SCRIPT

CONSIDER A CONDUIT WITH CROSS SECTION AREA, A.

VOLUME OF FLUID THAT PASSES THE AREA AT x IN TIME INTERVAL Δt IS

$$\Delta x A = \forall$$

FLOW RATE IS

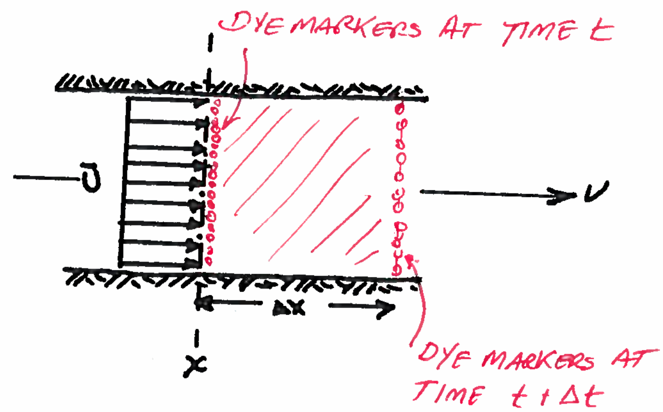
$$Q = \frac{\forall}{\Delta t} = \frac{\Delta x}{\Delta t} A$$

BOARD

CONTROL VOLUMES & CONTINUITY

VOLUMETRIC FLOW RATE

VOLUME OF FLUID CROSSING AN AREA PER UNIT OF TIME



SCRIPT

$$\frac{\Delta x}{\Delta t} = \bar{U} \text{ (IN DRAWING)}$$

$\bar{U}$ IS CALLED MEAN SECTION VELOCITY

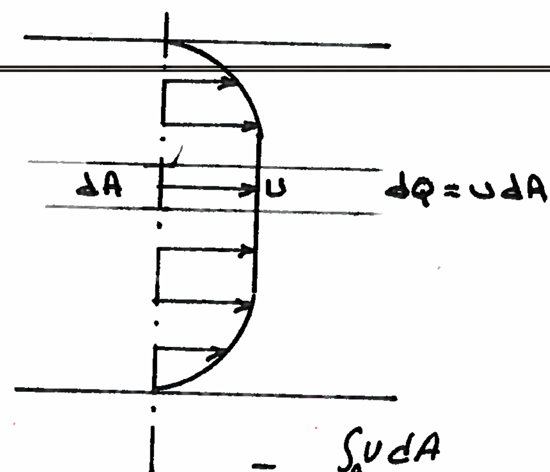
$$dQ = u\,dA$$

$$\int dQ = \int_A u\,dA$$

$$\bar{U} = \frac{\int_A u\,dA}{\int_A dA}$$

BOARD

IF VELOCITY VARIES ACROSS SECTION, THEN MEAN SECTION VELOCITY IS FOUND BY INTEGRATION



$$\bar{U} = \frac{\int_A u\,dA}{\int_A dA}$$

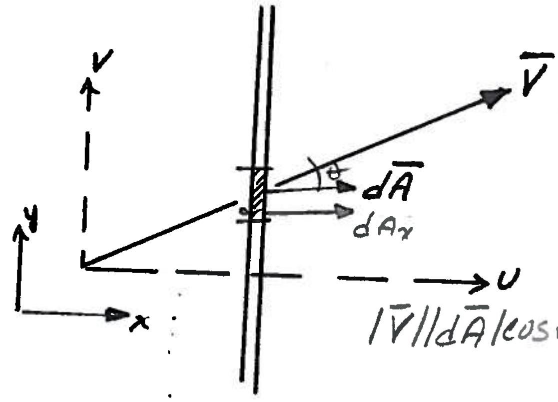
SCRIPT

FOR ARBITRARY ORIENTATION THE "INTEGRALS" ARE RESULT OF INNER PRODUCT OF VELOCITY VECTOR $\bar{V}$ AND AREA VECTOR $d\bar{A}$

SCALAR RESULT SHOWN IN PENCIL.

BOARD

OBSERVE THAT dA IS NORMAL TO U IN THIS DEFINITION



$|\bar{V}||d\bar{A}|\cos\theta = Q$

$$Q = \int_A \bar{V} \bullet d\bar{A} = \int_A u\,dA_x + \int_A v\,dA_y$$

SCRIPT

MASS OF FLUID THAT PASSES THE AREA AT x IN TIME INTERVAL Δt IS

$$\rho \Delta x A = \rho \forall$$

$$\frac{\rho \forall}{\Delta t} = \rho \frac{\Delta x}{\Delta t} A = \dot{m}$$

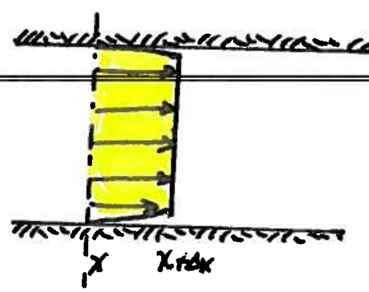
$$\frac{\Delta x}{\Delta t} \rightarrow \bar{U}$$
$\Delta t \to 0$

$$\dot{m} = \rho \bar{U} A$$

BOARD

MASS FLOW RATE

MASS OF FLUID CROSSING AN AREA PER UNIT OF TIME



NEARLY SAME EQUATION; DECIDEDLY THE SAME CONCEPT.

SCRIPT

THE "INTEGRALS" WILL BE CALLED THE "FLUX" INTEGRALS

IF HAVE TO PERFORM INTEGRATIONS, NEED TO CONSIDER HOW VELOCITY VARIES ACROSS SECTION

$$\bar{V}(d\bar{A}) \cdot d\bar{A}$$

REALLY WHAT'S GOING ON!

