CHAPTER 5

DIMENSIONAL ANALYSIS AND SIMILARITY

5.1 INTRODUCTION

This chapter treats the third and final method in our trio of techniques for studying fluid flows, dimensional analysis. The emphasis here is on the use of dimensional analysis to plan experiments and present data compactly, but many workers also use it in theoretical studies.

Basically, dimensional analysis is a method for reducing the number and complexity of experimental variables which affect a given physical phenomenon, using a sort of compacting technique. If a phenomenon depends upon n dimensional variables, dimensional analysis will reduce the problem to only k *dimensionless* variables, where the reduction $n - k = 1, 2, 3$, or 4, depending upon the problem complexity. Generally $n - k$ equals the number of different dimensions (sometimes called basic or primary or fundamental dimensions) which govern the problem. In fluid mechanics, the four basic dimensions are usually taken to be mass M, length L, time T, and temperature Θ, or an $MLT\Theta$ system for short. Sometimes one uses an $FLT\Theta$ system, with force F replacing mass.

Although its purpose is to reduce variables and group them in dimensionless form, dimensional analysis has several side benefits. The first is an enormous saving in time and money. Suppose one knew that the force F on a particular body immersed in a stream of fluid depended only on the body length L, the stream velocity V, the fluid density ρ, and the fluid viscosity μ; that is,

$$F = f(L, V, \rho, \mu) \tag{5.1}$$

Suppose further that the geometry and flow conditions are so complicated that our integral theories (Chap. 3) and differential equations (Chap. 4) fail to yield the solution for the force. Then we must find the function $f(L, V, \rho, \mu)$ experimentally.

Generally speaking, it takes about 10 experimental points to define a curve. To find the effect of body length in Eq. (5.1) we shall have to run the experiment for 10 lengths L. For each L we shall need 10 values of V, 10 values of ρ, 10 values of μ, making a grand total of 10^4, or 10,000, experiments. At \$5 per experiment—well, you see what we are getting into. However, with dimensional analysis, we can immediately reduce Eq. (5.1) to the equivalent form

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right)$$

or

$$C_F = g(\mathrm{Re}) \tag{5.2}$$

that is, the dimensionless *force coefficient* $F/\rho V^2 L^2$ is a function only of the dimensionless *Reynolds number* $\rho VL/\mu$. We shall learn exactly how to make this reduction in Secs. 5.2 and 5.4.

The function g is different mathematically from the original function f, but it contains all the same information. Nothing is lost in a dimensional analysis. And think of the saving: we can establish g by running the experiment for only 10 values of the single variable called the Reynolds number. We do not have to vary L, V, ρ, or μ separately but only the *grouping* $\rho VL/\mu$. This we do merely by varying velocity V in, say, a wind tunnel or drop test or water channel, and there is no need to build 10 different bodies or find 100 different fluids with 10 densities and 10 viscosities. The cost is now about \$50, maybe less.

A second side benefit of dimensional analysis is that it helps our thinking and planning for an experiment or theory. It suggests dimensionless ways of writing equations before we waste money on computer time to find solutions. It suggests variables which can be discarded; sometimes dimensional analysis will immediately reject variables, and sometimes it groups them off to the side, where a few simple tests will show them to be unimportant. Finally, dimensional analysis will often give a great deal of insight into the form of the physical relationship we are trying to study.

A third benefit is that dimensional analysis provides *scaling laws* which can convert data from a cheap, small *model* into design information for an expensive, large *prototype*. We do not build a million-dollar airplane and see whether it has enough lift force. We measure the lift on a small model and use a scaling law to predict the lift on the full-scale prototype airplane. There are rules we shall explain for finding scaling laws. When the scaling law is valid, we say that a condition of *similarity* exists between model and prototype. In the simple case of Eq. (5.2), similarity is achieved if the Reynolds number is the same for the model and prototype because the function g then requires the force coefficient to be the same also

$$\text{If } \mathrm{Re}_m = \mathrm{Re}_p \qquad \text{then} \qquad C_{Fm} = C_{Fp} \tag{5.3}$$

where subscripts m and p mean model and prototype, respectively. From the definition of force coefficient, this means that

$$\frac{F_p}{F_m} = \left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 \tag{5.4}$$

for data taken where $\rho_p V_p L_p/\mu_p = \rho_m V_m L_m/\mu_m$. Equation (5.4) is a scaling law: if you measure the model force at the model Reynolds number, the prototype force at the same Reynolds number equals the model force times the density ratio times the velocity ratio squared times the length ratio squared. We shall give more examples later.

