9.4.9 Determine the discharge in cubic meters per second measured by a 10-ft Parshall flume if the gage reading H_a is 1 m at a free-flow condition.

9.4.10 Determine the discharge through a 4-ft Parshall flume if the gage reading from well a is 1.0 m and from well b is 0.8 m.

9.4.11 A 40-ft Parshall flume measures $H_a = 1.0$ m and $H_b = 0.95$ m; determine the discharge.

10

HYDRAULIC SIMILITUDE AND MODEL STUDIES

Use of small models for studying the prototype hydraulic design can be dated at least to Leonardo da Vinci.* But the method developed for using the results of experiments conducted on a *scaled model* to predict quantitatively the behavior of a full-size hydraulic structure (or prototype) was realized only after the turn of this century. The principle on which the model studies are based comprises the theory of *hydraulic similitude*. The analysis of the basic relationship of the various physical quantities involved in the static and dynamic behaviors of water flow in a hydraulic structure is known as *dimensional analysis*.

All important hydraulic structures are now designed and built after certain preliminary model studies have been completed. Such studies may be conducted for any one or more of the following purposes:

* Leonardo da Vinci (1452–1519), a genius, Renaissance scientist, engineer, architect, painter, sculptor, and musician

1. The determination of the discharge coefficient of a large measurement structure, such as an overflow spillway or a weir.
2. The development of an effective method for energy dissipation at the outlet of a hydraulic structure.
3. The reduction of energy loss at the intake structure or at the transition section.
4. The development of an efficient, economic spillway or other type of flood-releasing structure for a reservoir.
5. The determination of the average time of travel in a temperature control structure, for example, in a cooling pond in a power plant.
6. The determination for the best cross section, location, and dimensions of the various components of the structure, such as the breakwater, the docks, and the locks, etc., in harbor and waterway design.
7. The determination of the dynamic behaviors of the floating, semi-immersible, and floor-installed structures in transportation and installation of offshore structures.

River models have also been extensively used in hydraulic engineering to determine

1. The pattern a flood wave travels through a river channel.
2. The effect of artificial structures, such as bends, levees, dikes, jetties, and training walls, on the sedimentation movements in the channel reach, as well as in the upstream and downstream channels.
3. The direction and force of currents in the channel or harbor and their effect on navigation and marine life.

10.1 DIMENSIONAL HOMOGENEITY

When a physical phenomenon is described by an equation or a set of equations, all terms in each of the equations must be kept dimensionally homogeneous. In other words, all terms in an equation must be expressed in the same units.

In fact, to derive a relationship among several parameters involved in a physical phenomenon, one should always check the equation for homogeneity of units. If all terms in an equation do not appear to have the same unit, then one can be sure that certain important parameters may be missing or misplaced.

Based on the physical understanding of the phenomenon and the concept of dimensional homogeneity, the solution of many hydraulic problems may be obtained. For example, we understand that the speed of surface wave propagation on water surface, C, is related to the gravitational acceleration, g, and the water depth, d. Generally, we may write

$$C = f(g, d) \tag{10.1}$$

The units of the physical quantities involved are indicated in the brackets,

$$C = [LT^{-1}]$$
$$g = [LT^{-2}]$$
$$d = [L]$$

Since the left-hand side of Equation (10.1) has the units of $[LT^{-1}]$, those units must then appear explicitly on the right-hand side. Thus, d and g must combine as a product and the function, f, must be the square root. We have

$$C = \sqrt{gd}$$

as discussed in Chapter 6, (Equation 6.11).

The dimensions of the physical quantities commonly used in hydraulic engineering are listed in Table 10.1.

TABLE 10.1 Dimensions of Physical Quantities Commonly Used in Hydraulic Engineering

Quantity	*Dimension*	*Quantity*	*Dimension*
Length	L	Force	MLT^{-2}
Area	L^2	Pressure	$ML^{-1}T^{-2}$
Volume	L^3	Shear Stress	$ML^{-1}T^{-2}$
Angle (radians)	None	Specific Weight	$ML^{-2}T^{-2}$
Time	T	Modulus of Elasticity	$ML^{-1}T^{-2}$
Discharge	L^3T^{-1}	Coefficient of Compressibility	$M^{-1}LT^2$
Linear Velocity	LT^{-1}	Surface Tension	MT^{-2}
Angular Velocity	T^{-1}	Momentum	MLT^{-1}
Acceleration	LT^{-2}	Angular Momentum	ML^2T^{-1}
Mass	M	Torque	ML^2T^{-2}
Moment of Inertia	ML^2	Energy	ML^2T^{-2}
Density	ML^{-3}	Power	ML^2T^{-3}
Viscosity	$ML^{-1}T^{-1}$	Kinematic Viscosity	L^2T^{-1}

10.2 PRINCIPLES OF HYDRAULIC SIMILITUDE

Similarity between hydraulic models and prototypes may be achieved in three basic forms

1. geometric similarity,
2. kinematic similarity,
3. dynamic similarity.

Geometric similarity implies similarity of form. The model is a geometric reduction of the prototype and is accomplished by maintaining a fixed ratio for all homologous lengths between the model and the prototype.

The physical quantities involved in geometric similarity are length, L, area, A, and volume, Vol. The ratio of homologous lengths in prototype and model is a constant and can be expressed as

$$\frac{L_p}{L_m} = L_r \tag{10.2}$$

An area, A, is the product of two homologous lengths; hence, the ratio of the homologous area is also a constant and can be expressed as

$$\frac{A_p}{A_m} = \frac{L_p^2}{L_m^2} = L_r^2 \tag{10.3}$$

A volume, Vol, is the product of three homologous lengths; the ratio of the homologous volume can be expressed as

$$\frac{\text{Vol}_p}{\text{Vol}_m} = \frac{L_p^3}{L_m^3} = L_r^3 \tag{10.4}$$

Example 10.1

A geometrically similar open channel model is constructed with a 5:1 scale. If the model measures a discharge of 0.2 m³/sec, determine the corresponding discharge in the prototype.