BOARD

AS WITH VOLUMETRIC FLOW RATE, IF VELOCITY VARIES THEN

$$\dot{m} = \int_A \rho \bar{V} \cdot d\bar{A}$$

KEY CONCEPTS:

$$Q = \int_A \bar{V} \cdot d\bar{A}$$ ← VECTOR INNER PRODUCT OF $\bar{V}$ & $d\bar{A}$

$$\dot{m} = \int_A \rho \bar{V} \cdot d\bar{A}$$

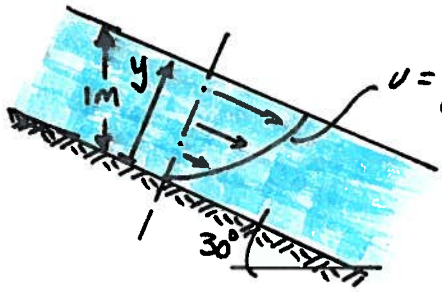
IF $\bar{V} \perp d\bar{A}$ THEN $\bar{V} \cdot d\bar{A} = V dA$

OTHERWISE NEED COMPONENTS

SCRIPT

BOARD

CHANNEL SHOWN IS 2m WIDE. WHAT IS VOLUMETRIC DISCHARGE?



KNOWN

$u(y) = y^{1/3}$ m/s

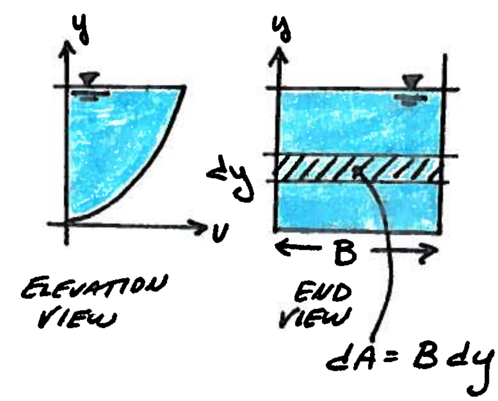
DISTANCE IN +Z AXIS 1m.

SLOPE 30°

UNKNOWN

Q

SOLUTION



1) DEPTH OF FLOW Y

$$y = 1\,m \cos 30° = 0.866m$$

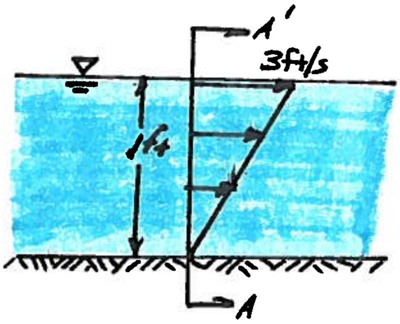
2) $Q = \int \vec{u}\, d\vec{A}$

$dA = B dy$

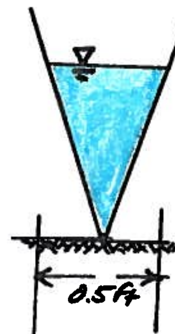
$$Q = \int_0^{0.866} y^{1/3} B\, dy = \frac{3}{4} y^{\frac{4}{3}} B \Big|_0^{0.866} = \left(\frac{3}{4}\right)(0.825)(2)$$

$$= 1.23 m^3/sec \leftarrow Q$$

SECTIONAL WATER VELOCITY IN V-CHANNEL VARIES LINEARLY WITH DEPTH FROM ZERO AT THE BOTTOM TO A MAXIMUM AT THE WATER SURFACE AS SHOWN. DETERMINE THE DISCHARGE IN THE CHANNEL



ELEVATION VIEW



SECTION A-A' END VIEW

KNOWN

$V(y)$

GEOMETRY

UNKNOWN

Q

GOVERNING EQUATION(S)

$$Q = \int_A \bar{v}\, d\bar{A}$$

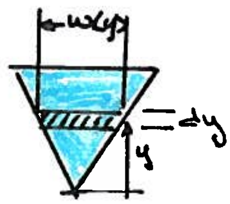
SOLUTION

① GEOMETRY

$@y=0,\ w=0$

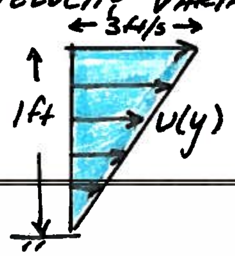
$@y=1,\ w=0.5$

$\therefore\ w(y) = 0.5y$



$$dA = w\,dy = 0.5y \cdot dy = \frac{y}{2}dy$$

② VELOCITY VARIATION



$@y=0,\ v=0$

$@y=1,\ v=3\text{ft/s}$

$$v(y) = \frac{3\text{ft}}{\text{s}} \cdot \frac{y}{1\text{ft}} = 3y$$

③ $$Q = \int_A v\,dA = \int_0^1 3y \cdot \frac{y}{2}dy = \int_0^1 \frac{3}{2}y^2 dy = \frac{3}{2} \cdot \frac{y^3}{3}\Big|_0^1$$

$$= \underline{\underline{\frac{1}{2}\ \text{ft}^3/\text{sec}}} \longleftarrow Q$$

SCRIPT

CONTROL VOLUME IS BASIS OF REYNOLD'S TRANSPORT THEOREM THAT ALLOWS ANALYSIS FROM EULERIAN REFERENCE FRAME RATHER THAN TRACKING INDIVIDUAL PARTICLES

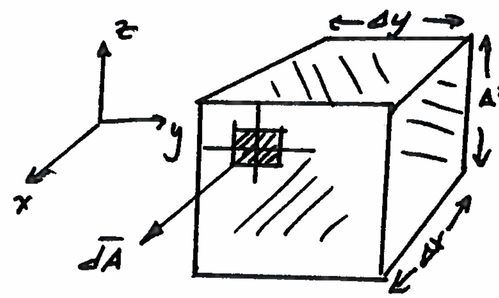
GOAL IS TO DESCRIBE FUNDAMENTAL LAWS OF MECHANICS IN INTEGRAL FORM

BOARD

CONTROL VOLUME ANALYSIS

A CONTROL VOLUME IS THE EQUIVALENT OF A FREE-BODY DIAGRAM IN DYNAMICS

CONTROL VOLUME IS SOME DEFINED AREA IN SPACE



THE BOUNDING SURFACE IS CALLED THE "CONTROL SURFACE"