Do you understand these introductory explanations? Be careful, learning dimensional analysis is like learning to play tennis: there are levels of the game. We can establish some ground rules and do some fairly good work in this brief chapter, but dimensional analysis in the broad view has many subtleties and nuances which only time and practice and maturity enable one to master. Although dimensional analysis has a firm physical and mathematical foundation, considerable art and skill are needed to use it effectively.

EXAMPLE 5.1 A copepod is a water crustacean approximately 1 mm in diameter. We want to know the drag force on the copepod when it moves slowly in fresh water. A scale model 100 times larger is made and tested in glycerin at $V = 30$ cm/s. The measured drag on the model is 1.3 N. For similar conditions, what are the velocity and drag of the actual copepod in water? Assume that Eq. (5.1) applies and the temperature is 20°C.

solution From Table 1.3 the fluid properties are:

Water (prototype):	$\mu_p = 0.001 \text{ kg/(m} \cdot \text{s)}$	$\rho_p = 999 \text{ kg/m}^3$
Glycerin (model):	$\mu_m = 1.5 \text{ kg/(m} \cdot \text{s)}$	$\rho_m = 1263 \text{ kg/m}^3$

The length scales are $L_m = 100$ mm and $L_p = 1$ mm. We are given enough model data to compute the Reynolds number and force coefficient

$$\text{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \frac{(1263 \text{ kg/m}^3)(0.3 \text{ m/s})(0.1 \text{ m})}{1.5 \text{ kg/(m} \cdot \text{s)}} = 25.3$$

$$C_{Fm} = \frac{F_m}{\rho_m V_m^2 L_m^2} = \frac{1.3 \text{ N}}{(1263 \text{ kg/m}^3)(0.3 \text{ m/s})^2(0.1 \text{ m})^2} = 1.14$$

Both these numbers are dimensionless, as you can check. For conditions of similarity, the prototype Reynolds number must be the same, and Eq. (5.2) then requires the prototype force coefficient to be the same

$$\text{Re}_p = \text{Re}_m = 25.3 = \frac{999 V_p(0.001)}{0.001}$$

or

$$V_p = 0.0253 \text{ m/s} = 2.53 \text{ cm/s} \qquad \textit{Ans.}$$

$$C_{Fp} = C_{Fm} = 1.14 = \frac{F_p}{999(0.0253)^2(0.001)^2}$$

or

$$F_p = 7.31 \times 10^{-7} \text{ N} \qquad \textit{Ans.}$$

It would obviously be difficult to measure such a tiny drag force.

Historically, the first person to write extensively about units and dimensional reasoning in physical relations was Euler in 1765. Euler's ideas were far ahead of his time, as were those of Joseph Fourier, whose 1822 book, "Analytical Theory of Heat," outlined what is now called the principle of dimensional homogeneity and even developed some similarity rules for heat flow. There were no further significant advances until Lord Rayleigh's book in 1877, "Theory of Sound," which proposed a "method of dimensions" and gave several examples of dimensional analysis. The final breakthrough which established the method as we know it today is generally credited to E. Buckingham in 1914 [24], whose paper outlined what is now called the *Buckingham pi theorem* for describing dimensionless parameters (see Sec. 5.4). However, it is now known that a Frenchman, A. Vaschy in 1892, and a Russian, D. Riabouchinsky in 1911, had independently published papers reporting results equivalent to the pi theorem. Following Buckingham's paper, P. W. Bridgman published a classic book in 1922 [1] outlining the general theory of dimensional analysis. The subject continues to be controversial because there is so much art and subtlety in using dimensional analysis. Thus, since Bridgman there have been at least 20 books published on the subject [1–20]. There will probably be more, but seeing the whole list might make some fledgling authors think twice. Nor is dimensional analysis limited to fluid mechanics or even engineering. Specialized books have been written on the application of dimensional analysis to metrology [21], astrophysics [22], and even economics [23]. The most recent book listed [20] is one of the most interesting and will intrigue even the most experienced user of dimensional analysis.

5.2 THE PRINCIPLE OF DIMENSIONAL HOMOGENEITY

In making the remarkable jump from the five-variable Eq. (5.1) to the two-variable Eq. (5.2), we were exploiting a rule which is almost a self-evident axiom in physics. This rule, the *principle of dimensional homogeneity*, can be stated as follows:

> If an equation truly expresses a proper relationship between variables in a physical process, it will be *dimensionally homogeneous*; i.e., each of its additive terms will have the same dimensions.

All the equations which are derived from the theory of mechanics are of this form. For example, consider the relation which expresses the displacement of a falling body

$$S = S_0 + V_0 t + \tfrac{1}{2}gt^2 \tag{5.5}$$

Each term in this equation is a displacement, or length, and has dimensions $\{L\}$. The equation is dimensionally homogeneous. Note also that any consistent set of units can be used to calculate a result.