Solution

The velocity ratio between the prototype and the model is

$$\frac{V_p}{V_m} = \frac{\frac{L_p}{T}}{\frac{L_m}{T}} = \frac{L_p}{L_m} = L_r = 5$$

Since in geometric similarity, time in model and prototype remains unscaled. The area ratio between the prototype and the model is

$$\frac{A_p}{A_m} = \frac{L_p^2}{L_m^2} = L_r^2 = 25$$

Accordingly, the discharge ratio is

$$\frac{Q_p}{Q_m} = \frac{A_pV_p}{A_mV_m} = (25)(5) = 125$$

Thus, the corresponding discharge in the prototype is

$$Q_p = 125Q_m = 125 \cdot 0.2 = 25 \text{ m}^3/\text{sec}$$

Kinematic similarity implies similarity in motion. Kinematic similarity between a model and the prototype is attained if the homologous moving particles have the same velocity ratio along geometrically similar paths. The kinematic similarity involves the scale of time as well as length. The ratio of times required for homologous particles to travel homologous distances in a model and its prototype is

$$\frac{T_p}{T_m} = T_r \tag{10.5}$$

The velocity V is defined in terms of distance per unit time; thus, the ratio of velocities can be expressed as

$$\frac{V_p}{V_m} = \frac{\frac{L_p}{T_p}}{\frac{L_m}{T_m}} = \frac{\frac{L_p}{L_m}}{\frac{T_p}{T_m}} = \frac{L_r}{T_r} \tag{10.6}$$

The acceleration, a, is defined in terms of length per unit time square; thus, the ratio of homologous acceleration is

$$\frac{a_p}{a_m} = \frac{\frac{L_p}{T_p^2}}{\frac{L_m}{T_m^2}} = \frac{\frac{L_p}{L_m}}{\frac{T_p^2}{T_m^2}} = \frac{L_r}{T_r^2} \tag{10.7}$$

The discharge Q, is expressed in terms of volume per unit time; thus,

$$\frac{Q_p}{Q_m} = \frac{\frac{L_p^3}{T_p}}{\frac{L_m^3}{T_m}} = \frac{\frac{L_p^3}{L_m^3}}{\frac{T_p}{T_m}} = \frac{L_r^3}{T_r} \tag{10.8}$$

Kinematic models constructed for hydraulic machinery may frequently involve angular displacement, θ, expressed in radians, which is equal to the tangential displacement, L, divided by the length of radius, R, of the curve at the point of tangency. The ratio of angular displacements may be expressed as

$$\frac{\theta_p}{\theta_m} = \frac{\dfrac{L_p}{R_p}}{\dfrac{L_m}{R_m}} = \frac{\dfrac{L_p}{L_m}}{\dfrac{R_p}{R_m}} = \frac{\dfrac{L_p}{L_m}}{\dfrac{L_p}{L_m}} = 1 \tag{10.9}$$

The angular velocity, N, in revolutions per minute, is defined as angular displacement per unit time; thus, the ratio

$$\frac{N_p}{N_m} = \frac{\dfrac{\theta_p}{T_p}}{\dfrac{\theta_m}{T_m}} = \frac{\dfrac{\theta_p}{\theta_m}}{\dfrac{T_p}{T_m}} = \frac{1}{T_r} \tag{10.10}$$

The angular acceleration, α, is defined as angular displacement per unit time square,

$$\frac{\alpha_p}{\alpha_m} = \frac{\dfrac{\theta_p}{T_p^2}}{\dfrac{\theta_m}{T_m^2}} = \frac{\dfrac{\theta_p}{\theta_m}}{\dfrac{T_p^2}{T_m^2}} = \frac{1}{T_r^2} \tag{10.11}$$

Example 10.2

A 10:1 scale model is constructed to study the flow motions in a cooling pond. If the designed discharge from the power plant is 200 m^3/sec and the model can accommodate a maximum flow rate of 0.1 m^3/sec, determine the time ratio.

Solution

The length ratio between the prototype and the model is

$$L_r = \frac{L_p}{L_m} = 10$$

and the discharge ratio is $Q_r = \dfrac{200}{0.1} = 2000$, and

$$Q_r = \frac{Q_p}{Q_m} = \frac{\dfrac{L_p^3}{T_p}}{\dfrac{L_m^3}{T_m}} = \left(\frac{L_p}{L_m}\right)^3 \cdot \left(\frac{T_m}{T_p}\right) = L_r^3 \cdot T_r^{-1}$$

Substituting the length ratio into the discharge ratio gives the time ratio

$$T_r = \frac{T_p}{T_m} = \frac{L_r^3}{Q_r} = \frac{(10)^3}{2000} = 0.5$$

and

$$T_m = 2T_p$$

A unit time period measured in the model is equivalent to two periods of time in the prototype pond.

Dynamic similarity implies similarity in forces involved in motion. Dynamic similarity between a model and its prototype is attained if the ratio of homologous forces in the model and prototype is kept at a constant value, or

$$\frac{F_p}{F_m} = F_r \tag{10.12}$$

Many hydrodynamic phenomena may involve several different kinds of forces in action. Since models are usually built to simulate the prototype on reduced scales, they usually are not capable of simulating all the forces simultaneously. In practice, a model is designed to study the effects of only a few *dominant forces*. Dynamic similarity requires that the ratios of these forces be kept the same between the model and the prototype. Hydraulics phenomena governed by the various types of force ratios are discussed in Sections 10.3, 10.4, 10.5, and 10.6. Since force is equal to mass, M, multiplied by acceleration, a, and since mass is equal to density, ρ, times volume, Vol, the force ratio may be expressed as

$$\begin{aligned} \frac{F_p}{F_m} &= \frac{M_p a_p}{M_m a_m} = \frac{\rho_p \,\mathrm{Vol}_p\, a_p}{\rho_m \,\mathrm{Vol}_m\, a_m} \\ &= \frac{\rho_p}{\rho_m} \cdot \frac{L_p^3}{L_m^3} \cdot \frac{\dfrac{L_p}{T_p^2}}{\dfrac{L_m}{T_m^2}} = \frac{\rho_p}{\rho_m} \cdot \frac{L_p^4}{L_m^4} \cdot \frac{1}{\dfrac{T_p^2}{T_m^2}} \\ &= \rho_r L_r^4 T_r^{-2} \end{aligned} \tag{10.13}$$

and the ratio of homologous masses, force divided by acceleration, may be expressed as

$$\frac{M_p}{M_m} = \frac{\dfrac{F_p}{a_p}}{\dfrac{F_m}{a_m}} = \frac{\dfrac{F_p}{F_m}}{\dfrac{a_p}{a_m}} = F_r T_r^2 L_r^{-1} \tag{10.14}$$