SCRIPT

PRINCIPLE IDEA IS TO EXPRESS FOLLOWING IN INTEGRAL FORM

1) CONSERVATION OF MASS

$$\left.\frac{dm}{dt}\right|_{sys} = 0$$

2) CONSERVATION OF MOMENTUM

$$m\left.\frac{d\bar{V}}{dt}\right|_{sys} = \Sigma \bar{F}$$

3) CONSERVATION ANGULAR MOMENTUM

$$m\left.\frac{d\bar{\omega}}{dt}\right|_{sys} = \Sigma \bar{r} \times \bar{F}$$

BOARD

$d\bar{A}$ IS THE OUTWARD POINTING AREA VECTOR

$d\bar{A}_{BACK} = -\Delta y \Delta z \underline{i}$, $d\bar{A}_{FRONT} = \Delta y \Delta z \underline{i}$

$d\bar{A}_{LEFT} = -\Delta x \Delta z \underline{j}$, $d\bar{A}_{RIGHT} = \Delta x \Delta z \underline{j}$

$d\bar{A}_{BOT} = -\Delta x \Delta y \underline{k}$, $d\bar{A}_{TOP} = \Delta x \Delta y \underline{k}$

SCRIPT

4) CONSERVATION OF ENERGY

$$\frac{dQ}{dt} - \frac{dW}{dt} = \left.\frac{dE}{dt}\right|_{SYS}$$

Q - HEAT INTO.

W - WORK DONE BY.

5) ENTROPY

$$\left.\frac{dS}{dt}\right|_{SYS} \geq \frac{1}{T}\frac{dQ}{dt}$$

T - TEMP. ABS.

S - ENTROPY.

BOARD

TO USE C.V. ANALYSIS THE SYSTEM EQUATIONS ARE CONVERTED TO VOLUME VARIATION EQUATIONS

EXTENSIVE PROPERTY - THROUGHOUT ENTIRE <u>MASS</u> OF FLUID

INTENSIVE PROPERTY - AMOUNT OF PROPERTY PER UNIT MASS

START WITH MASS ITSELF

EXTENSIVE PROPERTY IS MASS m.

MASS PER UNIT MASS $\frac{m}{m} = 1$

MASS PER UNIT VOLUME $\frac{m}{V} = \rho$

SCRIPT

RECALL EXTENSIVE AND INTENSIVE PROPERTIES

BOARD

FUNDAMENTAL RELATIONSHIP IS

$$\underbrace{B_{SYS}}_{\text{EXTENSIVE PROPERTY}} = \int \underbrace{\beta}_{\text{INTENSIVE PROPERTY}} dm = \int \beta \rho \, dV$$

$$\int \underbrace{\beta}_{\frac{m}{M}} \underbrace{\rho}_{\frac{M}{V}} \, dV$$

$$\int \frac{m}{m} \cdot \frac{m}{V} \cdot dV = m!$$

SCRIPT

CONSIDER THE CONTROL VOLUME SHOWN,

THE MASS m IS MOVING IN A VELOCITY FIELD $\bar{V}=\bar{V}(x,y,z,t)$

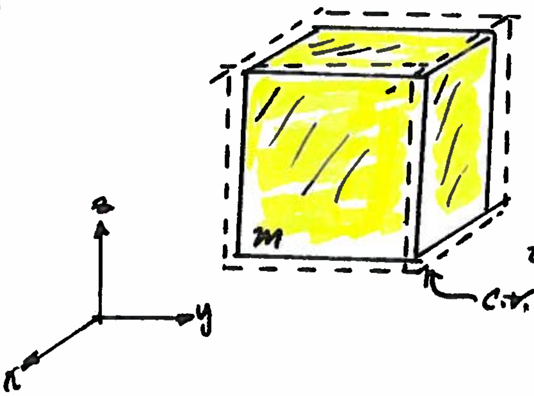
AT TIME t THE MASS IS COMPLETELY ENCLOSED BY THE C.∀.

BOARD

IN OUR CONSERVATION OF MASS CASE

$$\frac{dB}{dt}\bigg|_{sys} = \frac{d}{dt}\left(\int \beta \rho \, d\forall\right)$$

USE REYNOLDS TRANSPORT THEOREM TO CHANGE THE R.H.S INTO VOLUME BASED TERMS.

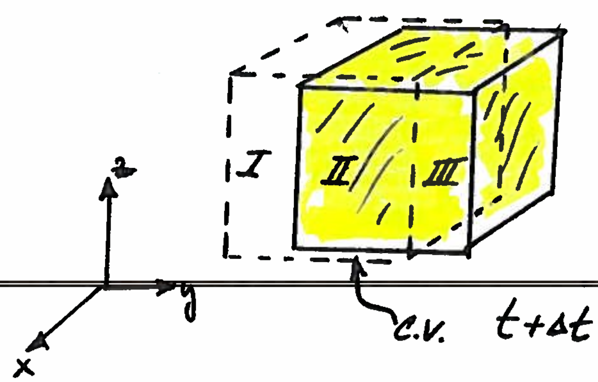


SCRIPT

AT TIME $t+\Delta t$ SOME MASS HAS LEFT THE CONTROL VOLUME.

DEFINITION OF DERIVATIVE AS LIMIT OF DIFFERENCE QUOTIENT PROVIDES A GUIDE

BOARD



$$\frac{dB}{dt}\bigg|_{system} = \lim_{\Delta t \to 0} \frac{B_{t+\Delta t} - B_t}{\Delta t}$$

SCRIPT

NOW CONSIDER THE VARIOUS PARTS TO TRACK MASS FROM t TO $t+\Delta t$

BOARD

FROM THE SKETCH

$$B_t = B_{c.v._t}$$

$$B_{t+\Delta t} = (B_{II} + B_{III})_{t+\Delta t}$$
$$= (B_{c.v.} - B_I + B_{III})_{t+\Delta t}$$

NOW IN TERMS OF INTENSIVE PROPERTIES

$$\frac{dB}{dt} = \lim_{\Delta t \to 0} \frac{\int_{cv} \beta\rho d\forall\big|_{t+\Delta t} - \int_I \beta\rho d\forall\big|_{t+\Delta t} + \int_{III} \beta\rho d\forall\big|_{t+\Delta t} - \int_{cv} \beta\rho d\forall\big|_t}{\Delta t}$$

SCRIPT

THE FIRST TERM IS THE TIME RATE OF CHANGE OF β IN THE CONTROL VOLUME.