Consider Bernoulli's equation for incompressible flow

$$\frac{p}{\rho} + \tfrac{1}{2}V^2 + gz = \text{const} \tag{5.6}$$

Each term, including the constant, has dimensions of velocity squared, or $\{L^2T^{-2}\}$. The equation is dimensionally homogeneous and gives proper results for any consistent set of units.

Students count on dimensional homogeneity and use it to check themselves when they cannot quite remember an equation during an exam. For example, which is it:

$$S = \tfrac{1}{2}gt^2? \qquad \text{or} \qquad S = \tfrac{1}{2}g^2t? \tag{5.7}$$

By checking the dimensions, we reject the second form and back up our faulty memory. We are exploiting the principle of dimensional homogeneity (PDH), and this chapter simply exploits it further.

Equations (5.5) and (5.6) also illustrate some other factors that often enter into a dimensional analysis: dimensional variables, dimensional constants, and pure constants.

Dimensional variables are the quantities which actually vary during a given case and would be plotted against each other to show the data. In Eq. (5.5), they are S and t, in Eq. (5.6) they are p, V, and z. All have dimensions, and all can be nondimensionalized as a dimensional-analysis technique.

Dimensional constants may vary from case to case but are held constant during a given run. In Eq. (5.5) they are S_0, V_0, and g, and in Eq. (5.6) they are ρ, g, and C. They all have dimensions and conceivably could be nondimensionalized, but they are normally used to help nondimensionalize the variables in the problem.

Pure constants have no dimensions and never did. They arise from mathematical manipulations. In both Eqs. (5.5) and (5.6) they are $\frac{1}{2}$ and the exponent 2, both of which came from an integration: $\int t\,dt = \frac{1}{2}t^2$, $\int V\,dV = \frac{1}{2}V^2$. Other common dimensionless constants are π and e.

Note that integration and differentiation of an equation may change the dimensions but not the homogeneity of the equation. For example, integrate or differentiate Eq. (5.5):

$$\int S\,dt = S_0t + \tfrac{1}{2}V_0t^2 + \tfrac{1}{6}gt^3 \tag{5.8a}$$

$$\frac{dS}{dt} = V_0 + gt \tag{5.8b}$$

In the integrated form (5.8a) every term has dimensions of $\{LT\}$, while in the derivative form (5.8b) every term is a velocity $\{LT^{-1}\}$.

Finally, there are some physical variables that are naturally dimensionless by virtue of their definition as ratios of dimensional quantities. Some examples are strain (change in length per unit length), Poisson's ratio (ratio of transverse strain to longitudinal strain), and specific gravity (ratio of density to standard water density). All angles are dimensionless (ratio of arc length to radius) and should be taken in radians for this reason.

The motive behind dimensional analysis is that any dimensionally homogeneous equation can be written in an entirely equivalent nondimensional form which is more compact. The exact details are spelled out in Sec. 5.4 as the pi theorem. For example, Eq. (5.5) is handled by defining dimensionless variables

$$S^* = \frac{S}{S_0} \qquad t^* = \frac{V_0 t}{S_0} \tag{5.9a}$$

or

$$S^{**} = \frac{gS}{V_0^2} \qquad t^{**} = \frac{gt}{V_0} \tag{5.9b}$$

Notice that there were two ways to nondimensionalize the variables. This is quite common; sometimes there are three or more ways. Which way is best? Usually neither, it is a matter of taste, custom, and the user's choice. You must accept the fact that there are several equivalent formulations of most dimensional analysis problems, all of which are correct.

There are two don'ts involved in operations like Eq. (5.9). First, *don't* nondimensionalize variables upside down:

$$S^* = \frac{S_0}{S} \qquad t^* = \frac{S_0}{V_0 t} \tag{5.10}$$

These are dimensionless, no question about it. But with the constants in the top and the variables in the bottom, there will be singularities where S and $t = 0$, the plots will look funny, users of your data will be confused, and the supervisor will be angry. It is not a good idea. Put your most important variable in the numerator and use parametric constants in the denominator.

Second, don't—repeat, *don't*—mix your variables (S, t) together in one definition:

$$S^* = \frac{V_0 t}{S} \tag{5.11}$$

This is beautiful and intriguing, but you will have mathematical problems and vexing presentation problems also. This idea sometimes works in an advanced technique called *similarity theory* (see, for example, Ref. 11), but it should not be used in dimensional analysis.