Since work is equal to force multiplied by distance, the ratio of homologous work in dynamic similarity is

$$\frac{\overline{W}_p}{\overline{W}_m} = \frac{F_p L_p}{F_m L_m} = F_r L_r \tag{10.15}$$

Power is the time rate of doing work; thus, the ratio of homologous powers in the model and prototype is

$$\frac{P_p}{P_m} = \frac{\dfrac{\overline{W}_p}{T_p}}{\dfrac{\overline{W}_m}{T_m}} = \frac{\overline{W}_p}{\overline{W}_m} \cdot \frac{1}{\dfrac{T_p}{T_m}} = \frac{F_r L_r}{T_r} \tag{10.16}$$

Example 10.3

A 59,680-w (80-hp) pump is used to power a water supply system. The model constructed to study the system has an 8:1 scale. If the velocity ratio is 2:1, determine the power needed for the model pump.

Solution

By substituting the length ratio into the velocity ratio, the time ratio is obtained

$$V_r = \frac{L_r}{T_r} = 2; \qquad L_r = 8$$

$$T_r = \frac{L_r}{2} = \frac{8}{2} = 4$$

The same fluid is used in the model and the prototype, unless otherwise specified; then $\rho_r = 1$, and the force ratio can be calculated, from Equation (10.13)

$$F_r = \rho_r L_r^4 T_r^{-2} = \frac{(1)(8)^4}{(4)^2} = 256$$

The power ratio is, from Equation (8.15)

$$P_r = \frac{F_r L_r}{T_r} = \frac{(256)(8)}{(4)} = 512$$

And the power required for the model pump is

$$P_m = \frac{P_p}{P_r} = \frac{59{,}680}{512} = 116.6 \text{ w} = 0.156 \text{ hp}$$

Example 10.4

The model designed to study the prototype of a hydraulic machine must

1. be geometrically similar,
2. have the same discharge coefficient defined as $Q/A\sqrt{2gH}$,
3. have the same ratio of peripheral speed to the water discharge velocity, $\omega D/(Q/A)$.

Determine the scale ratios in terms of discharge Q, head H, diameter D, and rotational angular velocity ω.

Solution

It is important to recognize that although the energy head, H, is expressed in units of length, it is not necessarily modeled as a linear length dimension. To have the same ratio of peripheral speed to the water discharge velocity, we have,

$$\frac{\omega_p D_p}{Q_p/A_p} = \frac{\omega_m D_m}{Q_m/A_m}$$

or,

$$\frac{\omega_r D_r A_r}{Q_r} = \frac{T_r^{-1} L_r L_r^2}{L_r^3 T_r^{-1}} = 1$$

To have the same discharge coefficient, we have,

$$\frac{Q_p/(A_p\sqrt{2gH_p})}{Q_m/(A_m\sqrt{2gH_m})} = \frac{Q_r}{A_r\sqrt{(gH)_r}} = 1$$

or,

$$\frac{L_r^3 T_r^{-1}}{L_r^2 (gH)_r^{1/2}} = 1$$

From which, we get,

$$(gH)_r = \frac{L_r^2}{T_r^2}$$

Since the gravitational acceleration, g, is the same for model and prototype, we may write,

$$H_r = L_r^2 T_r^{-2}$$

The other ratios asked for are

Discharge ratio: $Q_r = L_r^3 T_r^{-1}$
Diameter ratio: $D_r = L_r$, and
Angular speed ratio: $\omega_r = T_r^{-1}$

PROBLEMS

10.2.1 A 1-m-long, 1:30 model is used to study the wave force on a prototype of a sea wall structure. If the total wave force measured on the model is 2.27 N and the velocity scale is 1:10, determine the force per unit length of the prototype.

10.2.2 The moment exerted on a gate structure is studied in a laboratory water tank with a 1:125 scale model. If the moment measured on the model is 1.5 N · m on the 1-m long gate arm, determine the moment exerted on the prototype.

10.2.3 A 1:100 scale model is constructed to study a gate prototype that is designed to drain a reservoir. If the model reservoir is drained in 5.2 min, how long should it take to drain the prototype?

10.2.4 An overflow spillway is designed to have a 100-m long crest to carry the discharge of 1150 m^3/sec under a permitted maximum head of 3 m. The operation of the prototype spillway is studied by a 1:50 scale model in a hydraulic laboratory.

(a) Determine the model discharge based on Equation (8.9).
(b) The model velocity measured at the end (toe) of the spillway is 3 m/sec. Determine the corresponding velocity in the prototype.
(c) What are the model and prototype Froude numbers at the toe?
(d) If the force on a bucket energy dissipator at the toe of the spillway is measured to be 37.5 N, determine the corresponding force on the prototype.

10.2.5 If a 1:5 scale model, at 1200 rpm, is used to study the prototype of a centrifugal pump that produces 1 m^3/sec at 30-m head when rotating at 400 rpm, determine the model discharge and head.

10.2.6 A 1:25 scale model is being designed to study a prototype hydraulic structure. The velocity ratio between the model and the prototype is 1:5, and the measurement accuracy is required to be within 1% of the total force. Determine the accuracy of the force measurement in the model if the expected force on the prototype is 45,000 N. [Hint: use Equation (8.9)].

