SCRIPT

~~THE NE~~
THIS MODULE INTRODUCES BERNOULLI EQUATION WHICH IS FUNDAMENTAL IN APPLIED FLUID MECHANICS.

THE MODULE ALSO EXAMINES LAGRANGIAN AND EULERIAN REFERENCE FRAMES IN CONTEXT OF VARIATIONAL CALCULUS AS APPLIED TO FLUIDS.

(START WITH FLOW PATTERNS

BOARD

EULER'S EQUATION
BERNOULLI EQUATION & PRESSURE VARIATION

FLOW PATTERNS & VISUALIZATION

IN REAL FLUIDS, MARKERS SUCH AS DYE; SMOKE; HEAT ARE USED TO SEE (VISUALIZE) HOW THINGS FLOW.

THE MARKERS ARE TRACERS; TRACER HYPOTHESIS IS THAT THE TRACER MOVES WITH THE HOST FLUID.

SCRIPT

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TIMELINE IS A LINE FORMED BY MARKING ADJACENT PARTICLES AT SOME INSTANT

PATHLINE IS THE TRAJECTORY OF A PARTICULAR FLUID PARTICLE

STREAKLINE IS THE TRAJECTORY OF MANY PARTICLES THAT ALL PASS THROUGH A COMMON POINT IN SPACE

STREAMLINE IS A LINE IN A FLOW FIELD THAT IS TANGENT TO VELOCITY. NO FLOW CROSSES A STREAMLINE

(3D EQUIVALENT IS CALLED A STREAMTUBE)

SCRIPT

STEADY FLOW,

STREAMLINES, STREAKLINES & PATHLINES ARE COINCIDENT.

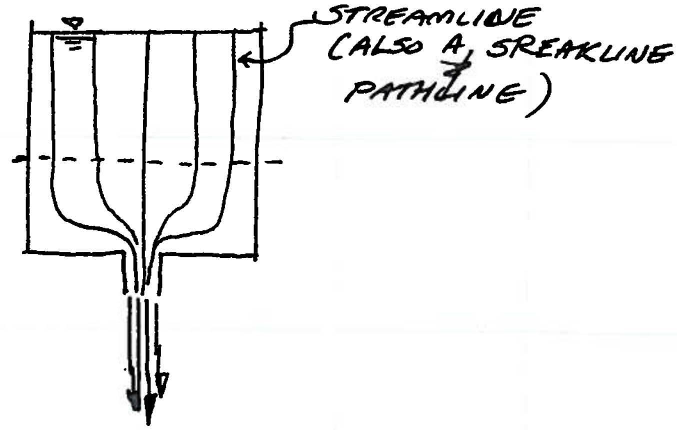
UNSTEADY FLOW,

LOOK DIFFERENT

ASK: IS THE FLOW FIELD IN SKETCH UNIFORM OR NON-UNIFORM?

HOW ABOUT ABOVE DASHED LINE?

BOARD



UNIFORM FLOW IS A FLOW FIELD WHERE VELOCITY DOES NOT CHANGE ALONG A STREAMLINE (VELOCITY DOES NOT VARY WITH POSITION)

NON-UNIFORM FLOW IS A FLOW FIELD WHERE VELOCITY DOES VARY WITH POSITION

SCRIPT

FLOW DIMENSIONS ARE CLASSIFIED BY HOW MANY SPACE COORDINATES ARE REQUIRED TO SPECIFY THE VELOCITY FIELD.

ALL REAL FLOWS ARE 3D.

USEFUL ENGINEERING ANALYSIS IS POSSIBLE WITH 2D AND 1D APPROXIMATIONS.

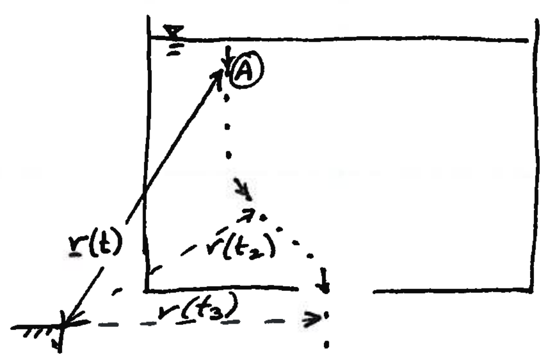
DISCUSSION OF P121 OF DIMENSIONALITY

BOARD

VELOCITY FIELD

Lagrangian – CONSIDER AN INDIVIDUAL FLUID PARTICLE

Eulerian – CONSIDER A POINT IN SPACE



$$\underline{V}_i = \frac{d\underline{r}_i}{dt}$$ (Lagrangian defn.)