Now try our definitions (5.9) in Eq. (5.5):

$$S_0 S^* = S_0 + V_0 \frac{S_0 t^*}{V_0} + \tfrac{1}{2} g \left(\frac{S_0 t^*}{V_0}\right)^2 \tag{5.12a}$$

$$\frac{V_0^2 S^{**}}{g} = S_0 + V_0 \frac{V_0 t^{**}}{g} + \tfrac{1}{2} g \left(\frac{V_0 t^{**}}{g}\right)^2 \tag{5.12b}$$

These still have dimensions of length, but if we divide through and isolate a dimensionless variable, for example, S^* or S^{**}, the PDH guarantees that *all* terms will be dimensionless. Thus divide (5.12*a*) by S_0 and divide (5.12*b*) by V_0^2/g

$$S^* = 1 + t^* + \frac{1}{2}\frac{gS_0}{V_0^2}t^{*2} \qquad (5.13a)$$

$$S^{**} = \frac{gS_0}{V_0^2} + t^{**} + \tfrac{1}{2}t^{**2} \qquad (5.13b)$$

These are both dimensionless equations, equivalent to each other and equivalent in every respect to the original Eq. (5.5). They are plotted in Fig. 5.1. Which form do you feel is better and more effective? You are asked to explain your choice in Prob. 5.1.

Whereas Eq. (5.5) was of the form

$$S = f(t, S_0, V_0, g) \qquad (5.14)$$

and involved five dimensional quantities, Eqs. (5.13) are each of the form

$$S' = g(t', \alpha) \qquad \alpha = \frac{gS_0}{V_0^2} \qquad (5.15)$$

and involve only three dimensionless quantities. The parameter α commonly occurs in processes affected by gravity and is a form of the Froude number (see Table 5.2).

