U}{a}$$

$$\text{Specific-heat ratio } \gamma = \frac{c_p}{c_v} \tag{5.40}$$

The Mach number is named after Ernst Mach (1838–1916), an Austrian physicist. The effect of γ is only slight to moderate, but Ma exerts a strong effect on compressible-flow properties if it is greater than about 0.3. These effects are studied in Chap. 9.

Oscillating Flows

If the flow pattern is oscillating, a seventh parameter enters through the inlet boundary condition. For example, suppose that the inlet stream is of the form

$$u = U \cos \omega t \tag{5.41}$$

Nondimensionalization of this relation results in

$$\frac{u}{U} = u^* = \cos\left(\frac{\omega L}{U} t^*\right) \tag{5.42}$$

The argument of the cosine contains the new parameter

$$\text{Strouhal number St} = \frac{\omega L}{U} \tag{5.43}$$

The dimensionless forces and moments, friction, and heat transfer, etc., of such an oscillating flow would be a function of both Reynolds and Strouhal number. This parameter is named after V. Strouhal, a German physicist who experimented in 1878 with wires singing in the wind.

Some flows which you might guess to be perfectly steady actually have an oscillatory pattern which is dependent on the Reynolds number. An example is the periodic vortex shedding behind a blunt body immersed in a steady stream of velocity U. Figure 5.2*a* shows an array of alternating vortices shed from a circular cylinder immersed in a steady crossflow. This regular, periodic shedding is called a *Kármán vortex street*, after T. von Kármán, who explained it theoretically in 1912. The shedding occurs in the range $10^2 < \text{Re} < 10^7$, with an average Strouhal number $\omega d/2\pi U \approx 0.21$. Figure 5.2*b* shows measured shedding frequencies.

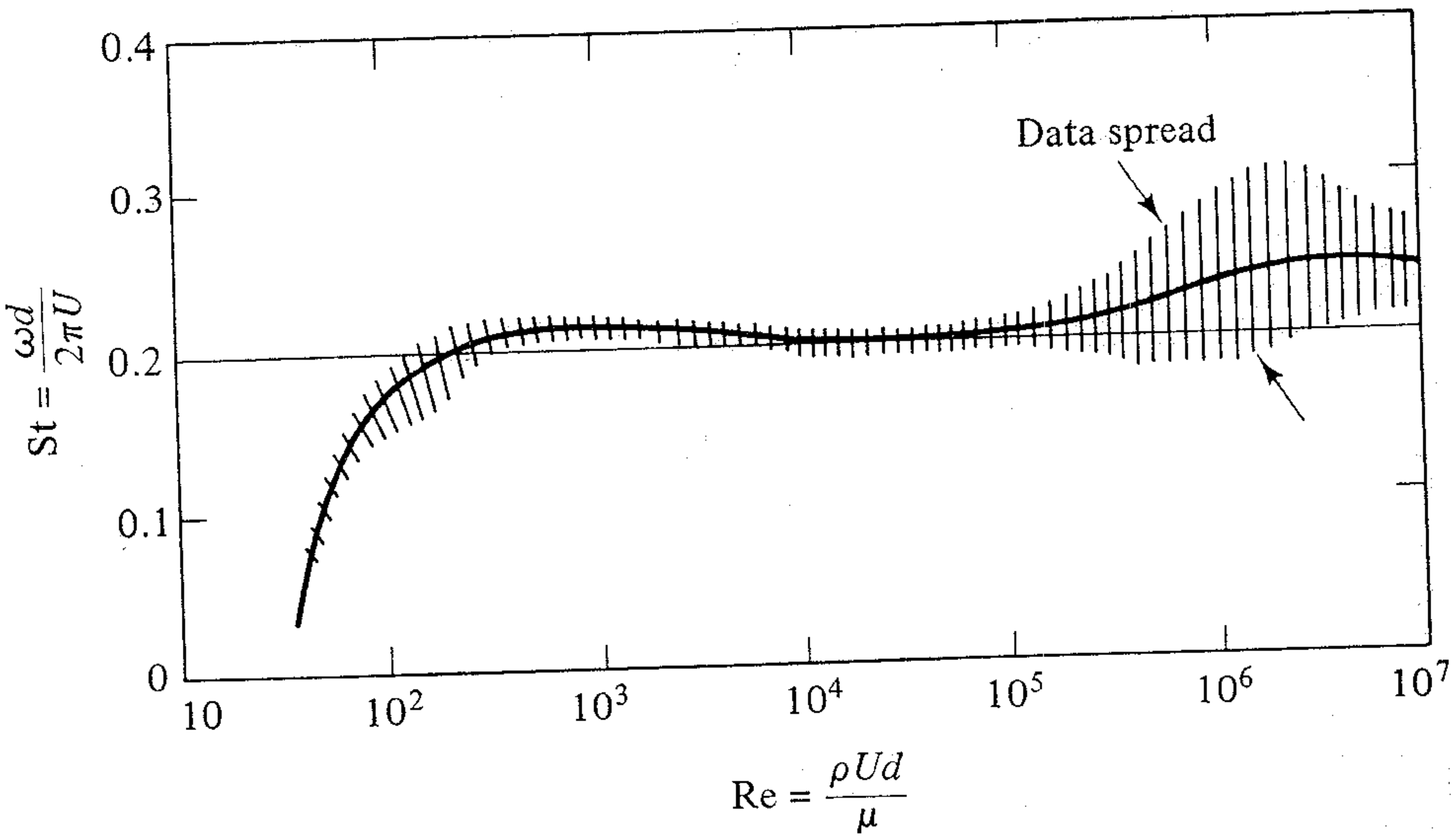
Resonance can occur if a vortex shedding frequency is near a body structural-vibration frequency. Electric transmission wires sing in the wind, undersea mooring lines gallop at certain current speeds, and slender structures flutter at critical wind or vehicle speeds. A striking example is the disastrous failure of the Tacoma Narrows suspension bridge in 1940, when wind-excited vortex shedding caused resonance with the natural torsional oscillations of the bridge.

