CE 5333 – Special Studies in Water Resources
Essay 2.1 Dimensional Analysis and Similitude

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1 Introduction

Engineering problems involving fluid mechanics rely upon data acquired by experiment. In most cases the empirical data[1] apply to general enough situations that engineers need them [the data] in normal design practice. These data are made available by publication in journals, textbooks, and handbooks. Examples of such data are friction loss coefficients for pipes, valves, and other closed conduit fittings; drag coefficients for simple geometric shapes; frictional loss terms for open conduits, etc.

However, many practical problems have unusual geometry or such unusual flow conditions that studies on a replica of the of the situation at a different scale are required to predict flow patterns, pressure variation, and frictional loss behavior. When such tests are conducted, the replica is called the *model* and the full-scale situation is called the *prototype*. The model is usually smaller than the prototype for economic reasons — a model operated at full scale is called a testbed.

Testbeds are common in in aeronautical engineering. There are usually several scale models to answer specific questions, then the first prototype (aircraft) is built. The first one is called a testbed, and usually uses proven engines and other features in a effort to guarantee flight — as experience is gained the testbed is converted into a true prototype.

1.1 Need for Dimensional Analysis

Engineering of fluid systems is more heavily dependent on empirical results that is structural engineering, chemical engineering, machine design, or electrical engineering because the analytical tools are not capable of yielding exact solutions to many problems in fluid systems[2].

Exact solutions are available for all hydrostatic problems and many laminar-flow problems, but the most general equations solved on supercomputers only yield fair approximations for

[1]Empirical and experimental are used as synonyms in this sentence.

[2]Many important aspects of these other fields are indeed more amenable to analysis, but important questions in these other fields are also addressed by dimensional analysis. The sentence is not meant to imply that any field is better understood than any other.

turbulent-flow problems — hence a continued need for experimental methods, if for no other reason that to verify the computer solutions[3].

For comparison of model studies and for correlating results into design generalizations researchers should employ dimensionless parameters. To appreciate the advantage of using dimensionless parameters, and example using flow through an inverted nozzle is presented.

Figure 1 is a sketch of a relatively unusual orifice plate in a pipe that is flowing full. In the sketch a nozzle type device constricts the flow, but notice the direction of flow depicted in the sketch is opposite to the direction we would usually associate a nozzle. In this example consider that in practice we may have flow the "correct way" most of the time and the "wrong way" (depicted in this sketch) part of the time — hence there is some practical need for the knowledge of how a device performs when the flow is the "wrong" direction.

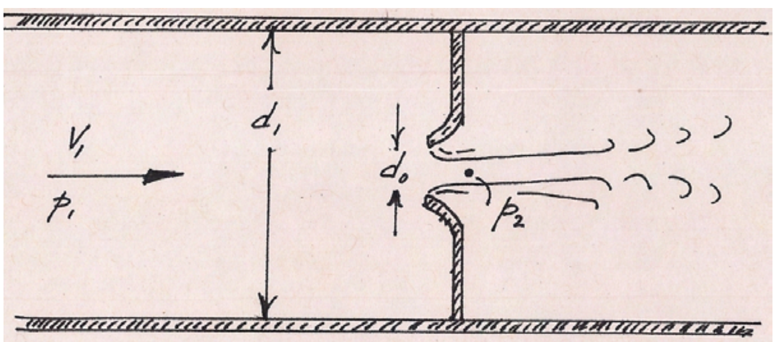

Figure 1: Flow through an inverted and submerged nozzle.

A test procedure would involve testing several orifices, each with a different throat diameter d_0. The pressure difference $p_1 - p_2$ is the value of interest for different velocities V_1, liquid density ρ, and different throat diameters d_0.

A reasonable inclination would be to make various measurements at different values of V_1, d_0, and ρ. The goal would be to plot results in a design chart, something similar to Figure 2

It would not take very long to realize that such measurements would involve a tremendous amount of work — so we need a better scheme[4]. A reasonable guess is that the Bernoulli

[3]The Boeing 757 and the Boeing Joint Strike Fighter entry were both entirely computer designed and tested. The testbeds for both these vehicles functioned, but not to the extent anticipated by the computer programs. The 757 testbed had engine failure at a certain angle of attack — had the machine gone into production before the testbed experiment, many would have crashed.