SCRIPT

FIELD FROM THE MATHEMATICAL DESCRIPTION OF A FIELD.

VARIOUS INTEGRAL RULES APPLY TO FIELDS, AND THESE ARE USED LATER.

MAGNETIC FIELDS
ELECTRIC FIELDS
GRAVITATIONAL FIELDS

ARE EXAMPLES OF OTHER "THINGS" THAT ARE FIELDS.

IDEA IS THAT IF YOU KNOW LOCATION, YOU KNOW BEHAVIOR.

BOARD

THE VELOCITY OF EACH PARTICLE ALL EXPRESSED AT ONCE IS CALLED THE <u>VELOCITY FIELD</u>

HENCE

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$$

IS A PARAMETRIC POSITION VECTOR

$$\underline{V} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$$

IS THE VELOCITY FIELD

(ALSO POSSIBLY PARAMETRIC IN TIME AND SPACE)

$$u(t) = \frac{dx}{dt}\Big|_{x,y,z,t} \qquad v(t) = \frac{dy}{dt}\Big|_{x,y,z,t} \qquad w(t) = \frac{dz}{dt}\Big|_{x,y,z,t}$$

SCRIPT

MORE ON LAGRANGIAN & EULERIAN REFERENCE FRAMES

PG 116-117 DO A NICE JOB COMPARING THE TWO REFERENCE FRAMES.

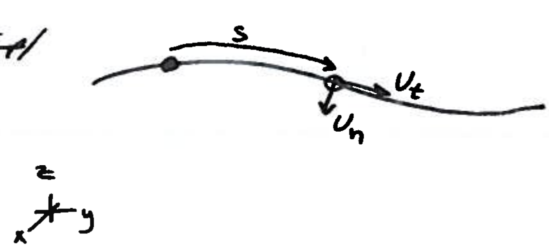
~~EULERIAN IS SOMETIMES CALLED INERTIAL REFERENCE FRAME, ALTHOUGH THIS IS NOT STRICTLY TRUE.~~

BOARD

VELOCITY FIELDS ARE REPRESENTED IN CARTESIAN COORDINATES OR ~~STREAM~~ PATHLINE COORDINATES

$$\underline{V} = u(x,y,z,t)\underline{i} + v(x,y,z,t)\underline{j} + w(x,y,z,t)\underline{k}$$

OR



TANGENTIAL & NORMAL NEED TO KNOW THE PATH. USUALLY OBVIOUS WHEN TO USE THE τ, η SYSTEM INSTEAD OF x-y.

SCRIPT

UNIFORM FLOW OCCURS IN THINGS WITH RELATIVELY STRAIGHT PATHLINES (STREAMLINES), AND CONSTANT CROSS SECTION GEOMETRIES.

PIPES –
CHANNELS OF CONSTANT – GEOMETRY
AQUIFERS (LIKE SAND FILLED PIPES)

IF GEOMETRY CHANGES LIKE IN A NOZZLE, NON-UNIFORM FLOW.

VELOCITY CHANGES (CONVECTIVE ACCELERATION)

BOARD

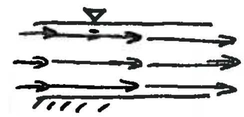
UNIFORM FLOW

$$\left.\frac{\partial V}{\partial s}\right|_{r} = 0$$

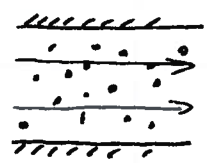
ALONG THE PATH NO CHANGE IN $\underline{V}$

EXAMPLES

UNIFORM IN PIPE



UNIFORM OPEN CHANNEL



UNIFORM FLOW IN AQUIFER

SCRIPT

LAMINAR FLOW IS WHERE FLUIDS IN ADJACENT LAYERS MOVE SMOOTHLY WITH RESPECT TO EACH OTHER.