Other Dimensionless Parameters

We have discussed seven important parameters in fluid mechanics, and there are others. Four additional parameters arise from nondimensionalization of the energy equation (4.75) and its boundary conditions. These four (Prandtl number, Eckert number, Grashof number, and wall-temperature ratio) are listed in Table 5.2 just in case you fail to solve Prob. 5.31. Another important and rather sneaky parameter is the wall-roughness ratio ϵ/L (in Table 5.2).[1] Slight changes in surface roughness have a striking effect in the turbulent flow or high-Reynolds-number range, as we shall see in Chap. 6.

[1] Roughness is easy to overlook because it is a slight geometric effect which does not appear in the equations of motion.

(a)



(b)

Fig. 5.2 Vortex shedding from a circular cylinder. (*a*) Vortex street behind a circular cylinder (*from Ref. 28, courtesy of U.S. Naval Research Laboratory*); (*b*) experimental shedding frequencies (*data from Refs. 26 and 27*).

This book is primarily concerned with Reynolds-, Mach-, and Froude-number effects, which dominate most flows. Note that we discovered all these parameters (except ϵ/L) simply by nondimensionalizing the basic equations without actually solving them.

If the reader is not satiated with the 12 parameters given in Table 5.2, Ref. 29 contains a list of over 300 dimensionless parameters in use in engineering.