IT IS REFERRED TO AS THE "VOLUME INTEGRAL"

BOARD

$$\frac{dB}{dt}\Big|_{sys} = \left.\lim_{\Delta t \to 0} \frac{\int_{cv} \beta\rho d\forall\big|_{t+\Delta t} - \int_{cv} \beta\rho d\forall\big|_t}{\Delta t}\right\} \frac{d}{dt}\int_{cv} \beta\rho d\forall$$

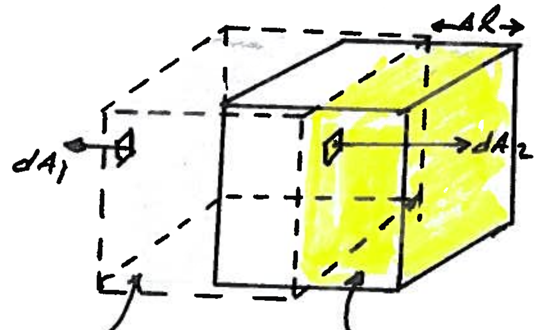
$$+ \lim_{\Delta t \to 0} \frac{\int_{III} \beta\rho d\forall\big|_{t+\Delta t}}{\Delta t}$$

$$- \lim_{\Delta t \to 0} \frac{\int_{I} \beta\rho d\forall\big|_{t+\Delta t}}{\Delta t}$$

SCRIPT

WE NEXT APPLY THE DIVERGENCE THEOREM TO RELATE THE FLUXES OF β ACROSS THE FACES OF THE C.S.

BOARD



$$\underbrace{-\int_{I} \beta\varphi d\forall \Big/_{t+\Delta t} = -\int_{C.S.} \beta\varphi \Delta l \, dA_1}_{\text{Gauss' Divergence Theorem}} \qquad \underbrace{\int_{III} \beta\varphi d\forall \Big/_{t+\Delta t} = \int_{C.S.} \beta\varphi \Delta l \, dA_2}_{\text{Gauss' Divergence Theorem}}$$

SCRIPT

IN THE LIMIT $\frac{\Delta l}{\Delta t}$ IS JUST THE FLUID VELOCITY ACROSS THE FACE WITH dA_2 NORMAL VECTOR

BOARD

$$\lim_{\Delta t \to 0} \frac{\int_{III} \beta\varphi d\forall \Big/_{t+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \int_{C.S.} \beta\varphi \frac{\Delta l}{\Delta t} dA_2$$

$$= \int_{C.S.} \beta\varphi V_2 \, dA_2$$

SIMILARILY FOR THE I FACE

$$\lim_{\Delta t \to 0} \frac{-\int_{I} \beta\varphi d\forall \Big/_{t+\Delta t}}{\Delta t} = -\int_{C.S.} \beta\varphi V_1 \, dA_1$$

NOW EXAMINE RELATIONSHIP BETWEEN dA & V

SCRIPT

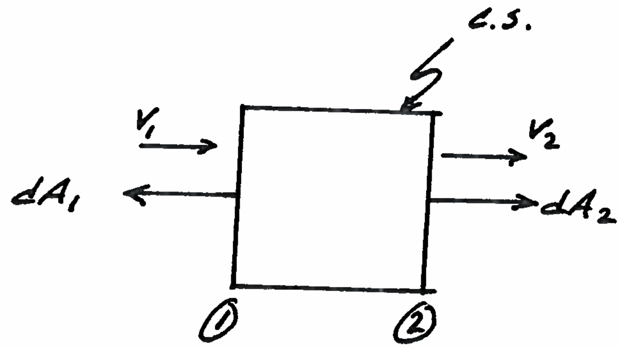
V_1 & dA_1 ARE IN OPPOSITE DIRECTIONS.

V_2 & dA_2 ARE IN SAME DIRECTION

V_1 & V_2 ARE IN SAME DIRECTION

IN TERMS OF VECTOR CALCULUS WE EXPRESS THIS TYPE OF RELATIONSHIP AS:

BOARD



$$-V_1\,dA_1 = \underline{V}\cdot \underline{dA}\Big|_{1}$$

$$V_2\,dA_2 = \underline{V}\cdot \underline{dA}\Big|_{2}$$

SCRIPT

COLLECT THESE TERMS INTO THE ORIGINAL EQUATION AND WE HAVE THE REYNOLDS TRANSPORT THEOREM.

BOARD

$$\frac{dB}{dt}\Big|_{SYS.} = \frac{d}{dt}\int_{CV} \beta\rho\, d\forall + \int_{C.S.} \beta\rho(\underline{V}\cdot \underline{dA})$$

SO FOR CONSERVATION OF MASS

$$B = M,\quad \beta = \frac{M}{M} = 1$$

$$\frac{dM}{dt}\Big|_{SYS} = 0 = \frac{d}{dt}\int_{CV} \rho\, d\forall + \int_{C.S.} \rho(\underline{V}\cdot \underline{dA})$$

SCRIPT

CONSERVATION OF MASS STATES THAT THE RATE OF ACCUMULATION IS BALANCED BY THE NET INFLOW.