TURBULENT FLOW IS WHEN FLUID MIXES BETWEEN THESE LAYERS

BOARD

LAMINAR (LAYERS)

TURBULENT

~~ONE DIMENSIONAL FLOWS~~

VISCOUS AND INVISCID FLOW

VISCOUS: SHEAR STRESSES IMPACT FLOW

INVISCID: SHEAR STRESSES SMALL ENOUGH TO IGNORE

SCRIPT

DIFFERENT FLOW REGIONS

APPROACH IS NEARLY INVISCID

BOUNDARY LAYER, VISCID FLOW ALONG A WALL

SEPARATION WHERE FLOW LEAVES THE WALL

WAKE, SEPARATED FLOW BEHIND A BODY

DRAG FORCE GOES UP WHEN THERE IS SEPARATION.

PREVENTION IS IMPOSSIBLE; BUT CAN BE DELAYED (SEE PHOTO)

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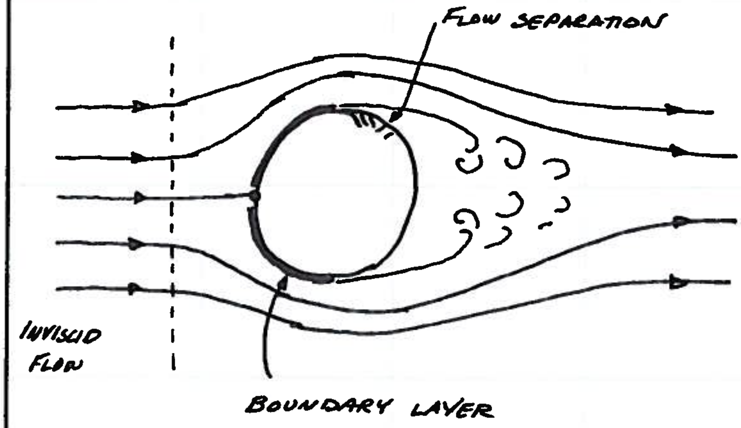
BOUNDARY LAYER, WAKE, POTENTIAL FLOW REGIONS



SCRIPT

ACCELERATION OCCURS WHEN THERE IS A SPEED AND/OR DIRECTION CHANGE

LOOK AT ONE DIRECTION

$$a_x = \frac{dV}{dx} = \underbrace{\frac{\partial u}{\partial t}}_{\text{LOCAL ACCELERATION}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{CONVECTIVE ACCELERATION}}$$

BOARD

ACCELERATION $\qquad \underline{F} = m\underline{a}$

$$\underline{a} = \frac{d\underline{V}}{dt}$$

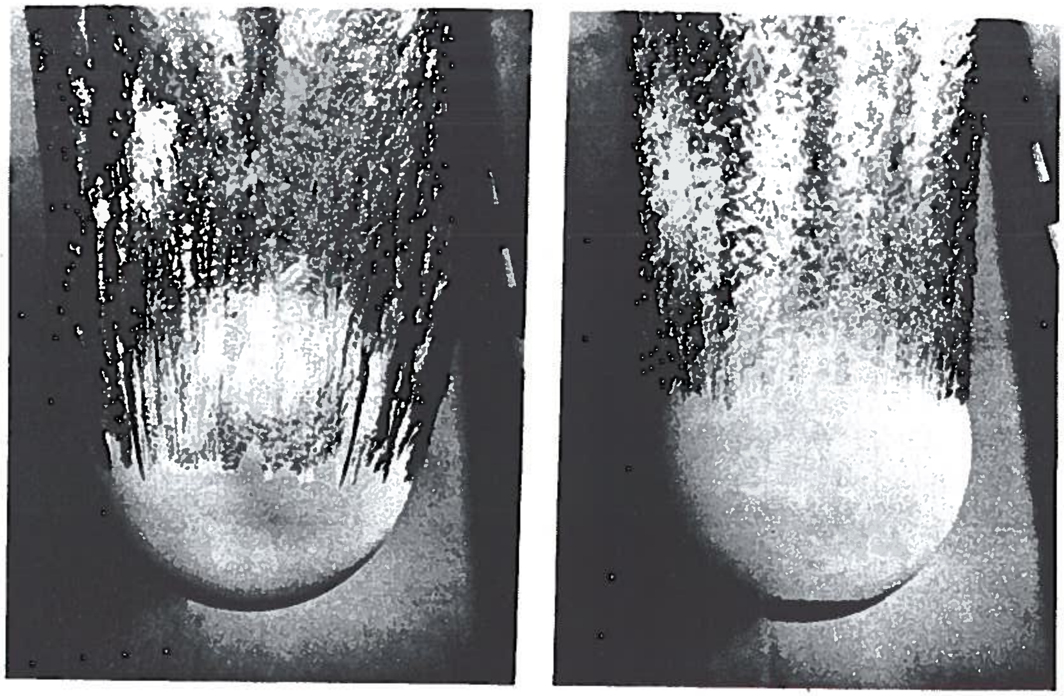
IN LAGRANGIAN FRAME:

$$\underline{a} = \frac{du}{dt}\underline{i} + \frac{dv}{dt}\underline{j} + \frac{\partial w}{\partial t}\underline{k}$$