Table 5.2

DIMENSIONLESS GROUPS IN FLUID MECHANICS

Parameter	*Definition*	*Qualitative ratio of effects*	*Importance*
Reynolds number	$\text{Re} = \dfrac{\rho U L}{\mu}$	$\dfrac{\text{Inertia}}{\text{Viscosity}}$	Always
Mach number	$\text{Ma} = \dfrac{U}{a}$	$\dfrac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude number	$\text{Fr} = \dfrac{U^2}{gL}$	$\dfrac{\text{Inertia}}{\text{Gravity}}$	Free-surface flow
Weber number	$\text{We} = \dfrac{\rho U^2 L}{\Upsilon}$	$\dfrac{\text{Inertia}}{\text{Surface tension}}$	Free-surface flow
Cavitation number (Euler number)	$\text{Ca} = \dfrac{p - p_v}{\rho U^2}$	$\dfrac{\text{Pressure}}{\text{Inertia}}$	Cavitation
Prandtl number	$\text{Pr} = \dfrac{\mu c_p}{k}$	$\dfrac{\text{Dissipation}}{\text{Conduction}}$	Heat convection
Eckert number	$\text{Ec} = \dfrac{U^2}{c_p T_0}$	$\dfrac{\text{Kinetic energy}}{\text{Enthalpy}}$	Dissipation
Specific heat ratio	$\gamma = \dfrac{c_p}{c_v}$	$\dfrac{\text{Enthalpy}}{\text{Internal energy}}$	Compressible flow
Strouhal number	$\text{St} = \dfrac{\omega L}{U}$	$\dfrac{\text{Oscillation}}{\text{Mean speed}}$	Oscillating flow
Roughness ratio	$\dfrac{\epsilon}{L}$	$\dfrac{\text{Wall roughness}}{\text{Body length}}$	Turbulent, rough walls
Grashof number	$\text{Gr} = \dfrac{\beta \Delta T g L^3 \rho^2}{\mu^2}$	$\dfrac{\text{Buoyancy}}{\text{Viscosity}}$	Natural convection
Temperature ratio	$\dfrac{T_w}{T_0}$	$\dfrac{\text{Wall temperature}}{\text{Stream temperature}}$	Heat transfer

5.4 THE PI THEOREM[1]

The power-product method outlined in Sec. 5.2 is sufficient to derive the dimensionless groups involved in any dimensional-analysis problem, but if the equations are few and the variables many, the algebra of finding the free exponents is laborious and the results rather arbitrary (see, for example, Example 5.5). In 1914 E. Buckingham [24] gave an alternate procedure now called the *Buckingham pi theorem*. The term pi comes from the mathematical notation Π, meaning a prod-

[1] This section may be omitted without loss of continuity.

uct of variables. The dimensionless groups found from the theorem are power products denoted by Π_1, Π_2, Π_3, etc. The method allows the pis to be found in sequential order without resorting to free exponents.

The first part of the pi theorem explains what reduction in variables to expect:

> If a physical process satisfies the PDH and involves n dimensional variables, it can be reduced to a relation between only k dimensionless variables or Π's. The reduction $j = n - k$ equals the maximum number of variables which do not form a pi among themselves and is always less than or equal to the number of dimensions describing the variables.

Take the specific case of force on an immersed body: Eq. (5.1) contains five variables F, L, U, ρ, and μ described by three dimensions (MLT). Thus $n = 5$ and $j \leq 3$. Therefore it is a good guess that we can reduce the problem to k pis, with $k = n - j \geq 5 - 3 = 2$. And this is exactly what we obtained: two dimensionless variables, $\Pi_1 = C_F$ and $\Pi_2 = \text{Re}$. On rare occasions it may take more pis than this minimum (see Example 5.4).

The second part of the theorem shows how to find the pis one at a time:

> Find the reduction j, then select j variables which do not form a pi among themselves.[1] Each desired pi group will be a power product of these j variables plus one additional variable which is assigned any convenient nonzero exponent. Each pi group thus found is independent.

To be specific, suppose that the process involves five variables

$$v_1 = f(v_2, v_3, v_4, v_5) \tag{5.44}$$

Suppose that there are three dimensions (MLT) and we search around and find that indeed $j = 3$. Then $k = 5 - 3 = 2$ and we expect, from the theorem, two and only two pi groups. Pick out three convenient variables which do *not* form a pi and suppose these turn out to be v_2, v_3, and v_4. Then the two pi groups are formed by power products of these three plus one additional variable

$$\Pi_1 = (v_2)^a(v_3)^b(v_4)^c v_1 = M^0L^0T^0 \qquad \Pi_2 = (v_2)^a(v_3)^b(v_4)^c v_5 = M^0L^0T^0 \tag{5.45}$$

Here we have arbitrarily chosen v_1 and v_5, the added variables, to have unit exponents. Equating exponents of the various dimensions is guaranteed by the theorem to give unique values of a, b, and c for each pi. And they are independent because only Π_1 contains v_1 and only Π_2 contains v_5. It is a very neat system once you get used to the procedure. We shall illustrate it with several examples.