BOARD

CONSERVATION OF MASS

$$\underbrace{\frac{d}{dt}\int_{CV} \rho \, d\forall}_{\text{RATE OF ACCUMULATION OF MASS IN CONTROL VOLUME}} + \underbrace{\int_{C.S.} \rho(\underline{V}\cdot \underline{dA})}_{\text{RATE OF NET MASS INFLOW}} = 0$$

RATE OF ACCUMULATION OF MASS IN CONTROL VOLUME

RATE OF NET MASS INFLOW ($\dot{\text{INFLOW}}$ + $\dot{\text{OUTFLOW}}$)

⇕ ⇕

RATE OF CHANGE OF STORAGE + $\dot{\text{OUTFLOW}}$ − $\dot{\text{INFLOW}}$ = 0

OR

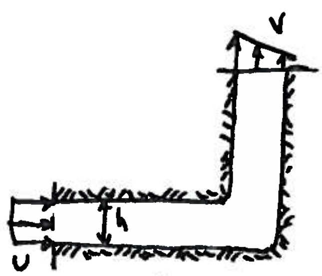
$\dot{\text{INFLOW}}$ − $\dot{\text{OUTFLOW}}$ = RATE CHANGE OF STORAGE

SCRIPT

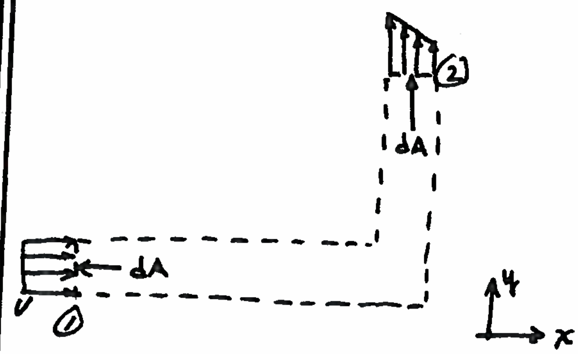
BOARD

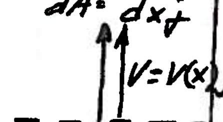
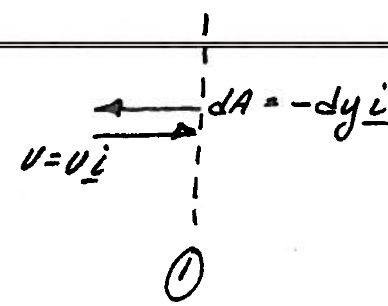
EXAMPLE

WATER ENTERS A 2D-CHANNEL OF CONSTANT CROSS SECTION, ~~H~~=WIDTH, ~~H~~=DEPTH, AND UNIFORM VELOCITY U. CHANNEL MAKES A 90° BEND THAT DISTORTS THE FLOW TO PRODUCE THE LINEAR PROFILE AT THE EXIT WITH $V_{MAX} = 2V_{MIN}$. FIND V_{MIN}



SKETCH

SCRIPT	BOARD
DEFINE THE C.V. INCLUDE A COORDINATE SYSTEM - NEEDED TO WRITE THE INTEGRALS. ASSUMED UNIT DEPTH	DEFINE THE C.V. 

SCRIPT	BOARD
APPLY CONTINUITY	$\frac{d}{dt}\int_{CV} \varphi \, d\forall + \int_{C.S.} \varphi \, \underline{V} \cdot d\underline{A} = 0$
	$= 0$ φ = CONSTANT $\frac{d\forall}{dt} = 0$
	
ANALYZE THE C.S. TO FIGURE OUT THE FLUX INTEGRALS	
	$\int_{C.S.} \varphi \underline{V} \cdot d\underline{A} = \int_{1} \varphi (V \cdot dA) + \int_{2} \varphi (V \cdot dA)$

SCRIPT

BOARD

$$\int_{①} \rho(V\cdot dA) = \int_{①} \rho\, u\,\underline{i}\,(-dy\,\underline{i}) = -\rho\int_0^h u\,dy$$

$$\int_{②} \rho(V\cdot dA) = \int_{②} \rho\,(v(x)\,j \cdot dx\, j)$$

$$= \rho\int_0^h v(x)\,dx$$

$$v(x) = 2U_{min} - \frac{U_{min}\,x}{h} \quad \Big\}\text{given in prob. statement}$$

$$\int_{①} + \int_{②} = -\rho u h + \rho\int_0^h 2U_{min} - \frac{U_{min}}{h}x\,dx$$

$$= -\rho u h + \rho 2U_{min} h - \frac{\rho U_{min} h^2}{2h} = 0$$

SCRIPT

BOARD

$$\therefore -U + 2U_{min} - \frac{U_{min}}{2} = 0$$

$$U_{min} = \frac{2U}{3}$$

IMPORTANT CONCEPTS

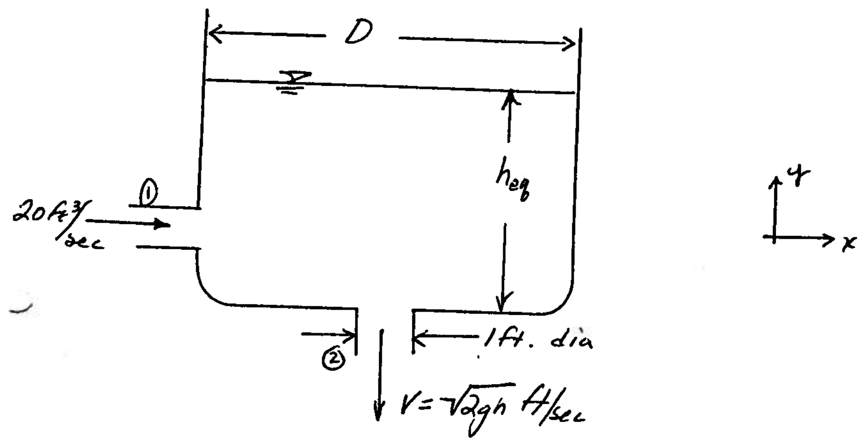
i) DRAW THE c.v.

ii) DRAW THE $\underline{dA}$; WRITE AS VECTOR COMPONENTS IN CORRECT DIRECTION

iii) $V\cdot dA$: FIGURE THE INNER PRODUCTS.

iv) IF $\frac{d\forall}{dt} = 0$ AND ρ = CONSTANT, THE VOLUME INTEGRAL VANISHES.

The open tank shown has a constant inflow of 20ft³/sec. A 1.0 ft diameter drain provides a variable outflow velocity $V_{out} = \sqrt{2gh}$ ft/sec. What is equilibrium depth in the tank?

