THE BERNOULLI EQUATION AND PRESSURE VARIATION

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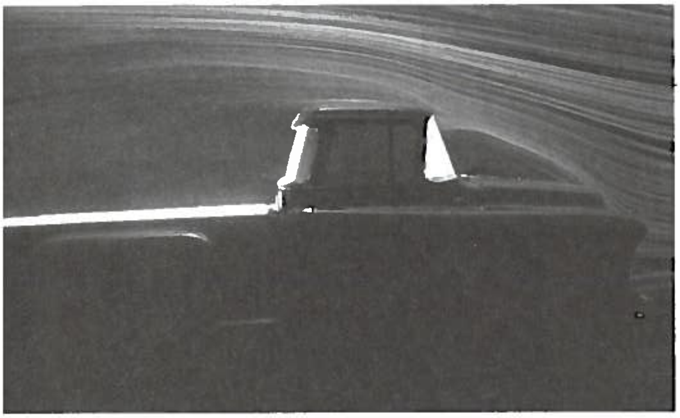


FIGURE 4.1

This photo shows flow over a model truck in a wind-tunnel. The purpose of the study was to compare the drag force on various designs of tonneau covers. The study was done by Stephen Lyda while he was an undergraduate engineering student. (Photo by Stephen Lyda)

Chapter Road Map

This chapter describes flowing fluids, introduces t Bernoulli equation, and describes pressure variat in flowing fluids.

Learning Objectives

STUDENTS WILL BE ABLE TO

- Describe streamlines, streaklines, and pathlines. Explai how these ideas differ. (§4.1)
- Describe velocity and the velocity field. (§4.2)
- Describe the Eulerian and Lagrangian approaches. (§
- Describe flowing fluids using the concepts introduced i section §4.3.
- Define acceleration. Sketch the direction of the accele vector of a fluid particle. Define local acceleration anc convective acceleration. (§4.4)
- Apply Euler's equation to describe pressure variations.
- Apply the Bernoulli equation along a streamline. (§4.6
- Define static pressure and kinetic pressure. Explain hov measure velocity using a Pitot-static tube. (§4.7)
- Define the rate-of-rotation and vorticity. Define an irrotc flow. (§4.8)
- Apply the Bernoulli equation in an irrotational flow. (§
- Define the pressure coefficient. Sketch the pressure var for flow around a circular cylinder. (§4.10)
- Calculate the pressure variation in a rotating flow. (§4

4.1 Describing Streamlines, Streaklines, and Pathlines

To visualize and describe flowing fluids, engineers use the streamline, streakline, and pathli
Hence, these topics are introduced in this section.

Pathlines and Streaklines

The **pathline** is the path of a fluid particle as it moves through a flow field. For example, wh the wind blows a leaf, this provides an idea about what the flow is doing. If we imagine that leaf is tiny and attached to a particle of air as this particle moves, then the motion of the l will reveal the motion of the particle. Another way to think of a pathline is to imagine atta ing a light to a fluid particle. A time exposure photograph taken of the moving light would the pathline. One way to reveal pathlines in a flow of water is to add tiny beads that are ne trally buoyant so that bead motion is the same as motion of fluid particles. Observing th beads as they move through the flow reveals the pathline of each particle.

The **streakline** is the line generated by a tracer fluid, such as a dye, continuously injec into a flow field at a starting point. For example, if smoke is introduced into a flow of air, t resulting lines are streaklines. Streaklines are shown in Fig. 4.1. These streaklines were p duced by vaporizing mineral oil on a vertical wire that was heated by passing an electri current through the wire.