Typically, there are six steps involved:

1. List and count the n variables involved in the problem. If any important variables are missing, dimensional analysis will fail.
2. List the dimensions of each variable according to $MLT\Theta$ or $FLT\Theta$. A list is given in Table 5.1.

[1] Make a clever choice here because all pis will contain these j variables in various groupings.

3. Find j. Initially guess j equal to the number of different dimensions present and look for j variables which do not form a pi product. If no luck, reduce j by 1 and look again. With practice, you will find j rapidly.
4. Select j variables which do not form a pi product. Make sure they please you and have some generality if possible, because they will then appear in every one of your pi groups. Pick density or velocity or length. Do not pick surface tension, for example, or you will form six different independent Weber-number parameters and thoroughly annoy your colleagues.
5. Add one additional variable to your j variables and form a power product. Algebraically find the exponents which make the product dimensionless. Try to arrange for your output or *dependent* variables (force, pressure drop, torque, power) to appear in the numerator and your plots will look better. Do this sequentially, adding one new variable each time, and you will find all $n - j = k$ desired pi products.
6. Write the final dimensionless function and check your work to make sure all pi groups are dimensionless.

EXAMPLE 5.6 Repeat the development of Eq. (5.2) from Eq. (5.1), using the pi theorem.

solution *Step 1.* Write the function and count variables:

$$F = f(L, U, \rho, \mu) \qquad \text{there are five variables } (n = 5)$$

Step 2. List dimensions of each variable. From Table 5.1

F	L	U	ρ	μ
$\{MLT^{-2}\}$	$\{L\}$	$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$

Step 3. Find j. No variable contains the dimension Θ, and so j is less than or equal to 3 (MLT). We inspect the list and see that L, U, and ρ cannot form a pi group because only ρ contains mass and only U contains time. Therefore j does equal 3, and $n - j = 5 - 3 = 2 = k$. The pi theorem guarantees for this problem that there will be exactly two independent dimensionless groups.

Step 4. Select j variables. The group L, U, ρ we found to prove that $j = 3$ will do fine.

Step 5. Combine L, U, ρ with one additional variable, in sequence, to find the two pi products.

First add force to find Π_1. You may select *any* exponent on this additional term as you please, to place it in the numerator or denominator to any power. Since F is the output, or dependent, variable, we select it to appear to the first power in the numerator

$$\Pi_1 = L^a U^b \rho^c F = (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

Equate exponents:

Length: $\qquad a + b - 3c + 1 = 0$

Mass: $c + 1 = 0$

Time: $-b \quad - 2 = 0$

We can solve explicitly for

$$a = -2 \qquad b = -2 \qquad c = -1$$

Therefore

$$\Pi_1 = L^{-2}U^{-2}\rho^{-1}F = \frac{F}{\rho U^2 L^2} = C_F \qquad \textit{Ans.}$$

This is exactly the right pi group as in Eq. (5.2). By varying the exponent on F, we could have found other equivalent groups such as $UL\rho^{1/2}/F^{1/2}$.

Finally, add viscosity to L, U, and ρ to find Π_2. Select any power you like for viscosity. By hindsight and custom, we select the power -1 to place it in the denominator:

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a(LT^{-1})^b(ML^{-3})^c(ML^{-1}T^{-1})^{-1} = M^0L^0T^0$$

Equate exponents:

Length: $a + b - 3c + 1 = 0$

Mass: $c - 1 = 0$

Time: $-b \quad + 1 = 0$

from which we find

$$a = b = c = 1$$

Therefore

$$\Pi_2 = L^1U^1\rho^1\mu^{-1} = \frac{\rho UL}{\mu} = \text{Re} \qquad \textit{Ans.}$$

We know we are finished; this is the second and last pi group. The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho UL}{\mu}\right) \qquad \textit{Ans.}$$

which is exactly what we found by the power-product method in Example 5.5.