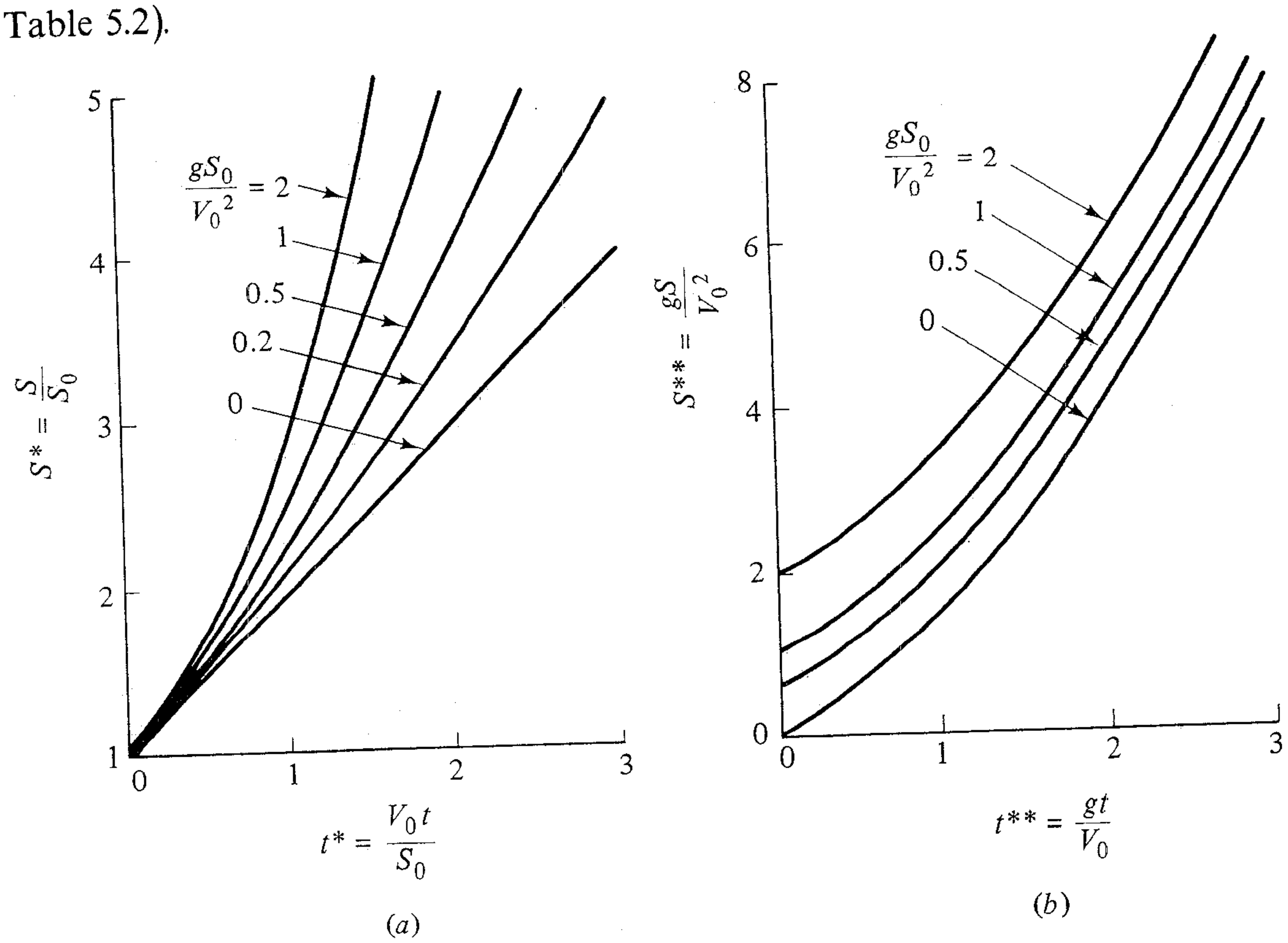

Fig. 5.1 Two entirely equivalent dimensionless forms of the falling-body equation (5.5): (*a*) Eq. (5.13*a*) and (*b*) Eq. (5.13*b*). Which form is more effective?

This example checks with our earlier statements about the dimensional-analysis technique. The original function of five variables is reduced to a dimensionless function of three variables. The reduction 5 − 3 = 2 should equal the number of dimensions ($MLT\Theta$) involved in the problem. Check our variables:

$$\{S, S_0\} = \{L\} \qquad \{t\} = \{T\} \qquad \{V_0\} = \{LT^{-1}\} \qquad \{g\} = \{LT^{-2}\} \tag{5.16}$$

As expected, there are only two dimensions involved, $\{L\}$ and $\{T\}$. This idea culminates in the pi theorem (Sec. 5.4).

The Power-Product Method

Take a last look at this example. Suppose that we were ignorant of the theory of dynamics and had to experiment to find the functional relationship Eq. (5.15). Since S is a length, the PDH tells us that the function f must be a length, so that t, S_0, V_0, and g must all combine in such a way as to eliminate time and leave only a length dimension. As Buckingham showed [24], the only way this can happen is for each term in f to be a grouping of products of *powers* of the quantities:

$$f_1 = (\text{const})(t)^a(S_0)^b(V_0)^c(g)^d \tag{5.17}$$

where the proportionality constant is dimensionless and a, b, c, and d are constant powers to be determined. Dimensionally, Eq. (5.17) must be length = length:

$$\{L\} = \{T\}^a\{L\}^b\{LT^{-1}\}^c\{LT^{-2}\}^d \tag{5.18}$$

Equating powers of length and time, we obtain two algebraic relations:

Length: $$1 = b + c + d \tag{5.19a}$$

Time: $$0 = a - c - 2d \tag{5.19b}$$

Since we have two equations in four unknowns, a, b, c, and d, any two can be written in terms of the other two. For example, let us solve for c and d in terms of a and b

$$c = 2 - a - 2b \qquad d = a + b - 1 \tag{5.20}$$

Equation (5.17) becomes

$$f_1 = (\text{const})\,\frac{V_0^2}{g}\left(\frac{gt}{V_0}\right)^a\left(\frac{gS_0}{V_0^2}\right)^b \tag{5.21}$$

Now, since f_1 is a "typical" term in f, while a and b are entirely arbitrary, Eq. (5.21) is trying to tell us that f can vary in an *arbitrary* fashion with the two parameters gt/V_0 and gS_0/V_0^2. Both parameters are dimensionless, and that is no accident; it is a consequence of the *power-product* method of grouping. We conclude that Eq.

(5.14) is equivalent to the dimensionless functional relation

$$\frac{gS}{V_0^2} = F\left(\frac{gt}{V_0}, \frac{gS_0}{V_0^2}\right) \qquad \text{or} \qquad S^{**} = F(t^{**}, \alpha) \tag{5.22}$$

We have just made a complete dimensional analysis of the physical process represented by Eq. (5.14). The dimensional-analysis technique does not tell us the form of the function F, which we would have to find by experiment or by theory, as in Eq. (5.13*b*).

This power-product method of solving for undetermined exponents illustrates the arbitrariness of the final groups obtained. Had we decided instead to solve for, say, b and c in terms of a and d, we would have obtained

$$b = 1 - a + d \qquad c = a - 2d \tag{5.23}$$

Substitution into Eq. (5.17) would give

$$f_1 = (\text{const})(S_0)\left(\frac{V_0 t}{S_0}\right)^a \left(\frac{gS_0}{V_0^2}\right)^d$$

which we may interpret as suggesting the new dimensionless functional relationship

$$\frac{S}{S_0} = F\left(\frac{V_0 t}{S_0}, \frac{gS_0}{V_0^2}\right) \qquad \text{or} \qquad S^{*} = F(t^{*}, \alpha) \tag{5.24}$$

Thus a perfectly legitimate alternate solution for the exponents gives the second, or alternate, form for the relationship. You must learn to accept the alternate choices which dimensional analysis gives to a typical analysis.