- Referenced to particle's current position

IN EULERIAN FRAME

$$\underline{a} = \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]\underline{i} + \left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]\underline{j} + \left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right]\underline{k}$$



(*a*) (*b*)

Fig. 7.14 Strong differences in laminar and turbulent separation on an 8.5-in bowling ball entering water at 25 ft/s: (*a*) smooth ball, laminar boundary layer; (*b*) same entry, turbulent flow induced by patch of nose-sand roughness. (*U.S. Navy photograph, Ordnance Test Station, Pasadena annex.*)

SCRIPT

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FLUID PARTICLE KINEMATICS (LAGRANGIAN)

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$$

$$\underline{V}(t) = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$$

$$\underline{a}(t) = \frac{du}{dt}\underline{i} + \frac{dv}{dt}\underline{j} + \frac{dw}{dt}\underline{k}$$

WHERE

$$u(t) = \frac{dx}{dt}, \quad v(t) = \frac{dy}{dt}, \quad w(t) = \frac{dz}{dt}$$

REFERENCE IS ALWAYS THE CURRENT PARTICLE POSITION

SCRIPT

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FLUID ELEMENT MECHANICS (EULERIAN)

$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ (IDENTIFIES A POINT IN SPACE)

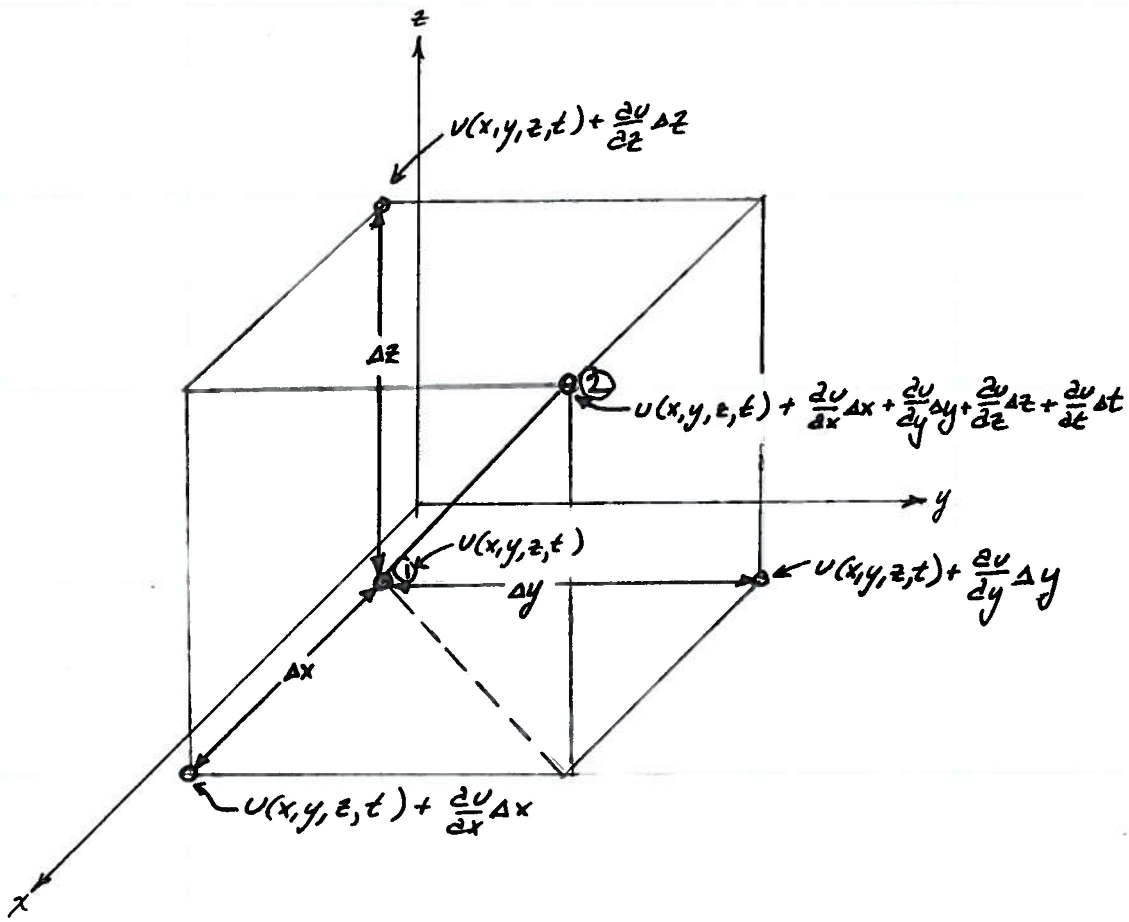
$$\underline{V}(x,y,z,t) = \left.\frac{dx}{dt}\right|_{x,y,z,t}\underline{i} + \left.\frac{dy}{dt}\right|_{x,y,z,t}\underline{j} + \left.\frac{dz}{dt}\right|_{x,y,z,t}\underline{k}$$