10.2.7 Sedimentation in a river section 5 km long is to be studied in a lab channel 20 m long. A time ratio of 4 will be used and the prototype discharge is 75 m^3/sec. Select a suitable length scale and determine the model discharge.

10.2.8 A 1:20 scale model of a prototype energy dissipation structure is constructed to study force distribution and water depths. A velocity ratio of 7.75 is used. Determine the force ratio and prototype discharge if the model discharge is 300 ℓ/sec.

10.2.9 A 1:50 scale model is used to study the power requirements of a prototype submarine. The model will be towed at a speed 50 times greater than the speed of the prototype in a tank filled with sea water. Determine the conversion ratios from the prototype to the model for the following quantities: (a) time, (b) force, (c) energy, (d) power.

10.3 PHENOMENA GOVERNED BY VISCOUS FORCE—REYNOLDS NUMBER LAW

Water in motion always involves inertial forces. When the inertial forces and the viscous forces can be considered to be the only forces that govern the motion, the ratio of these forces acting on homologous particles in a model and its prototype is defined by the Reynolds number law

$$N_R = \frac{\text{inertial force}}{\text{viscous force}} \tag{10.17}$$

The inertial force defined by Newton's second law of motion, $F = ma$, can be expressed by the ratio in Equation (10.13):

$$F_r = M_r \frac{L_r}{T_r^2} = \rho_r L_r^4 T_r^{-2} \tag{10.13}$$

The viscous force defined by Newton's law of viscosity,

$$F = \mu\left(\frac{dV}{dL}\right) \cdot A$$

may be expressed by

$$F_r = \frac{\mu_p \left(\frac{dV}{dL}\right)_p A_p}{\mu_m \left(\frac{dV}{dL}\right)_m A_m} = \mu_r L_r^2 T_r^{-1} \tag{10.18}$$

where μ is the viscosity and V denotes the velocity.

Equating values of F_r from Equations (10.13) and (10.18), we get

$$\rho_r L_r^4 T_r^{-2} = \mu_r L_r^2 T_r^{-1}$$

from which

$$\frac{\rho_r L_r^4 T_r^{-2}}{\mu_r L_r^2 T_r^{-1}} = \frac{\rho_r L_r^2}{\mu_r T_r} = \frac{\rho_r L_r V_r}{\mu_r} = 1 \tag{10.19}$$

Rearranging the above equation, we may write

$$\frac{\left(\dfrac{\rho_p L_p V_p}{\mu_p}\right)}{\left(\dfrac{\rho_m L_m V_m}{\mu_m}\right)} = 1$$

or

$$\frac{\rho_p L_p V_p}{\mu_p} = \frac{\rho_m L_m V_m}{\mu_m} = N_R \tag{10.20}$$

Equation (10.20) states that when the inertial force and the viscous force are considered to be the only forces governing the motion of the water, the Reynolds number of the model and the prototype must be kept at the same value.

If the same fluid is used in both the model and the prototype, the scale ratios for many physical quantities can be derived based on the Reynolds number law. These quantities are listed in Table 10.2.

TABLE 10.2 Scale Ratios for Reynolds Number Law (for Water in Both Model and Prototype, $\rho_r = 1$, $\mu_r = 1$)

Geometric Similarity		*Kinematic Similarity*		*Dynamic Similarity*	
Length	L_r	Time	L_r^2	Force	1
Area	L_r^2	Velocity	L_r^{-1}	Mass	L_r^3
Volume	L_r^3	Acceleration	L_r^{-3}	Work	L_r
		Discharge	L_r	Power	L_r^{-1}
		Angular Velocity	L_r^{-2}		
		Angular Acceleration	L_r^{-4}		

Example 10.5

In order to study a transient process, a model is constructed at a 10:1 scale. Water is used in the prototype, and it is known that viscous forces are the dominant ones. Compare the time and force ratios, if the model uses

(a) water,

(b) oil 5 times more viscous than water, with $\rho_{oil} = 0.8\rho_{water}$.

Solution

(a) From Table 10.2

$$T_r = L_r^2 = (10)^2 = 100$$

$$V_r = L_r^{-1} = (10)^{-1} = 0.1$$

$$F_r = 1$$

(b) From the Reynolds number law,

$$\frac{\rho_p L_p V_p}{\mu_p} = \frac{\rho_p L_m V_m}{\mu_m}$$

we have

$$\frac{\rho_r L_r V_r}{\mu_r} = 1$$

Since the ratios of viscosity and density are, respectively,

$$\mu_r = \frac{\mu_p}{\mu_m} = \frac{\mu_{water}}{\mu_{oil}} = \frac{\mu_{water}}{5\mu_{water}} = 0.2$$

$$\rho_r = \frac{\rho_p}{\rho_m} = \frac{\rho_{water}}{\rho_{oil}} = \frac{\rho_{water}}{0.8\rho_{water}} = 1.25$$

From the Reynolds number law

$$V_r = \frac{\mu_r}{\rho_r L_r} = \frac{(0.2)}{(1.25)(10)} = 0.016$$

The time ratio is

$$T_r = \frac{L_r}{V_r} = \frac{\rho_r L_r^2}{\mu_r} = \frac{(1.25)(10)^2}{(0.2)} = 625$$

The force ratio, then, is

$$F_r = \frac{\rho_r L_r^4}{T_r^2} = \frac{(\rho_r L_r^4)}{\left(\dfrac{\rho_r^2 L_r^4}{\mu_r^2}\right)} = \frac{\mu_r^2}{\rho_r} = \frac{(0.2)^2}{1.25} = 0.032$$

This example demonstrates the importance of selecting the model fluid. The properties of the liquid used in the model, especially the viscosity, greatly affect the performance in the Reynolds number models.

PROBLEMS

10.3.1 A Reynolds number scale model is used to study the operation of a prototype hydraulic device. The model is built on a 1:5 scale and uses water at 20°C. The prototype discharges 11.5 m^3/sec of water at 90°C temperature. Determine the model discharge.

10.3.2 The moment exerted on a ship's rudder is studied with a 1:20 scale model in a water tunnel using the same temperature as the river water. If the torque measured on the model is 10 N·m for a water tunnel velocity of 20 m^3/sec, determine the corresponding torque and speed for the prototype.