A list of dimensions in fluid mechanics is given in Table 5.1.

EXAMPLE 5.2 At low velocities (laminar flow), the volume flux Q through a small-bore tube is a function only of the pipe radius r, the fluid viscosity μ, and the pressure drop per unit pipe length dp/dx. Using the power-product method, rewrite the suggested relationship $Q = f(r, \mu, dp/dx)$ in dimensionless form.

solution First list the dimensions of the variables (Table 5.1):

$$Q = \{L^3T^{-1}\} \qquad r = \{L\} \qquad \mu = \{ML^{-1}T^{-1}\} \qquad \frac{dp}{dx} = \{ML^{-2}T^{-2}\}$$

Since Q is a volume flux and we assume dimensional homogeneity, the function f must be a volume flux. Assume a power product

$$f_1 = (\text{const})(r)^a(\mu)^b\left(\frac{dp}{dx}\right)^c$$

or

$$\{L^3T^{-1}\} = \{L\}^a\{ML^{-1}T^{-1}\}^b\{ML^{-2}T^{-2}\}^c$$

Table 5.1

DIMENSIONS OF FLUID-MECHANICS QUANTITIES

		Dimensions	
Quantity	*Symbol*	$\{MLT\Theta\}$	$\{FLT\Theta\}$
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flux	Q	L^3T^{-1}	L^3T^{-1}
Mass flux	$\dot{m}$	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$\dot{\epsilon}$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Expansion coefficient	β	Θ^{-1}	Θ^{-1}

Equating respective exponents, we have:

Length: $$3 = a - b - 2c$$

Mass: $$0 = b + c$$

Time: $$-1 = -b - 2c$$

With three equations in three unknowns, the solution is

$$a = 4 \qquad b = -1 \qquad c = 1$$

There is no arbitrariness; only one power product can be formed

$$Q = (\text{const}) \frac{r^4}{\mu} \frac{dp}{dx} \qquad \textit{Ans.}$$

The constant is dimensionless. The laminar-flow theory of Chap. 6 shows the value of the constant to be $\pi/8$.

EXAMPLE 5.3 The propagation speed C of a water wave is assumed to be a function of water density ρ, depth h, wave length λ, and the acceleration of gravity g:

$$C = f(\rho, h, \lambda, g) \qquad (1)$$

Rewrite Eq. (1) in dimensionless form, using the power-product method. Did we make a false assumption?

solution List the dimensions of the quantities

$$C = \{LT^{-1}\} \qquad \rho = \{ML^{-3}\} \qquad h = \{L\} \qquad \lambda = \{L\} \qquad g = \{LT^{-2}\}$$

The function f must be a speed, and the power product is

$$f_1 = (\text{const})(\rho)^a(h)^b(\lambda)^c(g)^d \qquad (2)$$

or

$$\{LT^{-1}\} = \{ML^{-3}\}^a\{L\}^b\{L\}^c\{LT^{-2}\}^d$$

Equate exponents:

Length: $\quad 1 = -3a + b + c + d$

Mass: $\quad 0 = a$

Time: $\quad -1 = -2d$

There are three equations in four unknowns. The solution is

$$a = 0 \qquad d = \tfrac{1}{2} \qquad c = -b + \tfrac{1}{2}$$

Equation (2) becomes

$$f_1 = (\text{const})(\rho)^0(h)^b(\lambda)^{-b+1/2}(g)^{1/2} = (\text{const})(g\lambda)^{1/2}\left(\frac{h}{\lambda}\right)^b \qquad (3)$$

As discussed in Eq. (5.21), this implies that f may vary arbitrarily with h/λ, so that Eq. (1) is equivalent to

$$\frac{C}{(g\lambda)^{1/2}} = F\left(\frac{h}{\lambda}\right) \qquad \textit{Ans.}$$

We falsely assumed the density to be important, but dimensional analysis shows it to disappear on dimensional grounds. Surface-wave theory shows the function F to be

$$F = \left(\frac{1}{2\pi}\tanh\frac{2\pi h}{\lambda}\right)^{1/2} \qquad (4)$$

but this of course cannot be determined by dimensional analysis alone.