$$u(x,y,z,t) = \left.\frac{dx}{dt}\right|_{x,y,z,t}$$

$$\underline{a}(x,y,z,t) = \left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)\underline{i}$$
$$+ \left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)\underline{j}$$
$$+ \left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)\underline{k}$$

LOCAL ⟵ ⟶ CONVECTIVE

CONSIDER THE X-COMPONENT OF VELOCITY $u(t) = u(x,y,z,t)$



FOR THE PARTICLE AT ① MOVING TO ②, THE PARTICLE TRANSLATES IN X, Y, Z DIRECTIONS A DISTANCE OF $\Delta x, \Delta y, \Delta z$ AND TIME ELAPSES A PERIOD OF Δt

THE VELOCITY AT ② IS

$$u(x,y,z,t) + \underbrace{\frac{\partial u}{\partial x}\Delta x}_{\text{TRANSLATION } \Delta x \text{ IN } +X \text{ DIRECTION}} + \underbrace{\frac{\partial u}{\partial y}\Delta y}_{\text{TRANSLATION } \Delta y \text{ IN } +Y \text{ DIRECTION}} + \underbrace{\frac{\partial u}{\partial z}\Delta z}_{\text{TRANSLATION } \Delta z \text{ IN } +Z \text{ DIRECTION}} + \overbrace{\frac{\partial u}{\partial t}\Delta t}^{\text{TRANSLATION } \Delta t \text{ IN TEMPORAL DIMENSION}}$$

SCRIPT

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VELOCITY AT ② IS ALSO

$u(t+\Delta t)$

$$\frac{u(t+\Delta t)-u(t)}{\Delta t}=\frac{\partial u}{\partial x}\frac{\Delta x}{\Delta t}+\frac{\partial u}{\partial y}\frac{\Delta y}{\Delta t}+\frac{\partial u}{\partial z}\frac{\Delta z}{\Delta t}+\frac{\partial u}{\partial t}\frac{\Delta t}{\Delta t}$$

$\lim_{\Delta t \to 0}$ LEFT SIDE IS $\frac{du}{dt}$

RIGHT SIDE IS

$$\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}+\frac{\partial u}{\partial t}$$

$$\frac{du}{dt}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}$$

SAME PATTERN ~~REPEATS~~ IN THE OTHER TWO DIRECTIONS

SCRIPT

BOARD

RECALL EULER'S EQUATION, DERIVED BACK WHEN SHOWED PRESSURE IS SCALAR.

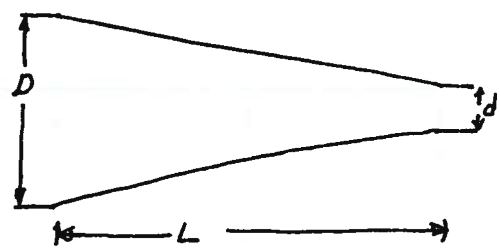
$$\rho g-\nabla p=\rho \underline{a}$$

INCLUDE g IN $\underline{a}$

$$-\nabla p=\rho \underline{a}$$

PRESSURE GRADIENT INVERSELY RELATED TO ACCELERATION

EXAMPLE



VELOCITY VARIES LINEARLY WITH TIME THROUGHOUT THE NOZZLE
V AT BASE IS $v = 2t$ AND AT THE TIP $v = 6t$

WHAT IS LOCAL ACCELERATION AT L/2 AT $t = 2$ SEC?