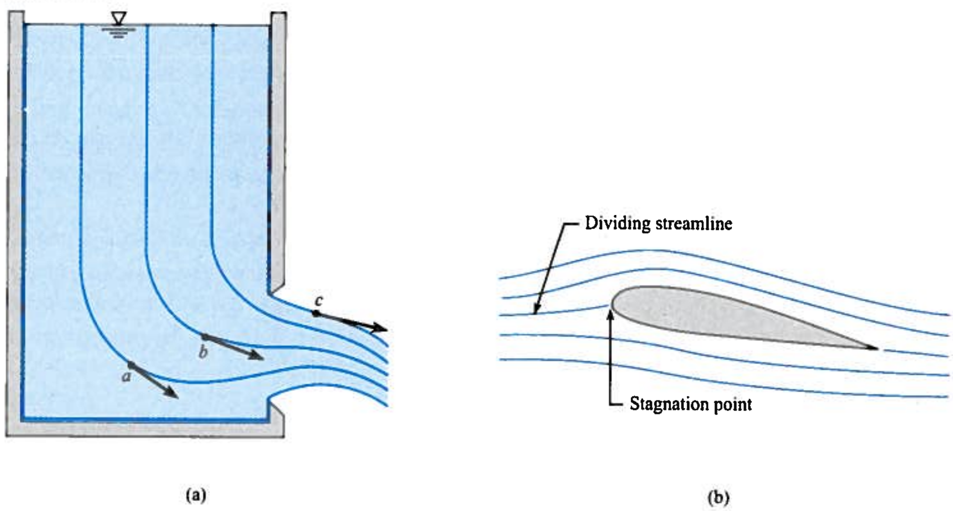
Streamlines

The **streamline** is defined as a line that is everywhere tangent to the local velocity vector.

EXAMPLE. The flow pattern for water draining through an opening in a tank (Fig. 4.2 can be visualized by examining streamlines. Notice that velocity vectors at points *a*, *b*, an are tangent to the streamlines. Also, the streamlines adjacent to the wall follow the conto of the wall because the fluid velocity is parallel to the wall. The generation of a flow patte is an effective way of illustrating the flow field.

Streamlines for flow around an airfoil (Fig. 4.2b) reveal that part of the flow goes over t airfoil and part goes under. The flow is separated by the **dividing streamline**. At the locati

FIGURE 4.2
(a) Flow through an opening in a tank.
(b) Flow over an airfoil section.

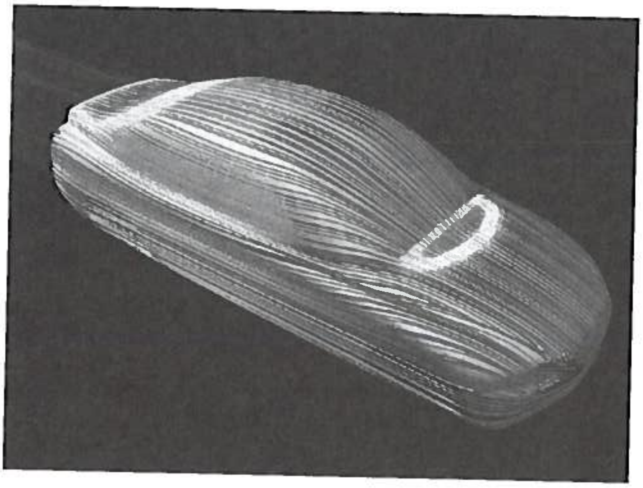


where the dividing streamline intersects the body, the velocity will be zero with respect to the body. This is called the stagnation point.

Streamlines for flow over an Volvo ECC prototype (Fig. 4.3) allow engineers to assess aerodynamic features of the flow and possibly change the shape to achieve better performance, such as reduced drag.

FIGURE 4.3

Predicted streamline pattern over the Volvo ECC prototype. (Courtesy of Analytical Methods, VSAERO software, Volvo Concept Center.)



Comparing Streamlines, Streaklines, and Pathlines

When flow is steady, the pathline, streakline, and streamline look the same so long as they all pass through the same point. Thus, the streakline, which can be revealed by experimental means, will show what the streamline looks like. Similarly, a particle in the flow will follow a line traced out of a streakline.

When flow is unsteady, then the streamline, streaklines, and pathlines look different. A captivating film entitled *Flow Visualization* (1) shows how and why the streamline, streakline, and pathline differ in unsteady flow.

EXAMPLE. To show how pathlines, streaklines, and streamlines differ in unsteady flow, consider a two-dimensional flow that initially has horizontal streamlines (Fig. 4.4). At a given time, t_0, the flow instantly changes direction, and the flow moves upward to the right at 45° with no further change. A fluid particle is tracked from the starting point, and up to time t_0, the pathline is the horizontal line segment shown on Fig. 4.4a. After time t_0, the particle continues to follow the streamline and moves up the right as shown in Fig. 4.4b. Both line segments constitute the pathline. Notice in Fig. 4.4b that the pathline (black dotted line) differs from an streamline for $(t < t_0)$ and any streamline for $(t > t_0)$. Thus, the pathline and the streamline are not the same.