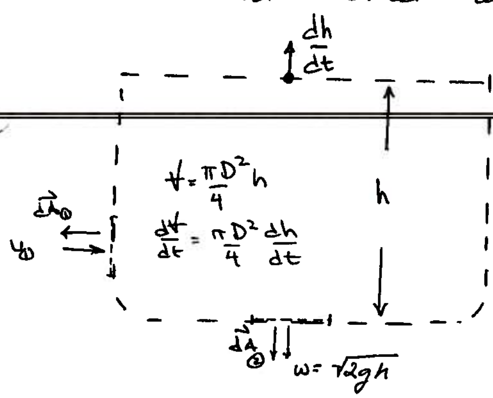


Continunity

$$0 = \frac{d}{dt}\int_{c.v.} \rho \, d\forall + \int_{c.s.} \rho \, (\vec{v} \cdot d\vec{A})$$

Control volume = Volume of water in tank at any time.

Control Surface = bounding surface, includes inlet at ① and outlet at ②



$$\frac{d}{dt}\int_{c.v.} \rho \, d\forall = \rho \frac{\pi D^2}{4}\frac{dh}{dt}$$

$$\int_{c.s. ①} \rho(\vec{v} \cdot d\vec{A}) = -u_① A_① = -20\,ft^3/sec$$

$$\int_{c.s. ②} \rho(\vec{v} \cdot d\vec{A}) = w_② A_② = \sqrt{2gh}\,\frac{\pi d^2}{4}$$

$\therefore$

$$0 = \rho \frac{\pi D^2}{4}\frac{dh}{dt} - 20\,ft^3/sec + \sqrt{2gh}\,\frac{\pi (1)^2}{4}$$

At equilibrium $\frac{dh}{dt} = 0$

$$\therefore \sqrt{2gh}\,\frac{\pi(1)^2}{4} = 20\,ft^3/sec$$

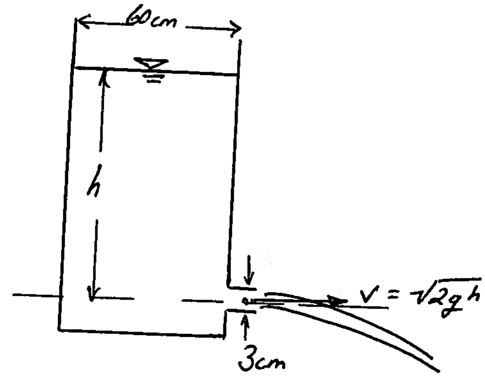
$$\sqrt{2gh} = \left[\left(\frac{20\,ft^3}{sec}\right)\frac{4}{\pi(1ft)^2}\right]^2$$

$$h = \left[\frac{(20)(4)}{\pi(1)^2}\right]^2 \Big/ 2(32.2)$$

$$\underline{\underline{= 10.1\,ft}}$$

4.82

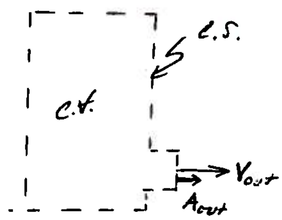
How long for the tank surface to drop from h=3m to h=30cm?



Solution

let c.v. = Volume of water in tank at any time, t.

$\forall = \frac{\pi D^2}{4} h.$ let c.s. be the bounding surface including the outlet pipe



Continuity

$$0 = \frac{d}{dt}\int_{c.v.} \rho \, d\forall + \int_{c.s.} \rho (\vec{v} \cdot d\vec{A})$$

$$\frac{d}{dt}\int_{c.v.} \rho \, d\forall = \rho \frac{d}{dt}\left(\frac{\pi D^2}{4} h\right) = \rho \frac{dh}{dt}\left(\frac{\pi D^2}{4}\right)$$

$$\int_{c.s.} \rho (\vec{v} \cdot d\vec{A}) = \rho (V_{out})(A_{out}) = \rho \sqrt{2gh} \, \frac{\pi d^2}{4}$$

$$\therefore \cancel{\rho} \frac{dh}{dt}\left(\frac{\cancel{\pi} D^2}{\cancel{4}}\right) = -\cancel{\rho} \sqrt{2gh} \, \frac{\cancel{\pi} d^2}{\cancel{4}}$$

$$\frac{dh}{dt} = -\sqrt{2g} \, \frac{d^2}{D^2} \sqrt{h}$$ Separate & Integrate

$$\frac{dh}{\sqrt{h}} = -\sqrt{2g}\frac{d^2}{D^2}dt$$

$$\int h^{-1/2}dh = \int -\sqrt{2g}\frac{d^2}{D^2}dt$$

$$2h^{1/2} = -\sqrt{2g}\frac{d^2}{D^2}t + C$$

$@\ h = h_0 \quad C = 2\sqrt{h_0}$

Evaluate constant of integration

Now solve for time

$$\frac{2h^{1/2} - 2\sqrt{h_0}}{-\sqrt{2g}\frac{d^2}{D^2}} = t$$

Substitute in numerical values

$$\frac{2(0.3)^{1/2} - 2(3)^{1/2}}{-(2(9.8))^{1/2}\left(\frac{0.03}{0.6}\right)^2} = t$$

$= 214\ sec.$

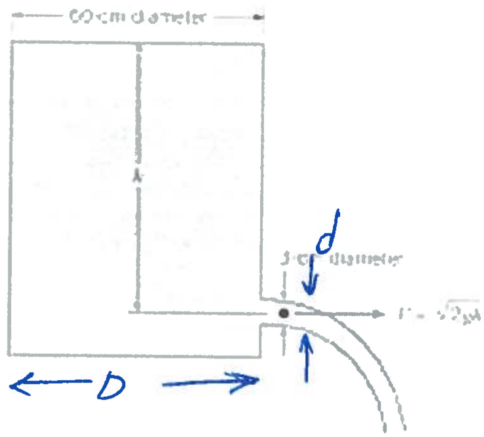
userone

CIVE 3434 Fluid Mechanics and Hydraulics
Fall 2005

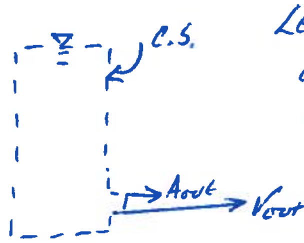
UNIVERSITY OF HOUSTON
9/19/2005

Exercise_04_02

Find the time for the tank to drain from $h=3$ meters to $h=0.5$ meters.



let C.V. = Volume of water in tank at any time t



Let C.S. be bounding surface including outlet pipe; location of free surface will move.