A Successful Application

Dimensional analysis is fun, but does it work? Yes; if all important variables are included in the proposed function, the dimensionless function found by dimensional analysis will collapse all the data onto a single curve or set of curves.

An example of the success of dimensional analysis is given in Fig. 5.3 for the measured drag on smooth cylinders and spheres. The flow is normal to the axis of the cylinder, which is extremely long, $L/d \to \infty$. The data are from many sources, for both liquids and gases, and include bodies from several meters in diameter down to fine wires and balls less than 1 mm in size. Both curves in Fig. 5.3*a* are entirely experimental; the analysis of immersed body drag is one of the weakest areas of modern fluid-mechanics theory. Except for some isolated digital-computer calculations, there is no theory for cylinder and sphere drag except *creeping flow*, $\mathrm{Re} < 1$.

The Reynolds number of both bodies is based upon diameter, hence the notation Re_d. But the drag coefficients are defined differently

$$C_D = \begin{cases} \dfrac{\text{drag}}{\frac{1}{2}\rho U^2 L d} & \text{cylinder} \\ \dfrac{\text{drag}}{\frac{1}{2}\rho U^2 \frac{1}{4}\pi d^2} & \text{sphere} \end{cases} \tag{5.46}$$

They both have a factor $\frac{1}{2}$ as a traditional tribute to Bernoulli and Euler, and both are based on the projected area, i.e., the area one sees when looking toward the body from upstream. The usual definition of C_D is thus

$$C_D = \frac{\text{drag}}{\frac{1}{2}\rho U^2 (\text{projected area})} \tag{5.47}$$

However, one should carefully check the definitions of C_D, Re, etc., before using data in the literature.

Figure 5.3*a* is for long, smooth cylinders. If wall roughness and cylinder length are included as variables, we obtain from dimensional analysis a complex three-parameter function

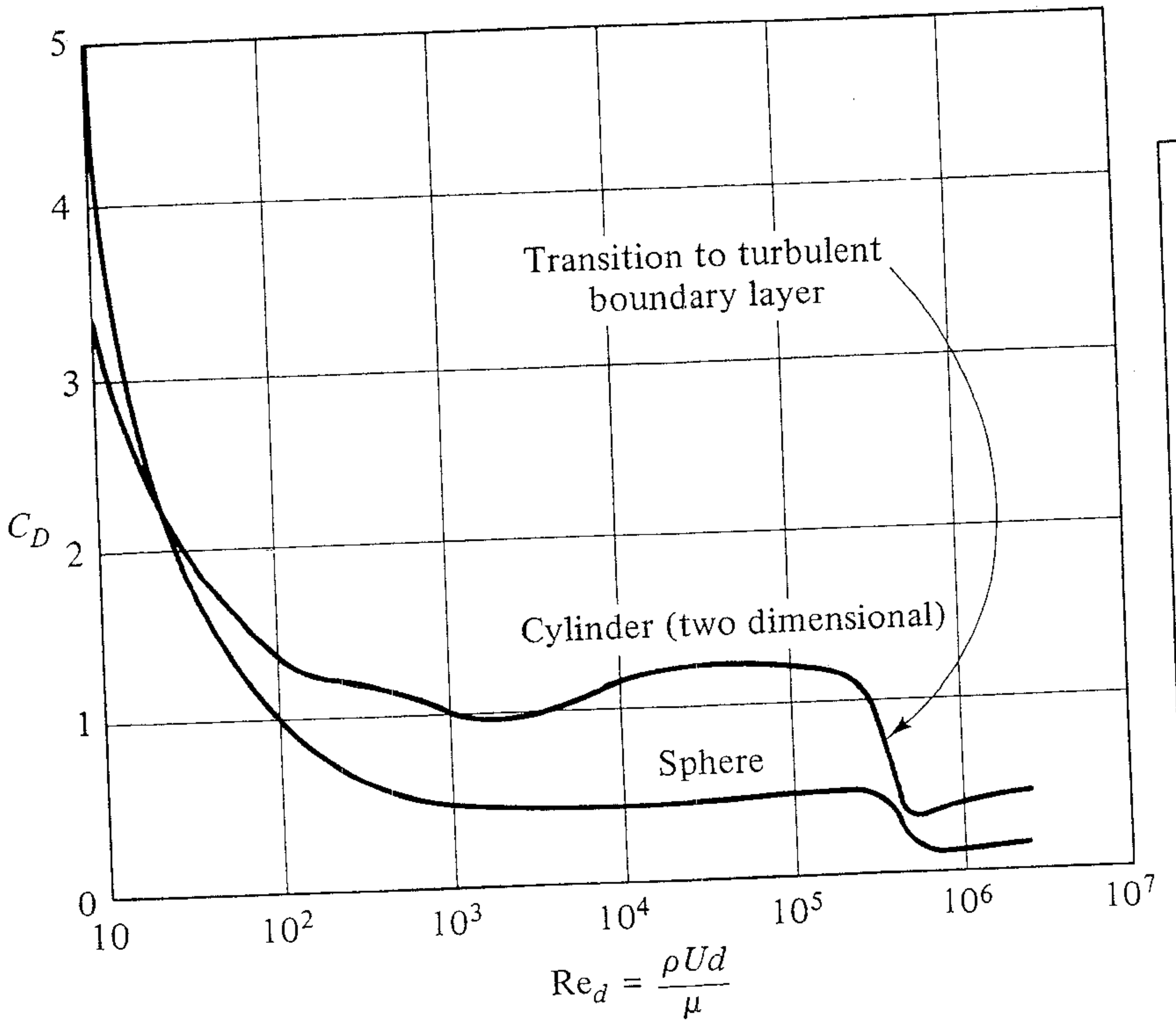
$$C_D = f\left(\mathrm{Re}_d, \frac{\epsilon}{d}, \frac{L}{d}\right) \tag{5.48}$$