Next consider the streakline by introducing black tracer fluid as shown in Figures 4.4c and d. As shown, the streakline in Fig. 4.4d differs from the pathline and from any streamline.

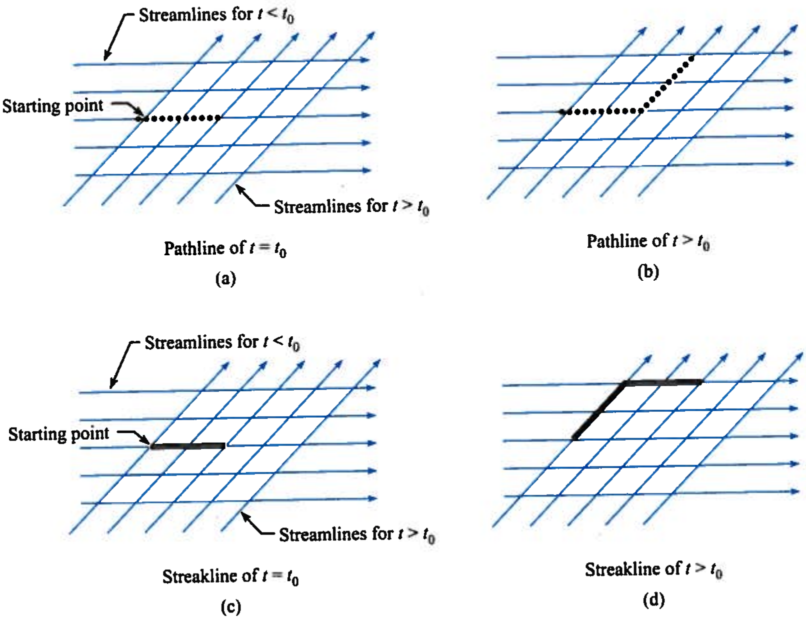


FIGURE 4.4
Streamlines, pathlines, and streakline for an unsteady flow field.

4.2 Characterizing Velocity of a Flowing Fluid

This section introduces *velocity* and the *velocity field*. Then, these ideas are used to introdu two alternative methods for describing motion.

- *Lagrangian approach*: Describes motion of a specified collection of matter.
- *Eulerian approach*. Describes motion at locations in space.

Describing Velocity

Velocity, a property of a fluid particle, gives the speed and direction of travel of the particle an instant in time. The mathematical definition of velocity is:

$$\mathbf{V}_A = \frac{d\mathbf{r}_A}{dt} \quad (4$$

where $\mathbf{V}_A$ is the velocity of particle A, and $\mathbf{r}_A$ is the position of particle A at time t.

EXAMPLE. When water drains from a tank (Fig. 4.5a), $\mathbf{V}_A$ gives the speed and directior travel of the particle at point A. The velocity $\mathbf{V}_A$ is the time rate of change of the vector $\mathbf{r}$

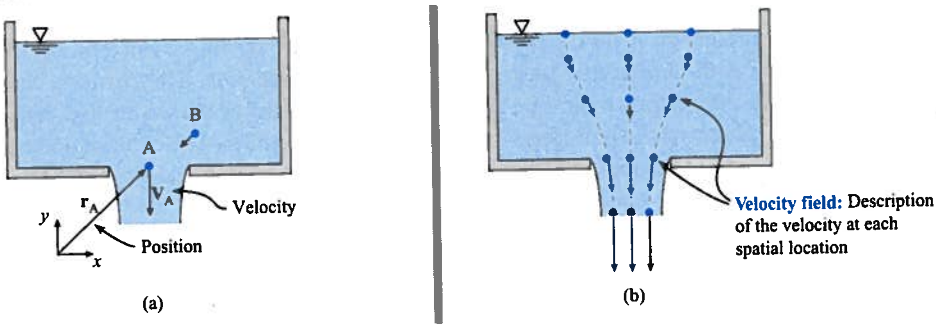


FIGURE 4.5
Water draining out of a tank. (a) The velocity of Particle A is the time derivative of the position. (b) The velocity field represents the velocity of each fluid particle throughout the region of flow.