EXAMPLE 5.4 Assume that the tip deflection δ of a solid circular cantilever beam is a function of the tip load P, beam radius r and length L, and material modulus of elasticity E: $\delta = f(P, r, L, E)$. Express this relation in dimensionless form using the power-product method. Is there something peculiar about the reduction in variables?

solution List the dimensions of the quantities in terms of mass, length, and time:

$$\{\delta\} = \{L\} \qquad \{P\} = \{MLT^{-2}\} \qquad \{r\} = \{L\} \qquad \{L\} = \{L\} \qquad \{E\} = \{ML^{-1}T^{-2}\}$$

The function f must be a deflection or length, and the suggested power product is

$$f_1 = (\text{const})(P)^a(r)^b(L)^c(E)^d \tag{2}$$

or

$$\{L\} = \{MLT^{-2}\}^a\{L\}^b\{L\}^c\{ML^{-1}T^{-2}\}^d$$

Equate exponents:

Length: $\qquad 1 = a + b + c - d$

Mass: $\qquad 0 = a + d$

Time: $\qquad 0 = -2a - 2d$

The solution is given in terms of a and c:

$$d = -a \qquad b = 1 - c - 2a$$

Equation (2) becomes

$$f_1 = (\text{const})(P)^a(r)^{1-c-2a}(L)^c(E)^{-a} = (\text{const})(r)\left(\frac{P}{Er^2}\right)^a\left(\frac{L}{r}\right)^c \tag{3}$$

By analogy with Eq. (5.21), the arbitrary exponents a and c imply a general function of two variables. We can now rewrite the original function $\delta = f(P, r, L, E)$ as

$$\frac{\delta}{r} = F\left(\frac{P}{Er^2}, \frac{L}{r}\right) \tag{4}$$

This is as far as raw dimensional analysis takes us, and it seems peculiar; there were five original variables and three different dimensions (MLT), so we expected $5 - 3 = 2$ dimensionless variables and we got three instead. This is because mass and time each affected Eq. (2) in the same manner, giving $a = -d$. The MLT dimensions were not independent, as we could see by expressing the variables in the FLT system:

$$\{\delta\} = \{L\} \qquad \{P\} = \{F\} \qquad \{r\} = \{L\} \qquad \{E\} = \{FL^{-2}\}$$

This shows that there are really only two different dimensions, force and length, which is characteristic of static structural problems.

We can "improve" Eq. (4) by taking advantage of a little physics, as Langhaar points out [8, p. 91]. For small elastic deflections, δ is proportional to the load P. Since P occurs in only one variable in Eq. (4), we can slip that variable outside to satisfy this proportionality

$$\frac{\delta}{r} = \frac{P}{Er^2} G\left(\frac{L}{r}\right)$$

or

$$\frac{\delta Er}{P} = G\left(\frac{L}{r}\right) \tag{5}$$

For a given "geometry" L/r, $\delta \propto P/Er \propto P/EL$. Beam-bending theory indicates that the function $G(L/r) = (\frac{4}{3}\pi)(L/r)^3$.

EXAMPLE 5.5 Try to develop Eq. (5.2) from Eq. (5.1) via the power-product method. Explain your curious "force coefficients."

solution Our basic function is $F = f(L, U, \rho, \mu)$. The list of dimensions can be made for these five variables:

$$\{F\} = \{MLT^{-2}\} \qquad \{L\} = \{L\} \qquad \{U\} = \{LT^{-1}\} \qquad \{\rho\} = \{ML^{-3}\} \qquad \{\mu\} = \{ML^{-1}T^{-1}\}$$

We expect no fewer than $5 - 3 = 2$ dimensionless variables. The suggested power product is for f equal to a force

$$f_1 = (\text{const})(L)^a(U)^b(\rho)^c(\mu)^d$$

or

$$\{MLT^{-2}\} = \{L\}^a\{LT^{-1}\}^b\{ML^{-3}\}^c\{ML^{-1}T^{-1}\}^d \tag{1}$$

Equate exponents:

Length: $1 = a + b - 3c - d$

Mass: $1 = c + d$

Time: $-2 = -b - d$

Solve for three unknowns in terms of the fourth. A variety of formulations will occur, depending upon which we choose to be the "free" exponent. If we choose d, then

$$a = 2 - d \qquad b = 2 - d \qquad c = 1 - d$$

Equation (1) becomes

$$f_1 = (\text{const})L^{2-d}U^{2-d}\rho^{1-d}\mu^d = (\text{const})(\rho U^2L^2)\left(\frac{\rho UL}{\mu}\right)^{-d} \tag{2}$$

As usual, the arbitrariness of d implies an arbitrary function of its argument. The original function can now be rewritten as

$$\frac{F}{\rho U^2L^2} = G\left(\frac{\rho UL}{\mu}\right) \quad \text{or} \quad C_F = G(\text{Re}) \qquad \textit{Ans.} \tag{3}$$

This is exactly Eq. (5.2). Since the theory of fluid forces on immersed bodies is still rather weak and qualitative, the function $G(\text{Re})$ is generally determined by experiment.