To describe this function completely would require 1000 or more experiments. Therefore it is customary to explore the length and roughness effects separately to establish trends.

The table with Fig. 5.3*a* shows the length effect with zero wall roughness. As length decreases, the drag decreases by up to 50 percent. Physically, the pressure is "relieved" at the ends as the flow is allowed to skirt around the tips instead of deflecting over and under the body.

Figure 5.3*b* shows the effect of wall roughness for an infinitely long cylinder. The sharp drop in drag occurs at lower Re_d as roughness causes an earlier transition to a turbulent boundary layer on the surface of the body. Roughness has the same effect on sphere drag, a fact which is exploited in sports by deliberate dimpling of golf balls to give them less drag at their flight $\mathrm{Re}_d \approx 10^5$.

Figure 5.3 is a typical experimental study of a fluid-mechanics problem, aided by dimensional analysis. As time and money and demand allow, the complete three-parameter relation (5.48) could be filled out by further experiments.

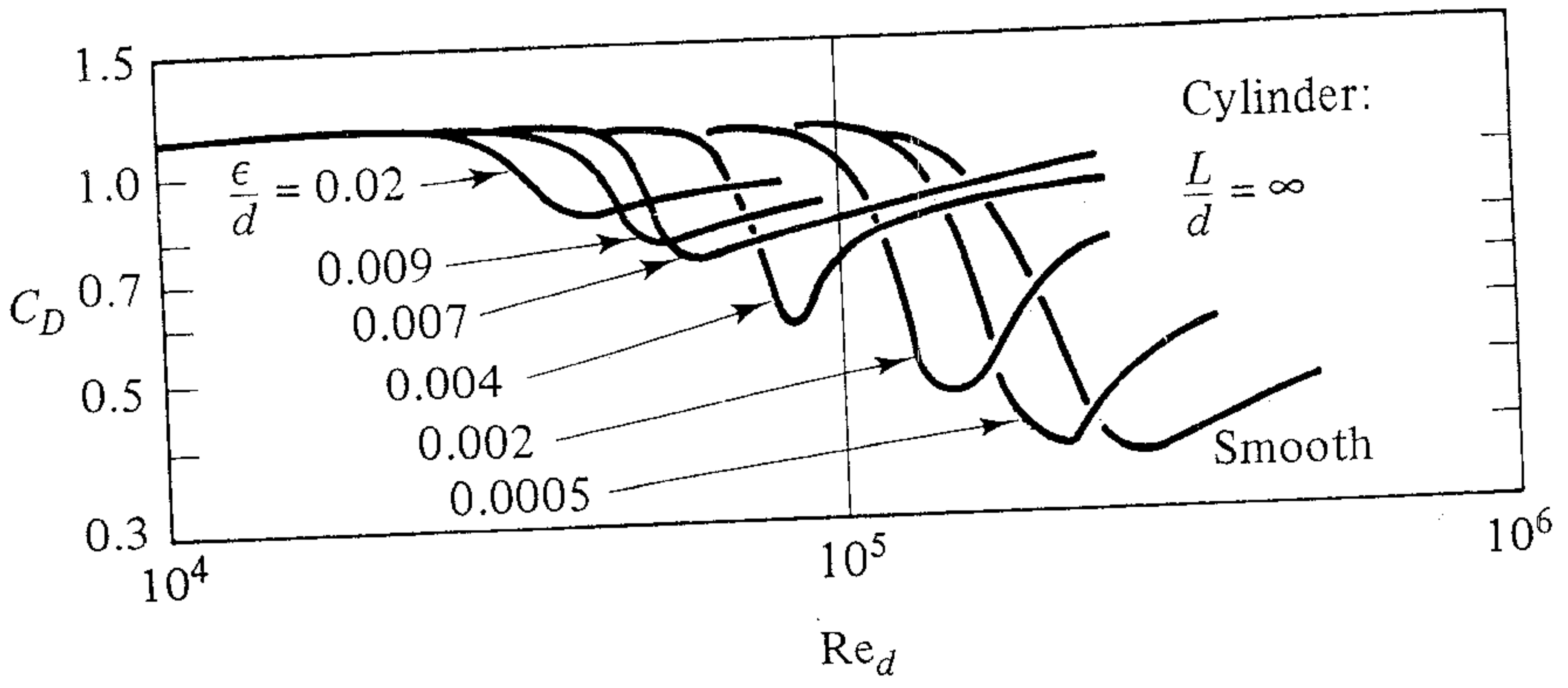


Cylinder length effect

$(10^4 < \mathrm{Re} < 10^5)$

L/d	C_D
∞	1.20
40	0.98
20	0.91
10	0.82
5	0.74
3	0.72
2	0.68
1	0.64

(*a*)



(*b*)