SOLUTION

LINEAR VARIATION MEANS $\frac{dV}{dx} = \text{CONST}$

$$\therefore \quad V(x) = \frac{dV}{dx} x + V_0$$

$$V(0) = 2t = v_0$$

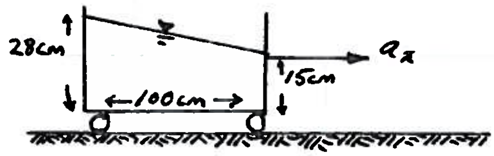
$$V(L) = 6t = \frac{dV}{dx} L + 2t \quad \therefore \quad \frac{dV}{dx} = \frac{4t}{L}$$

$$V(x) = \frac{4t}{L} x + 2t$$

$$a\left(x = \frac{L}{2}\right) = \frac{d}{dt}\left(V\left(\frac{L}{2}\right)\right) = \frac{d}{dt}\left(\frac{4t}{\cancel{L}} \frac{\cancel{L}}{2} + 2t\right) = \frac{d}{dt}(4t) = 4$$

$$\therefore a_l = 4 \, m/s^2$$

EXAMPLE



WHAT VALUE OF a_x IS REQUIRED TO MAINTAIN SHAPE SHOWN?

$$-\frac{dp}{dx} = \rho a_x$$

$$-dp = \rho a_x \, dx$$

$$-\int_{p_0}^{p} dp = \rho a_x \int_0^{100} dx = \rho a_x (x)\Big|_0^{100}$$

$$p - p_0 = -\rho a x \Big|_0^{100cm}$$

$$p_0 = \rho g_z \, 28cm$$

$$p = \rho g_z \, 15cm$$

$$\rho g_z (15cm - 28cm) = -\rho a (100cm - 0cm)$$

$$-13cm = -\frac{\rho a_x}{\rho g_z} (100cm)$$

$$-0.13 = -\frac{a_x}{g_z}$$

$$\therefore g_z \, 0.13 = a_x$$

$$(9.8 m/s^2)(0.13) = 1.27 \ m/s^2$$

SCRIPT

Valid when only forces are gravity & pressure

(hence inviscid flow)

(Shell out to examples 1 & 2

Textbook develops equation on a streamline – alternate derivation/plausibility is offered herein.

BOARD

"Euler's equation of motion for a fluid"

$\rho \underline{a} = \rho g - \nabla p$

Two "classical" ways to use within scope this class

- Uniform linear acceleration
- Constant angular velocity

Illustrative examples follow next 4 pgs.

"Bernoulli's equation"

Euler's equation, neglect viscous

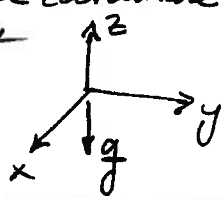
$\rho \underline{a} = \rho g - \nabla p$

SCRIPT

Start with Eulers equation.

Assume negligible viscosity effects.

Choose coordinate system so that



z is up, g is down.

Don't really need to do so, but eases the calculus a lot!

BOARD

Require that $g = -g\underline{k}$ (z is + up)

then

$\rho g - \nabla p = -\nabla(p + \rho g z)$ thus

$\rho \underline{a} = -\nabla(p + \rho g z)$

Require that ρg = constant (incompressible)

$$\frac{\rho \underline{a}}{\rho g} = -\frac{\nabla(p + \rho g z)}{\rho g}$$

$$\frac{\underline{a}}{g} = -\nabla\left(\frac{p}{\rho g} + z\right)$$

SCRIPT

Write ~~act~~ acceleration as total derivative.

note a_x
a_y
a_z

Terms, only showing a_x here!

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Invoke irrotational requirement
- In book derive on a streamline; eliminates need to consider vorticity

BOARD

Write $\underline{a}$ in differential form

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{du}{dt}$$

Require steady flow

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0$$

Then the component equations are:

$$\frac{1}{g}\left(u\frac{du}{dx} + v\frac{\partial u}{\partial y} + w\frac{du}{\partial z}\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\gamma} + z\right)$$

$$\frac{1}{g}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{dv}{\partial z}\right) = -\frac{d}{dy}\left(\frac{p}{\gamma} + z\right)$$

$$\frac{1}{g}\left(u\frac{dw}{\partial x} + v\frac{dw}{\partial y} + w\frac{\partial w}{dz}\right) = \frac{-\partial}{\partial z}\left(\frac{p}{\gamma} + z\right)$$