If we choose another free exponent, two different but related force coefficients will arise. For example, if we solve for a, b, and d in terms of c, the solution is

$$a = 1 + c \qquad b = 1 + c \qquad d = 1 - c$$

or

$$f_1 = (\text{const})(LU\mu)\left(\frac{\rho UL}{\mu}\right)^c$$

or

$$\frac{F}{LU\mu} = G_1\left(\frac{\rho UL}{\mu}\right) = G_1(\text{Re}) \tag{4}$$

The new force coefficient is not unique, but

$$\frac{F}{LU\mu} = \frac{F}{\rho U^2 L^2}\frac{\rho UL}{\mu} \equiv C_F \text{ Re} \tag{5}$$

The "correct" force coefficient is thus a matter of taste and custom. This particular choice $F/LU\mu$ is uncommon but very useful in highly viscous, "creeping" motion, for which it equals a pure constant (Chap. 7).

Further, if we had solved for a, c, and d in terms of b, we would have obtained

$$a = b \qquad c = b - 1 \qquad d = 2 - b$$

or

$$\frac{F\rho}{\mu^2} = G_2\left(\frac{\rho UL}{\mu}\right) = G_2(\text{Re}) \tag{6}$$

This force coefficient is not unique either

$$\frac{F\rho}{\mu^2} = \frac{F}{\rho U^2 L^2}\left(\frac{\rho UL}{\mu}\right)^2 \equiv C_F \text{ Re}^2 \tag{7}$$

Do you like it? This writer has never seen it used. It has some merit in nondimensionalizing F strictly with fluid physical properties, leaving the effect of velocity and size entirely on the right-hand side of Eq. (6). We shall suggest a use for it in Sec. 5.6.

If a dimensional analysis results in two or more dimensionless products, we always have this freedom to choose from a variety of algebraically related groups.

Some Peculiar Engineering Equations

The foundation of the dimensional-analysis method rests on two assumptions: (1) that the proposed physical relation is dimensionally homogeneous and (2) that all the relevant variables have been included in the proposed relation.

If a relevant variable is missing, dimensional analysis will fail, giving either algebraic difficulties or, worse, yielding a dimensionless formulation which does not resolve the process. A typical case is Manning's open-channel formula, discussed in Example 1.4:

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \tag{1.5}$$

Since V is velocity, R is a radius, and n and S are dimensionless, the formula is not dimensionally homogeneous. This should be a warning that (1) the formula changes if the *units* of V and R change and (2) if valid, it represents a very special case. Equation (1.5) predates the dimensional-analysis technique and is valid only for water in rough channels at moderate velocities and large radii in English units.

Such dimensionally inhomogeneous formulas abound in the hydraulics literature. Another example is the Hazen-Williams formula [25] for volume flux of water through a straight smooth pipe

$$Q = 61.9D^{2.63}\left(\frac{dp}{dx}\right)^{0.54} \tag{5.25}$$

where D is diameter and dp/dx the pressure gradient. Some of these formulas arise because numbers have been inserted for fluid properties and other physical data into perfectly legitimate homogeneous formulas. We shall not give the units of Eq. (5.25) to avoid encouraging its use.

On the other hand, some formulas are "constructs" which cannot be made dimensionally homogeneous. The "variables" they relate cannot be analyzed by the dimensional-analysis technique. Most of these formulas are raw empiricisms convenient to a small group of specialists. Here are three examples:

$$B = \frac{25{,}000}{100 - R} \tag{5.26}$$

$$S = \frac{140}{130 + \mathrm{API}} \tag{5.27}$$

$$0.0147D_E - \frac{3.74}{D_E} = 0.26t_R - \frac{172}{t_R} \tag{5.28}$$

Equation (5.26) relates the Brinell hardness B of a metal to its Rockwell hardness R. Equation (5.27) relates the specific gravity S of an oil to its density in degrees API. Equation (5.28) relates the viscosity of a liquid in D_E, or degrees Engler, to its viscosity t_R in Saybolt seconds. Such formulas have a certain usefulness when communicated between fellow specialists, but we cannot handle them here. Variables like Brinell hardness and Saybolt viscosity are not suited to an $MLT\Theta$ dimensional system.

5.3 NONDIMENSIONALIZATION OF THE BASIC EQUATIONS

We could use the power-product method of the previous section to analyze problem after problem, finding the dimensionless parameters which govern in each case. Textbooks on dimensional analysis [e.g., 7] do this. An alternate and very powerful technique is to attack the basic equations of flow from Chap. 4. Even though these equations cannot be solved in general, they will reveal basic dimensionless parameters, e.g., Reynolds number, in their proper form and