Fig. 5.3 The proof of practical dimensional analysis: drag coefficients of a cylinder and sphere: (*a*) drag coefficient of a smooth cylinder and sphere (data from many sources); (*b*) increased roughness causes earlier transition to a turbulent boundary layer.

EXAMPLE 5.7 The capillary rise h of a liquid in a tube varies with tube diameter d, gravity g, fluid density ρ, surface tension Υ, and the contact angle θ. (*a*) Find a dimensionless statement of this relation. (*b*) If $h = 3$ cm in a given experiment, what will h be in a similar case if diameter and surface tension are half as much, density is twice as much, and the contact angle is the same?

solution (*a*) *Step 1*. Write down the function and count variables

$$h = f(d, g, \rho, \Upsilon, \theta) \qquad n = 6 \text{ variables}$$

Step 2. List the dimensions (*FLT*) from Table 5.2:

h	d	g	ρ	Υ	θ
$\{L\}$	$\{L\}$	$\{LT^{-2}\}$	$\{FT^2L^{-4}\}$	$\{FL^{-1}\}$	None

Step 3. Find j. Several groups of three form no pi: Υ, ρ, and g or ρ, g, and d. Therefore $j = 3$, and we expect $n - j = 6 - 3 = 3$ dimensionless groups. One of these is obviously θ, which is already dimensionless:

$$\Pi_3 = \theta \qquad \textit{Ans. (a)}$$

If we chose carelessly to search for it using steps 4 and 5, we would still find $\Pi_3 = \theta$.

Step 4. Select j variables which do not form a pi group: ρ, g, d.