10.3.3 A 1:10 scale model of a water supply piping system is to be tested at 20°C to determine the total head loss in the prototype that carries water at 85°C. The prototype is designed to carry 5.0 m^3/sec discharge with 1-m diameter pipes. Determine the model discharge and model velocity. Discuss how losses determined from the model are converted to the prototype losses.

10.3.4 A submerged vehicle moves at 5 m/sec in the ocean. At what theoretical speed must a 1:10 model be towed for there to be dynamic similarity between the model and the prototype? Assume that the sea water and towing tank water are the same.

10.3.5 A structure is built underwater on the ocean floor where a strong current of 5 m/sec is measured. The structure is to be studied by a 1:25 model in a water tunnel using sea water at the same temperature as that measured in the ocean. What speed must the water tunnel provide in order to study the force load on the structure due to the current? If the required tunnel velocity is judged to be impractical, can the study be performed in a wide tunnel using air at 20°C? What would the corresponding air speed in the tunnel need to be?

10.4 PHENOMENA GOVERNED BY GRAVITY FORCE—FROUDE NUMBER LAW

When inertial force and gravity force are considered to be the only dominant forces in the fluid motions, the ratio of the inertial forces acting on the homologous elements of the fluid in the model and prototype can be defined by Equation (10.13)

$$\frac{F_p}{F_m} = \rho_r L_r^4 T_r^{-2} \qquad (10.13)$$

and the ratio of gravity forces, which is determined by the weight of the homologous fluid elements involved,

$$\frac{F_p}{F_m} = \frac{M_p g_p}{M_m g_m} = \frac{\rho_p g_p L_p^3}{\rho_m g_m L_m^3} = \rho_r g_r L_r^3 \qquad (10.21)$$

Equating the values from Equations (10.3) and (10.21), we get

$$\rho_r L_r^4 T_r^{-2} = \rho_r g_r L_r^3$$

Rearranging, we get

$$g_r L_r = \frac{L_r^2}{T_r^2} = V_r^2$$

or

$$\frac{V_r}{g_r^{1/2} L_r^{1/2}} = 1 \qquad (10.22)$$

From which

$$\frac{\left(\dfrac{V_p}{g_p^{1/2} L_p^{1/2}}\right)}{\left(\dfrac{V_m}{g_m^{1/2} L_m^{1/2}}\right)} = 1$$

Hence,

$$\frac{V_p}{g_p^{1/2} L_p^{1/2}} = \frac{V_m}{g_m^{1/2} L_m^{1/2}} = N_F \quad \text{(Froude number)} \qquad (10.23)$$

In other words, when the inertial force and the gravity force are considered to be the only forces that dominate the fluid motions, the Froude number of the model and the prototype should be kept at the same value.

If the same fluid is used in both the model and the prototype, and they are both subjected to the same gravitational force field, many physical quantities can be derived based on the Froude number law. These quantities are listed in Table 10.3.

TABLE 10.3 Ratios for the Froude Number Law ($g_r = 1$, $\rho_r = 1$)

Geometric Similarity		*Kinematic Similarity*		*Dynamic Similarity*	
Length	L_r	Time	$L_r^{1/2}$	Force	L_r^3
Area	L_r^2	Velocity	$L_r^{1/2}$	Mass	L_r^3
Volume	L_r^3	Acceleration	1	Work	L_r^4
		Discharge	$L_r^{5/2}$	Power	$L_r^{7/2}$
		Angular Velocity	$L_r^{-1/2}$		
		Angular Acceleration	L_r^{-1}		

Example 10.6

An open channel model 30 m long is built to satisfy Froude number law. What is the flow in the model for a prototype flood of 700 m^3/sec if the scale used is 20:1? Determine also the force ratio.

Solution

From Table 10.3, the discharge ratio is

$$Q_r = L_r^{5/2} = (20)^{2.5} = 1789$$

Thus, the model flow should be

$$Q_m = \frac{Q_P}{Q_r} = \frac{700\ \text{m}^3/\text{sec}}{1789} = 0.391\ \text{m}^3/\text{sec} = 391\ \ell/\text{sec}$$

The force ratio is

$$F_r = \frac{F_P}{F_m} = L_r^3 = (20)^3 = 8000$$

PROBLEMS

10.4.1 An overflow spillway with a 300-m crest is designed to discharge 3600 m^3/sec. A 1:20 model of the cross section of the dam is built in the laboratory flume 1 m wide. Calculate the required laboratory flow rate. Neglect viscosity and surface tension effects.

10.4.2 If a 1:1000 scale tidal basin model is used to study the operation of a prototype satisfying the Froude number law, what length of time in the model represents the period of one day in the prototype?

10.4.3 A ship 100 m long designed to travel at a top speed of 1 m/sec is to be studied in a towing tank with a 1:50 scale model. Determine what speed the model must be towed for (a) the Reynolds number law and (b) the Froude number law.

10.4.4 An overflow spillway is designed to be 100 m high and 120 m long, carrying a discharge of 1200 m^3/sec under an approaching head of 2.75 m. The spillway operation is to be analyzed by a 1:50 model in a hydraulic laboratory. Determine
(a) The model discharge,
(b) If the discharge coefficient at the model crest measures 2.12, what is the prototype crest discharge coefficient?
(c) If the velocity at the outlet of the model spillway measures 25 m/sec, what is the prototype velocity?
(d) If the U.S.B.R. Type II stilling basin, 50 m wide is used to dissipate the energy at the toe of the spillway, what is the energy dissipation in the model and in the prototype as measured in units of kilowatts?
(e) What is the efficiency of the dissipator in the prototype?

10.4.5 An energy dissipator is being designed to force a hydraulic jump at the end of a spillway channel discharging 400 m^3/sec. The initial depth in the 20-m wide prototype is expected to be 0.8 m. Determine the discharge of the 1:10 scale model and the velocity and force ratios between prototype and model.