Require $\text{curl}(\vec{V}) = \underline{0}$ (irrotational flow)

SCRIPT

Use property of irrotational flow to replace cross-terms

Substitute and observe mathematical structure.

$$u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial u^2}{\partial x}$$

Chain rule of ~~calculus~~ calculus

Make substitutions

BOARD

For $\text{curl}(\vec{V}) = \underline{0}$ then

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z};\quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x};\quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Substitute into component equations

$$\frac{1}{g}\left(u\frac{\partial u}{\partial x} + v\frac{dv}{dx} + w\frac{dw}{\partial x}\right) = -\frac{d}{\partial x}\left(\frac{p}{\gamma} + z\right)$$

$$\frac{1}{g}\left(u\frac{du}{dy} + v\frac{dv}{dy} + w\frac{dw}{dy}\right) = -\frac{d}{dy}\left(\frac{p}{\gamma} + z\right)$$

$$\frac{1}{g}\left(u\frac{\partial w}{\partial x} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) = -\frac{d}{dz}\left(\frac{p}{\gamma} + z\right)$$

Recall that $y\frac{dy}{dx} = \frac{1}{2}\frac{dy^2}{\partial x}$ so.

$$\frac{1}{g}\left(\frac{d}{dx}\left(\frac{U^2}{2} + \frac{V^2}{2} + \frac{W^2}{2}\right)\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\gamma} + z\right)$$

$$\frac{1}{g}\left(\frac{\partial}{dy}\left(\frac{U^2}{2} + \frac{V^2}{2} + \frac{W^2}{2}\right)\right) = -\frac{\partial}{dy}\left(\frac{p}{\gamma} + z\right)$$

NAME CLEVELAND DATE 16 FEB 14
COURSE CE3305 SHEET 3 OF 4

SCRIPT

REARRANGE

OBSERVE

$$\bar{V} = (u, v, w)$$

$$\bar{V} \cdot \bar{V} = u^2 + v^2 + w^2$$

$$|\bar{V}| = (\bar{V} \cdot \bar{V})^{1/2}$$

(L_2 NORM, EUCLEDIAN DISTANCE)

BOARD

$$\frac{1}{g}\left(\frac{\partial}{\partial z}\left(\frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2}\right)\right) = \frac{-\partial}{\partial z}\left(\frac{p}{\gamma} + z\right)$$

Rearrange

$$0 = -\frac{\partial}{\partial x}\left(\frac{p}{\gamma} + z + \frac{u^2 + v^2 + w^2}{2g}\right)$$

$$0 = \frac{-\partial}{\partial y}\left(\frac{p}{\gamma} + z + \frac{u^2 + v^2 + w^2}{2g}\right)$$

$$0 = -\frac{\partial}{\partial z}\left(\frac{p}{\gamma} + z + \frac{u^2 + v^2 + w^2}{2g}\right)$$

The sum $u^2 + v^2 + w^2$ is simply the magnitude of the velocity vector (speed) squared so that:

SCRIPT

SUBSTITUTE AND OBSERVE THAT IF

$$\frac{d}{dx}\xi = \frac{d}{dy}\xi = \frac{d}{dz}\xi = 0$$

THEN ξ = CONSTANT

BERNOULLI'S EQUATION VALID FOR:

STEADY; INCOMPRESSIBLE; IRROTATIONAL; NON-VISCOUS FLOW FIELD

BOARD

$$\frac{d}{dx}\left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) = \frac{d}{dy}\left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) =$$

$$\frac{d}{dz}\left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) = 0$$

Recall $\frac{dy}{dx} = 0$ if y = constant wrt x

$\therefore$ All three equations must equal the same constant, so anywhere in the flow field

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = C$$

must hold. Called Bernoulli's equation.

SCRIPT	BOARD
NOTE CAN BE ALONG A STREAMLINE OR IN ANY IRROTATIONAL FIELD	OFTEN WRITTEN ALONG A STREAMLINE OR IN AN IRROTATIONAL FLOW FIELD AS $\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}$ LATER WE WILL SEE THESE THREE TERMS CALLED "TOTAL HEAD" THEY HAVE UNITS OF LENGTH.
EXAMPLES 1,2,3 pgs 132 - 134	~~SHELL~~ ← SHELL OUT TO THREE EXAMPLES