[4]The same issue is true if we are only compiling published data, there would be a huge amount of work to collapse the data into such dimensional charts.

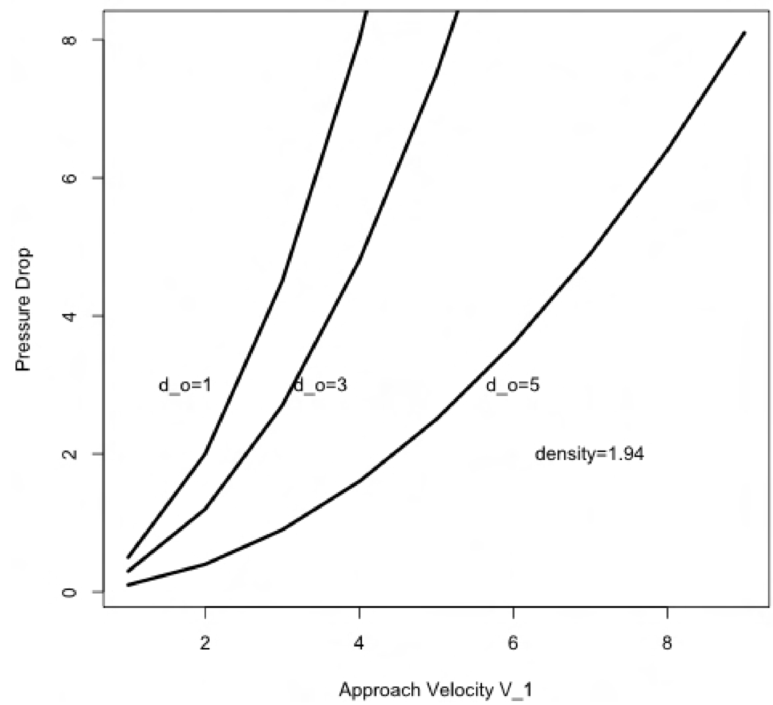

Figure 2: Relationship of Pressure Drop, Velocity, and Orifice Diameter for a particular liquid density.

equation might apply[5] — the conditions for use of the Bernoulli equation are that the liquid has low viscosity (like water), the streamlines converge in the direction of flow, and the flow is steady. These conditions certainly prevail in the region of interest in the example. If we knew the velocity in the nozzle we could solve directly for the pressure drop, but it is reasonable to assume separation will occur downstream of the throat, so we have no idea of the flow section diameter in the throat[6]. Thus one might determine the relationship between δp and the other variables experimentally[7]

If we analyze the Bernoulli equation and convert into dimensionless form (ratios where the dimensions cancel) we can insert coefficients for the unknowns and use these to guide our efforts. Equation 1 is the Bernoulli equation written for sections 1 and 2.

$$p_1 + \rho\frac{V_1^2}{2} = p_2 + \rho\frac{V_2^2}{2} \tag{1}$$

[5]or the energy equation variant.
[6]We do know that the throat section is smaller that the orifice diameter, but not how much.
[7]Oddly enough, it would be kind of hard to get a pressure measurement without disturbing the flow in the orifice, but V_2 could be measured by acoustic interferometery directly.

First isolate the pressure drop term.

$$p_1 - p_2 = \rho\frac{V_2^2}{2} - \rho\frac{V_1^2}{2} \tag{2}$$

Now divide both sides by the upstream velocity head.

$$\frac{p_1 - p_2}{\rho\frac{V_1^2}{2}} = \frac{\rho\frac{V_2^2}{2}}{\rho\frac{V_1^2}{2}} - \frac{\rho\frac{V_1^2}{2}}{\rho\frac{V_1^2}{2}} \tag{3}$$

Simplify the right hand side

$$\frac{p_1 - p_2}{\rho\frac{V_1^2}{2}} = \frac{V_2^2}{V_1^2} - 1 \tag{4}$$

We know from continuity that $\frac{V_2}{V_1} = \frac{A_1}{A_2}$ but we don't know A_2. We want to express the equation in terms of A_o anyway, because we can control the size of the orifice. Instead we invent a coefficient based on the following functional logic: $\frac{V_2}{V_1} = f(\frac{A_1}{A_0}) = f_1((\frac{d_1}{d_0})^2)$. In words we are stating that the velocity ratio is equal to a function of the area ratio. The area ratio is equal to a function of diameter ratio squared — this last statement is a known consequence of geometry.