10.4.6 A 1:25 model is built to study a stilling basin at the outlet of a steep spillway chute. The stilling basin consists of a horizontal floor (apron) with U.S.B.R. Type II baffles installed to stabilize the location of the hydraulic jump. The prototype has a rectangular cross section 25 m wide designed to carry a 75-m^3/sec discharge. The velocity immediately before the jump is 10 m/sec. Determine the following:
(a) The model discharge.
(b) The depth downstream of the jump in the prototype if the dynamic force measured on the model baffles is 16.2 N.
(c) The force on the baffles per unit width of the prototype channel.
(d) The energy dissipated in the basin.

10.5 PHENOMENA GOVERNED BY SURFACE TENSION—WEBER NUMBER LAW

Surface tension is a measure of energy level on the surface of a liquid body. The force is of primary importance in hydraulic engineering practice in the study of the motion of small surface waves or control of evaporation from a large body of water, such as a water storage tank or reservoir.

Surface tension, denoted by σ, is measured in terms of force per unit length. Hence, the force is $F = \sigma L$. The ratio of analogous surface tension forces in prototype and in model is

$$F_r = \frac{F_p}{F_m} = \frac{\sigma_p L_p}{\sigma_m L_m} = \sigma_r L_r \tag{10.24}$$

Equating the surface tension force ratio to the inertial force ratio [Equation (10.13)] gives

$$\sigma_r L_r = \rho_r \frac{L_r^4}{T_r^2}$$

Rearranging gives

$$T_r = \left(\frac{\rho_r}{\sigma_r}\right)^{1/2} L_r^{3/2} \tag{10.25}$$

By substituting for T_r the basic relationship of $V_r = L_r/T_r$, the Equation 10.25 may be rearranged to give

$$V_r = \frac{L_r}{\left(\frac{\rho_r}{\sigma_r}\right)^{1/2} L_r^{3/2}} = \left(\frac{\sigma_r}{\rho_r L_r}\right)^{1/2}$$

or

$$\frac{\rho_r V_r^2 L_r}{\sigma_r} = 1 \qquad (10.26)$$

Hence,

$$\frac{\rho_p V_p^2 L_p}{\sigma_p} = \frac{\rho_m V_m^2 L_m}{\sigma_m} = N_w \qquad \text{(Weber number)} \qquad (10.27)$$

In other words, the Weber number must be kept at the same value in the model and in the prototype for studying phenomena governed by surface tension force. If the same liquid is used in both model and prototype, then $\rho_r = 1.0$, and $\sigma_r = 1.0$, and Equation 10.27 can be simplified to

$$V_r^2 L_r = 1$$

or

$$V_r = L_r^{1/2} \qquad (10.28)$$

Since $V_r = L_r/T_r$, we may also write

$$\frac{L_r}{T_r} = L_r^{1/2}$$

Thus,

$$T_r = L_r^{3/2} \qquad (10.29)$$

PROBLEMS

10.5.1 A model is built to study the surface tension phenomenon in a reservoir. Determine the conversion ratios between the model and the prototype for the following quantities if the model is built with a 1:100 scale: (a) rate of flow, (b) energy, (c) pressure, (d) power. The same fluid is used in the model and the prototype.

10.5.2 A measuring device includes certain small glass tubes of a given geometry. To study the surface tension effect, a 5:1 scale model (larger than prototype) is built. Determine the discharge and force ratios.

10.5.3 Determine the surface tension of a liquid in the prototype if a time ratio of 2 is established with a 1:10 scale model. The surface tension of the liquid in the model is 150 dyn/cm. What is the force ratio?

10.6 PHENOMENA GOVERNED BY BOTH GRAVITY AND VISCOUS FORCES

In the case of surface vessels moving through water or the propagation of shallow water waves in open channels, both gravity and viscous forces may be important. The study of these phenomena requires that both the Froude number and Reynolds number laws be satisfied simultaneously. That is,

$$\frac{\rho_r L_r V_r}{\mu_r} = \frac{V_r}{(g_r L_r)^{1/2}}$$

Assuming that both the model and the prototype are affected by the earth's gravitational field, $g_r = 1$ and since $\nu = \mu/\rho$, the above relationship may be simplified to

$$\nu_r = L_r^{3/2} \qquad (10.30)$$

This requirement can only be met by choosing a special model fluid with a kinematic viscosity ratio to water equal to the three-half power of the scale ratio. In general, this requirement is difficult to meet. For example, a 1:10 scale model would require that the model fluid have a kinematic viscosity of 30 times less than that of water, which is obviously impossible.

However, two expedients may be available depending on the relative importance of the two forces in the particular phenomenon. In the case of ship resistance, the ship model may be built according to the Reynolds model law and may operate in a towing tank in accordance with the Froude number law. In the case of shallow water waves in open channels, empirical relationships such as Manning's formula [Equation (3.28)] may be used as an auxiliary condition for the wave measurements, according to the Froude number law.

10.7 MODELS FOR FLOATING AND SUBMERGED BODIES

Model studies for floating and submerged bodies are performed in order to obtain information on

1. the friction drag along the boundary of the moving vessel,
2. the form drag resulting from flow separation from the vessel boundary due to the boundary shape,

3. the force expended in the generation of gravity waves,
4. the stability of the body in withstanding the water waves and the wave forces on the body.

The first two forces are strictly viscous phenomena, and, therefore, the models should be designed according to the Reynolds number law. The third force is a gravity force governed phenomenon and, hence, must be analyzed by applying the Froude number law. All three measurements may be performed simultaneously in water in a towing tank. In analyzing the data, however, the friction forces and the form drag forces are first computed from the measurements by using known formulas and drag coefficients. The remaining force measured in towing the vessel through the water surface is the force expended in generating the gravity waves (wave resistance), and it is scaled up to the prototype values by the Froude number law.