Step 5. Add one additional variable in sequence to form the pis:

Add h:

$$\Pi_1 = \rho^a g^b d^c h = (FT^2L^{-4})^a (LT^{-2})^b (L)^c (L) = F^0 L^0 T^0$$

Solve for

$$a = b = 0 \qquad c = -1$$

Therefore

$$\Pi_1 = \rho^0 g^0 d^{-1} h = \frac{h}{d} \qquad \textit{Ans. (a)}$$

Finally add Υ, again selecting its exponent to be 1

$$\Pi_2 = \rho^a g^b d^c \Upsilon = (FT^2L^{-4})^a (LT^{-2})^b (L)^c (FL^{-1}) = F^0 L^0 T^0$$

Solve for

$$a = b = -1 \qquad c = -2$$

Therefore

$$\Pi_2 = \rho^{-1} g^{-1} d^{-2} \Upsilon = \frac{\Upsilon}{\rho g d^2} \qquad \textit{Ans. (a)}$$

Step 6. The complete dimensionless relation for this problem is thus

$$\frac{h}{d} = F\left(\frac{\Upsilon}{\rho g d^2}, \theta\right) \qquad \textit{Ans. (a)} \quad (1)$$

This is as far as dimensional analysis goes. Theory, however, establishes that h is proportional to Υ. Since Υ occurs only in the second parameter, we can slip it outside

$$\left(\frac{h}{d}\right)_{\text{actual}} = \frac{\Upsilon}{\rho g d^2} F_1(\theta) \qquad \text{or} \qquad \frac{h \rho g d}{\Upsilon} = F_1(\theta)$$

Example 1.13 showed theoretically that $F_1(\theta) = 4 \cos \theta$.

(*b*) We are given h_1 for certain conditions d_1, Υ_1, ρ_1, and θ_1. If $h_1 = 3$ cm, what is h_2 for $d_2 = \frac{1}{2} d_1$, $\Upsilon_2 = \frac{1}{2}\Upsilon_1$, $\rho_2 = 2\rho_1$, and $\theta_2 = \theta_1$? We know the functional relation, Eq. (1), must still hold at condition 2

$$\frac{h_2}{d_2} = F\left(\frac{\Upsilon_2}{\rho_2 g d_2^2}, \theta_2\right)$$

But

$$\frac{\Upsilon_2}{\rho_2 g d_2^2} = \frac{\frac{1}{2}\Upsilon_1}{2\rho_1 g(\frac{1}{2}d_1)^2} = \frac{\Upsilon_1}{\rho_1 g d_1^2}$$

Therefore, functionally,

$$\frac{h_2}{d_2} = F\left(\frac{\Upsilon_1}{\rho_1 g d_1^2}, \theta_1\right) = \frac{h_1}{d_1}$$

We are given a condition 2 which is exactly similar to condition 1, and therefore a scaling law holds

$$h_2 = h_1 \frac{d_2}{d_1} = (3 \text{ cm}) \frac{\frac{1}{2}d_1}{d_1} = 1.5 \text{ cm} \qquad \textit{Ans. (b)}$$

If the pi groups had not been exactly the same for both conditions, we would have to know more about the functional relation F to calculate h_2.

5.5 MODELING AND ITS PITFALLS

So far we have learned about dimensional homogeneity and two methods, the power product and the pi theorem, for converting a homogeneous physical relation into dimensionless form. This is straightforward mathematically, but there are certain engineering difficulties which need to be discussed.

First, we have more or less taken for granted that the variables which affect the process can be listed and analyzed. Actually, selection of the important variables requires considerable judgment and experience. The engineer must decide for example whether viscosity can be neglected. Are there significant temperature effects? Is surface tension important? What about wall roughness? Each pi group which is retained increases the expense and effort required. Judgment in selecting variables will come through practice and maturity; this book should provide some of the necessary experience.

Once the variables are selected and the dimensional analysis performed, the experimenter seeks to achieve *similarity* between the model tested and the prototype to be designed. With sufficient testing, the model data will reveal the desired dimensionless function between variables

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_k) \tag{5.49}$$

With Eq. (5.49) available in chart, graphical, or analytical form, we are then in a position to ensure complete similarity between model and prototype. A formal statement would be as follows:

> Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for model and prototype.