From this statement Equation 4 is rewritten as

$$C_p = \frac{p_1 - p_2}{\rho\frac{V_1^2}{2}} = [f_1((\frac{d_1}{d_0})^2)]^2 - 1 \tag{5}$$

The right hand side is solely a function of the diameter ratio (the left side is renamed the pressure coefficient C_p), so the equation is yet again rewritten as

$$C_p = [f_2(\frac{d_1}{d_0})] \tag{6}$$

At this point in the analysis, we have no idea the exact form of the function but we now have related the diameter ratio — a thing we can measure, to the pressure coefficient, also comprised of measurable terms. If one plots the pressure coefficient versus the diameter ratio the many different curves collapse into a single curve, as in Figure 3.

Figure 3 contains the same information as Figure 2, but all existing on a single curve. If we work in the dimensionless relationship we have achieved a considerable savings in work. To collect data for Figure 2 would take about 20 experiments per curve for a total of 60

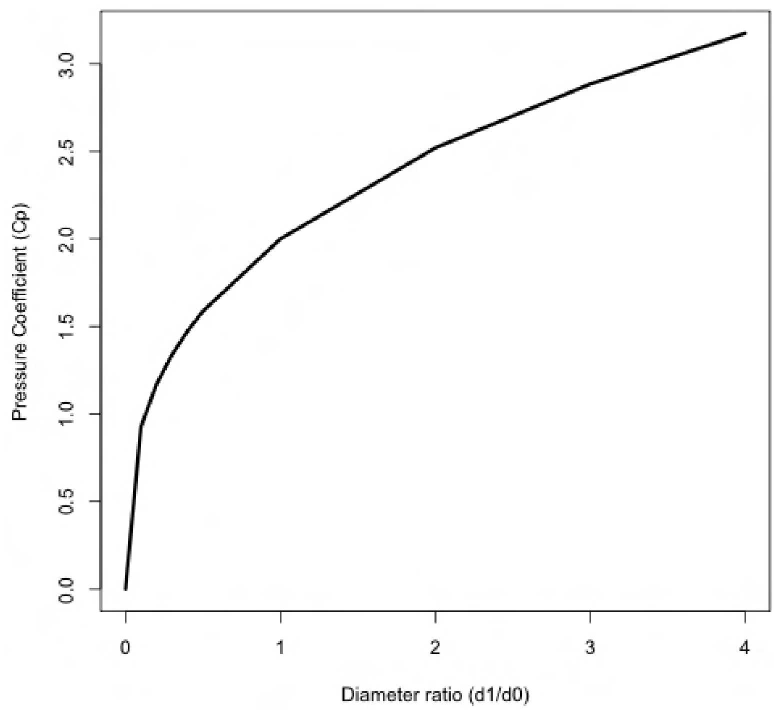

Figure 3: Relationship of Pressure Coefficient and Diameter Ratio.

experiments[8]. To collect data for Figure 3 would only require 20 experiments — after some kind of analysis to generate the interpolation between the points we have the same information from 20 experiments as we would have had for 60. Naturally we would want to vary the approach velocities in the experiments to make full use of the vertical range of the pressure coefficient graph (all the points clustered in three locations would be kind of pointless). The process of non-dimensionalizing the equation reduces the parameters from five $(p_1 - p_2, \rho, V_1, d_1, d_2)$ to two ($\frac{p_1-p_2}{\rho\frac{V_1^2}{2}}, \frac{d_1}{d_0}$).

In this example we had some idea about the governing equation, in many practical cases such knowledge is not available a-priori and we seek the structure by means of *dimensional analysis*.

[8]Typically 20 points are required to define an experimental curve. In many Civil Engineering studies we get by with far fewer, but for the sake of illustration assume we adhere to this common rule of thumb. The need for 20 experiments is not arbitrary; there is sound statistical reasoning behind the rule of thumb.