The analysis procedure is demonstrated in Example (10.7). For subsurface vessels, such as submarines, the effect of surface waves on the vessels may be neglected. Hence, the Froude number model is not needed. To study the stability and wave force on stationary offshore structures, the effect of inertial force must be taken into consideration. The inertial force, defined as $F_i = M' \cdot a$, can be calculated directly from the prototype dimensions. Here, M' is the mass of water displaced by the portion of the structure immersed below the waterline (also known as the *virtual mass*), and a is the acceleration of the water mass.

Example 10.7

A ship model with a maximum cross-sectional area of 0.78 m^2 immersed below the waterline, has a characteristic length of 0.9 m. The model is towed in a wave tank at the speed of 0.5 m/sec. For the particular shape of the vessel, it is found that the drag coefficient can be approximated by $C_D = (0.06/N_R^{0.25})$ for $10^4 \leq N_R \leq 10^6$, and $C_D = 0.0018$ for $N_R > 10^6$. The Froude number law is applied for the 1:50 model. During the experiment, a total force of 0.40 N is measured. Determine the total resistance force on the prototype vessel.

Solution

Based on the Froude number law, we may determine the velocity ratio,

$$V_r = L_r^{1/2} = (50)^{1/2} = 7.07$$

Hence, the corresponding velocity of the vessel is

$$V_p = V_r \cdot V_m = 0.5 \cdot 7.07 = 3.54 \text{ m/sec}$$

The Reynolds number for the model is

$$N_R = \frac{V_m \cdot L_m}{\nu} = \frac{0.5 \cdot 0.9}{1.003 \cdot 10^{-6}} = 4.49 \cdot 10^5$$

and the drag coefficient for the model is

$$C_{D_m} = \frac{0.06}{(4.49 \cdot 10^5)^{1/4}} = 0.0023$$

The drag force on a vessel is defined as $D = C_D(\frac{1}{2}\rho A \cdot V^2)$, where ρ is the water density, and A is the projected area of the immersed portion of the vessel on a plane normal to the direction of the motion. Thus, the model drag force can be calculated as

$$D_m = C_{D_m}(\tfrac{1}{2}\rho_m A_m \cdot V_m^2) = 0.0023 \cdot \tfrac{1}{2} \cdot 1000 \cdot 0.78 \cdot 0.5^2 = 0.2243 \text{ N}$$

The model wave resistance is the difference between the measured towing force and the drag force

$$F_w = 0.4 - 0.2243 = 0.1757 \text{ N}$$

For the prototype, the Reynolds number is

$$N_R = \frac{V_p \cdot L_p}{\nu_p} = \frac{V_p \cdot L_r \cdot L_m}{\nu_p} = 1.59 \cdot 10^8$$

The drag coefficient of the prototype vessel is $C_{D_p} = 0.0018$, and the drag force is

$$D_p = C_{D_p}(\tfrac{1}{2}\rho_p A_p \cdot V_p^2) = C_{D_p}(\tfrac{1}{2}\rho_p A_m \cdot L_r^2 \cdot V_p^2) = 21937 \text{ N}$$

The wave resistance on the prototype vessel is calculated by applying Froude number law (see Table 10.3)

$$F_{w_p} = F_{w_r} \cdot F_{w_m} = L_r^3 \cdot F_{w_m} = 21963 \text{ N}$$

Hence the total resistance force on the prototype is

$$F = D_p + F_{w_p} = 21937 + 21963 = 43900 \text{ N}$$

PROBLEMS

10.7.1 A ship 100 m long moves at 1.5 m/sec in freshwater at 15°C. A 1:100 scale model of the prototype ship is to be tested in a towing tank containing a liquid of specific gravity 0.9. What viscosity must this liquid have for both Reynolds and Froude number laws to be satisfied?

10.7.2 A 1:250 ship model is towed in a wave tank and a wave resistance of 10.7 N is measured. Determine the corresponding prototype wave resistance on the prototype.

10.7.3 A barge model 1 m long is tested in a towing tank at the speed of 1 m/sec. Determine the prototype velocity if the prototype is 150 m in length. The model has 2-cm draft and is 10 cm wide. The drag coefficient is $C_D = 0.25$ for $N_R > 5 \cdot 10^4$, and the towing force required to tow the model is 0.3 N. What force is required to tow the barge in waterways?

10.7.4 A concrete caisson 60 m wide, 120 m long, and 12 m high is to be towed in sea water in the longitudinal direction to an offshore construction site where it will be sunk. The calculated floating depth of the caisson is 8 m, with 4 m remaining above the water surface. A 1:100 model is built to study the operation of the prototype. If the model is towed in a wave tank using sea water, what is the model speed that corresponds to the prototype speed of 1.5 m/sec? The model study considers both the skin drag and form drag force (Reynolds number) and the resistance due to the generation of gravity waves in motion (Froude number).

10.8 OPEN CHANNEL MODELS

Open channel models may be used to study either the velocity discharge-slope relationship or the effect of flow patterns on the changes in bed configuration. For the former applications, relatively long reaches of the river channel can be modeled. A special example is the U.S. Army Engineers Waterways Experiment Station in Vicksburg, Mississippi, where the Mississippi River is modeled on one site. In these applications, where the influence of changes in bed configuration are only of secondary concern, a fixed-bed model may be used. Basically, this model is used in studying the velocity-slope relationship in a particular channel; therefore, the effect of bed roughness is important.

An empirical relation, such as the Manning equation [Equation (3.28)], may be used to assume the similarity between the prototype and the model

$$V_r = \frac{V_p}{V_m} = \frac{\dfrac{1}{n_p} R_{h_p}^{2/3} S_p^{1/2}}{\dfrac{1}{n_m} R_{h_m}^{2/3} S_m^{1/2}} = \frac{1}{n_r} R_{h_r}^{2/3} S_r^{1/2} \tag{10.31}$$

If the model is built with the same scale ratio for the horizontal dimensions ($\overline{X}$) and vertical dimensions ($\overline{Y}$), known as the *undistorted model*, then

$$R_{h_r} = \overline{X}_r = \overline{Y}_r = L_r \quad \text{and} \quad S_r = 1$$

and

$$V_r = \frac{1}{n_r} L_r^{2/3} \tag{10.32}$$

Since the Manning's roughness coefficient $n \propto R_h^{1/6}$ [Equation (6.3)], we may write

$$n_r \quad L_r^{1/6} \tag{10.33}$$

This model may frequently result in a model velocity so small (or, conversely, the model roughness will be so large) that realistic measurements cannot be made; or a model water depth may be so shallow that the physical characteristics of the flow may be altered. Such situations may be resolved by using a distorted model in which the vertical scale and the horizontal scale do not have the same value; usually, a smaller vertical scale ratio, $\overline{X}_r > \overline{Y}_r$. This means that

$$S_r = \frac{S_p}{S_m} = \frac{\overline{Y}_r}{\overline{X}_r} < 1$$

Hence, $S_m > S_p$, and the result is a larger slope for the model. The use of the Manning equation requires that the flow be fully turbulent in both model and prototype.