Continuity

$$0 = \frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho V \cdot dA$$

$$\frac{d}{dt}\int_{CV} \rho \, dV = \rho \frac{d}{dt}\left(\frac{\pi}{4}D^2 h\right) = \rho \frac{dh}{dt}\left(\frac{\pi D^2}{4}\right)$$

$$\int_{CS} \rho V \cdot dA = \rho (V_{out})(A_{out}) = \rho \sqrt{2gh}\, \frac{\pi d^2}{4}$$

$\therefore$

$$0 = \cancel{\rho}\frac{dh}{dt}\left(\frac{\pi D^2}{4}\right) + \cancel{\rho}\sqrt{2gh}\,\frac{\pi d^2}{4}$$

$$-\frac{dh}{dt} = \left(\frac{\cancel{\pi} d^2}{\cancel{4}}\right)\left(\frac{\cancel{4}}{\cancel{\pi} D^2}\right)\sqrt{2gh}$$

$$\frac{dh}{dt} = -\left(\frac{d}{D}\right)^2 \sqrt{2gh}$$

Separate

$$\frac{dh}{\sqrt{h}} = -\left(\frac{d}{D}\right)^2 \sqrt{2g}\, dt$$

Integrate.

$$\int \frac{dh}{\sqrt{h}} = -\left(\frac{d}{D}\right)^2 \sqrt{2g} \int dt$$

$$\int h^{-1/2} dh = 2h^{1/2} = -\left(\frac{d}{D}\right)^2 \sqrt{2g}\, t + C$$

Evaluate C at $t=0$, $h = h_0$ (known value)

$$2h_0^{1/2} = C$$

Now solve for time

$$\frac{2h^{1/2} - 2h_0^{1/2}}{-\left(\frac{d}{D}\right)^2 \sqrt{2g}} = t$$

Insert numerical values

$$\frac{2(0.5)^{1/2} - 2(3)^{1/2}}{-\left(2(9.8)\right)^{1/2}\left(\frac{0.03}{0.60}\right)} = t = 185 \text{ sec.}$$

CIVE 3434 Fluid Mechanics and Hydraulics
Fall 2005

Exercise_04_03

A tank containing oil is to be pressurized to decrease the draining time. The tank shown in the figure is 2 meters in diameter and 6 meters high (the top of the tank is closed). The oil is originally at a level of 5 meters. The oil has density 880 kg/m^3. The outlet port has a diameter of 2 cm and the outlet velocity is given by,

$$V_e = \sqrt{2gh + \frac{2p}{\rho}}$$

Where p is the gage pressure in the tank, ρ is the density of the oil, and h is the elevation of the surface above the hole. Assume that the temperature of the air in the tank remains constant (isothermal). The pressure in the air is given by

$$p = (p_o + p_{atm})\frac{(L-h_o)}{(L-h)} - p_{atm}$$

Where L is the height of the tank, p_{atm} is the atmospheric pressure, and the subscript o refers to the starting conditions. The initial pressure in the tank is 300 kPa and the atmospheric pressure is 100 kPa.

Applying continuity, the drainage equation is

$$\frac{dh}{dt} = -\frac{A_e}{A_T}\sqrt{2gh + \frac{2p}{\rho}}$$

Where A_e is the area of the outlet port and A_T is the area of the tank.Integrate this equation numerically to predict the oil depth with time (for 1-minute time intervals) for one hour of draining.

$$\frac{dh}{dt} = -\frac{A_e}{A_T}\sqrt{2gh + \frac{2p}{\varphi}} \leftarrow p = (p_0 + p_{atm})\frac{(L-h_0)}{(L-h)} - p_{atm}$$

$$\frac{dh}{dt} = -\frac{A_e}{A_T}\sqrt{2gh + \frac{2}{\varphi}\left[(p_0+p_{atm})\frac{(L-h_0)}{(L-h)} - p_{atm}\right]}$$

h = unknown, rest are constants

Recall from calculus

$$\frac{dh}{dt} \approx \frac{h(t+\Delta t) - h(t)}{\Delta t} \quad \left(\text{or } \frac{dh}{dt} = \lim_{\Delta t \to 0} \frac{h(t+\Delta t)-h(t)}{\Delta t}\right)$$

$$\therefore \underbrace{h(t+\Delta t) \approx h(t) + \frac{dh}{dt}\Delta t}$$

Use this update formula as basis of numerical approximation

<u>Known</u>

$h(t=0) = 5m$

$g = 9.8 m/s^2$

Note

$$\frac{A_e}{A_t} = \frac{\pi(0.02)^2}{4} \cdot \frac{4}{\pi(2.0)^2} = \frac{d^2}{D^2}$$

$A_e = \frac{(0.02m)^2}{4}\pi$

$A_T = \frac{(2.0m)^2}{4}\pi$

$p_0 = 300 kPa$

$p_{atm} = 100 kPa$

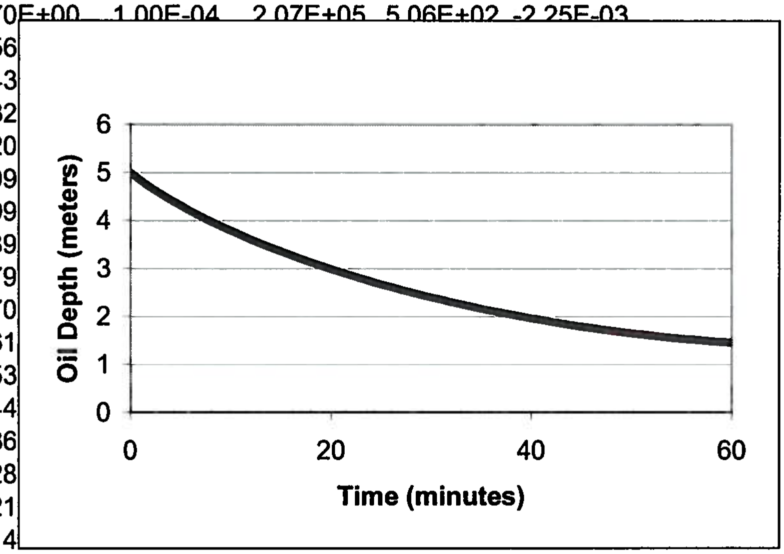
$L = 6m$

– spreadsheet using $\Delta t = 60$ sec attached.