1.2 Dimensions and Equations

Variables in engineering are expressed in terms of a limited number of basic dimensions. For most engineering problems these dimensions are: mass, length, time, and temperature. The units of all other dimensional variables can be expressed in terms of these units. For example, force is the product of mass and acceleration, thus the units of force can be expressed in terms of M,L, and T as

$$[F] = M\frac{L}{T^2} \tag{7}$$

In this expression the brackets mean “dimension of.” Recall from chemistry the bracket is also used to express activities and concentrations. In most instances the meaning will be obvious from context. In words Equation 7 is read as: “ The dimensions of F are the product of mass and length divided by time squared.”

Some other common terms are viscosity which has units of newton seconds per square meter ($N * s/m^2$), so the dimensions of viscosity are:

$$[\mu] = \frac{FT}{L^2} = \frac{M}{LT} \tag{8}$$

Pressure (normal stress) is by definition the ratio of force to area so the dimensions of pressure are:

$$[p] = \frac{F}{A} = \frac{M}{LT^2} \tag{9}$$

All equations must balance in magnitude (be in correctly scaled units). In addition all rational equations[9] must also be dimensionally homogeneous. The left hand side of the equation must have the same dimensions as the right hand side.

1.3 The Buckingham Π Theorem

The number of independent dimensionless groups of variables needed to correlate the variables in a given process is equal to $n - m$, where n is the number of variables involved and m is the number of basic dimensions included in the variables. This relationship is called the Buckingham Π theorem (Buckingham, 1915).

[9]Those from the laws of physics — not the same meaning as in the rational equation of hydrology

The dimensionless parameters are called Π, and they are collected into groups called π groups. If an equation describing a physical system has n dimensional variables and is expressed as

$$y_1 = f(y_2, y_3, \ldots, y_n) \tag{10}$$

then it can be rearranged and expressed in terms of $n - m$ dimensionless groups as

$$\pi_1 = \Phi(\pi_2, \pi_3, \ldots, \pi_{n-m}) \tag{11}$$

Thus if the drag force F of a liquid flowing past a sphere is known to be a function of velocity V, density ρ, viscosity μ, and diameter d, then these five variables and three fundamental dimensions (M, L, T) are involved. We will have $5 - 3 = 2$ basic groupings of variables that can be used to correlate experimental results in the form of

$$\pi_1 = \Phi(\pi_2) \tag{12}$$

Subsequent essays will illustrate two methods to identify these groups: a step-by-step method and a matrix solution method. The two methods are equivalent in end result.

2 Readings

1. Read pages 260 – 267 in White (1979). A copy of the necessary pages is on the server.
2. Read pages 325 – 335 in Hwang and Hita (1987). A copy of the necessary pages is on the server.

3 Exercises

1. Determine which of the following equations are dimensionally homogeneous.

   (a)
   $$Q = \frac{2}{3} C L \sqrt{2g} H^{3/2} \tag{13}$$

   where Q is discharge, C is a pure number (scaling coefficient, no dimension), L is length, g is gravitational acceleration, and H is head.

(b)

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \tag{14}$$

where V is velocity, n is a roughness term (length to the one-sixth power), R is the hydraulic radius (length), and S is slope.

(c)

$$h_f = f \frac{L}{D} \frac{V^2}{2g} \tag{15}$$

where h_f is is head loss, f is a dimensionless friction coefficient, L is length, D is diameter, V is velocity, and g is gravitational acceleration.

(d)

$$D = \frac{0.074}{Re^{0.2}} \frac{Bx\rho V^2}{2} \tag{16}$$

where D is drag force, Re is a Reynolds number (Vx/ν), B is width, x is length, ρ is mass density, and V is velocity.

2. Determine the dimensions of the following variables and combinations of variables in terms of length, mass, and time. (Use Newton's second law to convert force into mass.)

   (a) T torque

   (b) $\rho V^2/2$, where ρ is mass density and V is velocity.

   (c) $\sqrt{\tau/\rho}$, where τ is shear stress.

   (d) Q/ND^3, where Q is discharge, D is diameter, and N is angular speed of a pump.

References

Buckingham, E. (1915). Model experiments and the forms of empirical equations. *Transactions of American Society of Mechancial Engineers* (No. 37), 263.

Hwang, N. H. C. and C. E. Hita (1987). *Fundamentals of Hydraulic Engineering Systems.* San Diego, CA: Prentice Hall.

White, F. M. (1979). *Fluid Mechanics.* New York, NY: McGraw-Hill.