Open channel models involving problems of sediment transportation, erosion, or deposit require movable bed models. A movable channel bed consists of sand or other loose material that can be moved in response to the forces of the current at the channel bed. Normally, it is impractical to scale the bed material down to the model scale. A vertical scale distortion is usually employed on a movable bed model in order to provide a sufficient tractive force to induce bed material movement. Quantitative similarity is difficult to attain in movable bed models. For any sedimentation studies performed, it is important that the movable bed model be quantitatively verified by a number of field measurements.

Example 10.8

An open channel model is built to study the effects of tidal waves on sedimentation movement in a 10-km river reach (the reach meanders in an area 7 km long). The mean depth and width of the reach are 4 m and 50 m, respectively, and the discharge is 850 m³/sec. Manning's roughness coefficient $n_p = 0.035$. If the model is to be constructed in a laboratory room 18 m long, determine a convenient scale and the model discharge.

Solution

In surface wave phenomenon the gravitational forces are dominant. The Froude number law will be used for the modeling. The laboratory length will limit the horizontal scale

$$\overline{X}_r = \frac{L_p}{L_m} \quad \frac{7000}{18} \quad 389$$

We will use $L_r = 400$ for convenience.

It is judged reasonable to use a vertical scale of Y_r 80 (enough to meas ure surface gradients). Recall that the hydraulic radius is the characteristic dimension in open channel and that for a large width-to-depth ratio the hydraulic radius is roughly equal to the water depth. Thus, we can make the following approximation:

$$R_{h_r} = \overline{Y}_r = 80$$

Since

$$N_F = \frac{V_r}{g_r^{1/2} R_{h_r}^{1/2}} = 1 \tag{10.22}$$

Then

$$V_r = R_{h_r}^{1/2} = \overline{Y}_r^{1/2} = (80)^{1/2}$$

Using Manning's formula [or Equation (10.31)],

$$V_r = \frac{V_p}{V_m} = \frac{1}{n_r} R_{h_r}^{2/3} S_r^{1/2} \qquad S_r = \frac{\overline{Y}_r}{\overline{X}_r}$$

we have

$$n_r = \frac{R_{h_r}^{2/3} S_r^{1/2}}{V_r} = \frac{\overline{Y}_r^{2/3}\left(\dfrac{\overline{Y}_r}{\overline{X}_r}\right)^{1/2}}{\overline{Y}_r^{1/2}} = \frac{\overline{Y}_r^{2/3}}{\overline{X}_r^{1/2}} = \frac{(80)^{2/3}}{(400)^{1/2}} = 0.928$$

Hence,

$$n_m = \frac{n_p}{n_r} = \frac{0.035}{0.928} = 0.038$$

The discharge ratio is

$$Q_r = A_r \cdot V_r = \overline{X}_r \overline{Y}_r \overline{V}_r = \overline{X}_r \overline{Y}_r^{3/2} = (400)(80)^{3/2} = 286{,}217$$

Thus, the model discharge required is

$$Q_m = \frac{Q_p}{Q_r} = \frac{850}{286{,}217} = 0.003 \text{ m}^3/\text{sec} \equiv 3\ \ell/\text{sec}$$

In order to use the Manning formula, turbulent flow must be ensured in the model. To verify the turbulent flow condition in the model, it is necessary to calculate the value of the model Reynolds number.

The horizontal prototype velocity is

$$V_p = \frac{850 \text{ m}^3/\text{sec}}{4 \text{ m} \cdot 50 \text{ m}} = 4.25 \text{ m/sec}$$

Hence,

$$V_m = \frac{V_p}{V_r} = \frac{4.25}{(80)^{1/2}} = 0.475 \text{ m/sec}$$

The model Reynolds number is

$$N_R = \frac{V_m \overline{Y}_m}{\nu} = \frac{0.475 \cdot 0.05}{1.1 \cdot 10^{-6}} = 21{,}598$$

which is much greater than the critical Reynolds number (2,000). Hence, the flow is turbulent in the model.

PROBLEMS

10.8.1 A new laboratory site is available for modeling the channel of Example 10.8 so that the length is no longer a restriction, but the roughness coefficient of the material to be used in the movable bed is $n_m = 0.018$. Determine a convenient scale and the corresponding model velocity.

10.8.2 A ship channel model is built to study sedimentation control in the prototype which is 40 m wide and 7.5 m deep and carries a discharge of 300 m^3/sec. For a vertical scale of 1:65 and a roughness coefficient of $n_p = 0.03$, determine all the other needed ratios for the study if $n_m = 0.02$.

10.8.3 A 1:100 scale model is constructed to study the pattern of flow in a river reach. If the reach has a Manning's coefficient $n = 0.025$, what should be the corresponding value of n in the model? Discuss the errors that may result from the study if the value of n were not modeled.

10.8.4 Determine the value of n in the model in Problem 10.8.3 if the vertical scale is exaggerated to 1:25 distortion.

10.8.5 A 1:300 scale model is constructed to study the discharge-depth relationship in a river reach with Manning's coefficient $n = 0.031$. If the model discharges 52 ℓ/sec and has Manning's coefficient $n = 0.033$, determine an adequate vertical scale ratio and the flowrate for the prototype.