#_Worksheet for CIVE 3434 Tank Drain Problem
#_Input

h(t=0)	5	meters	initial depth
De	0.02	meters	outlet diameter
Dt	2	meters	tank diameter
p(0)	3.00E+05	Pa	initial pressure
patm	1.00E+05	Pa	atmospheric pressure
g	9.8	meters/sec/:	gravitational acceleration constant
L	6	meters	tank height
r	1000	kg/meter^3	oil density

Time (sec)	Time(min)	Depth (h)	(De/Dt)^2	p(t)	2gh+2p/rhc	dH/dt
0	0	5	1.00E-04	3.00E+05	6.98E+02	-2.64E-03
60	1	4.84E+00	1.00E-04	2.45E+05	5.85E+02	-2.42E-03
120	2	4.70E+00	1.00E-04	2.07E+05	5.06E+02	-2.25E-03
180	3	4.56				
240	4	4.43				
300	5	4.32				
360	6	4.20				
420	7	4.09				
480	8	3.99				
540	9	3.89				
600	10	3.79				
660	11	3.70				
720	12	3.61				
780	13	3.53				
840	14	3.44				
900	15	3.36				
960	16	3.28				
1020	17	3.21				
1080	18	3.14				
1140	19	3.06E+00	1.00E-04	3.62E+04	1.33E+02	-1.15E-03



EX_04_04

Repeat exercise 04_02 using the same kind of numerical model you developed for 04_03 and confirm that the analytical solution and the numerical solution are close.

From 04_02

$$\frac{dh}{dt} = -\left(\frac{d}{D}\right)^2 \sqrt{2gh}$$

$$\lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} = \frac{dh}{dt}$$

$$\therefore \frac{h(t+\Delta t) - h(t)}{\Delta t} \approx \frac{dh}{dt} = -\left(\frac{d}{D}\right)^2 \sqrt{2gh}$$

for small Δt

Thus

$$\underbrace{h(t+\Delta t) = h(t) + \Delta t\left[-\left(\frac{d}{D}\right)^2 \sqrt{2gh(t)}\right]}$$

Use this update equation as basis for numerical model

Known

$d = 0.03m$

$D = 0.6m$ find t until $h(t) = 0.5m$

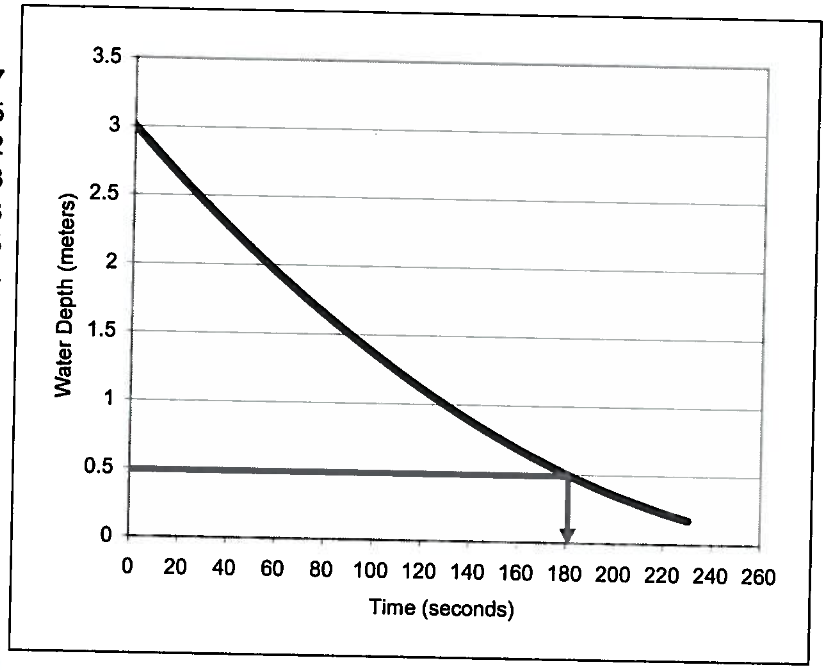
$h_0 = 3m$

#_Worksheet for CIVE 3434 time to drain

#_Input

d	0.03	outlet diameter
D	0.6	tank diameter
g	9.8	gravitational acceleration constant
h(0)	3	initial tank depth

Time(sec)	Depth (meters)	dH/dt
0	3	-0.01917
10	2.808297105	-0.01855
20	2.622820315	-0.01792
30	2.443573143	-0.0173
40	2.270559354	-0.01668
50	2.103783001	-0.01605
60	1.943248451	-0.01543
70	1.788960429	-0.0148
80	1.640924065	-0.01418
90	1.499144941	-0.01355
100	1.363629165	-0.01292
110	1.234383441	-0.0123
120	1.111415164	-0.01167
130	0.994732536	-0.01104
140	0.884344703	-0.01041
150	0.780261935	-0.00978
160	0.682495843	-0.00914
170	0.591059668	-0.00851
180	0.505968653	-0.00787
190	0.427240538	-0.00723
200	0.354896246	-0.00659
210	0.288960831	-0.00595
220	0.229464863	-0.0053
230	0.176446521	-0.00465



little larger than 180 secs.

Analytical solution was 185 seconds

~~Numerical & Analytical~~

Numerical

t= 185, depth = $\frac{0.50596+0.4272}{2} = 0.46$

~~are~~ ~~within~~ $\frac{5}{185} \times 100\% = 2.7\%$

Results are within 3% of each other.

(i.e. Relative error

$$\frac{\text{Numerical} - \text{Analytical}}{\text{Analytical}} \times 100\%$$