Velocity Field

A description of the velocity of each fluid particle in a flow is called a **velocity field**. In general each fluid particle in a flow has a different velocity. For example, particles A and B in Fig 4.5a have different velocities. Thus, the velocity field describes how the velocity varies with position (see Fig. 4.5b).

A velocity field can be described visually (Fig. 4.5b) or mathematically as shown by the following example.

EXAMPLE. A steady, two-dimensional velocity field in a corner is given by

$$\mathbf{V} = (2x\,\mathrm{s}^{-1})\mathbf{i} - (2y\,\mathrm{s}^{-1})\mathbf{j} \tag{4.2}$$

where x and y are position coordinates measured in meters, and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions, respectively.

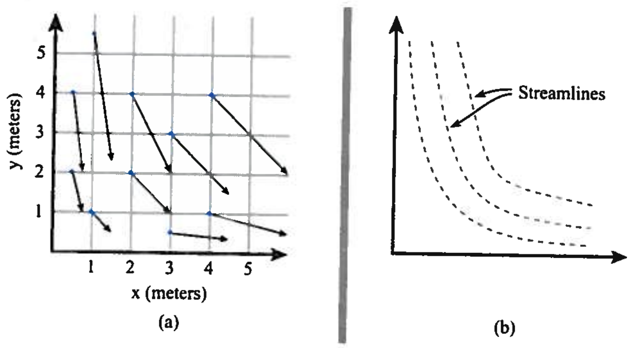
When a velocity field is given by an equation, a plot can help one visualize the flow. For example, select the location $(x, y) = (1, 1)$ and then substitute $x = 1.0$ meter and $y = 1.0$ meter into Eq. (4.2) to give the velocity as

$$\mathbf{V} = (2\ \mathrm{m/s})\mathbf{i} - (2\ \mathrm{m/s})\mathbf{j} \tag{4.3}$$

Plot this point and repeat this process at other points to create Fig. 4.6a. Last, one can use definition of the streamline (line that is everywhere tangent to the velocity vector) to create a streamline pattern (Fig. 4.6b).

FIGURE 4.6

The velocity field specified by Eq. (4.2): (a) velocity vectors, and (b) the streamline pattern.



Summary The velocity field describes the velocity of each fluid particle in a spatial region. The velocity field can be shown visually as in Figs. 4.5 and 4.6 or described mathematically as in Eq. 4.2.

The concept of a field can be generalized. A **field** is a mathematical or visual description of a variable as a function of position and time.

EXAMPLES. A pressure field describes the distribution of pressure at various points in space and time. A temperature field describes the distribution of temperature at various points in space and time.

A field can be scalar valued (e.g., temperature field, pressure field) or a field can be vect valued (e.g., velocity field, acceleration field).

✔CHECKPOINT PROBLEM 4.1

A velocity field is given as $\mathbf{V} = (ax + by)\mathbf{i}$ where $a = b = 2\ s^{-1}$ and (x, y) is the position in the fielc in meters. A particle moving in this field

a. Moves in the x-direction only

b. Moves in the y-direction only

c. Moves in both the x- and y-directions.

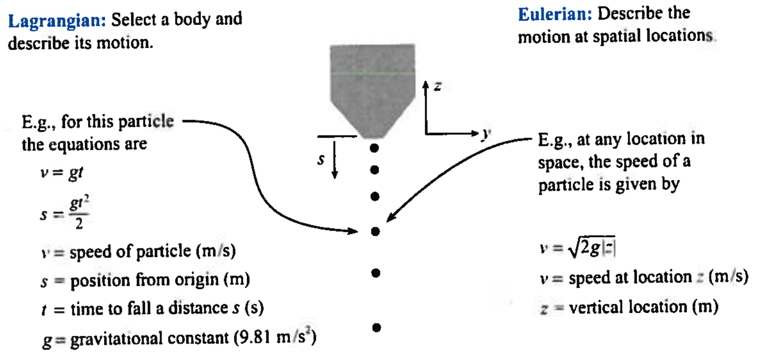
The Eulerian and Lagrangian Approaches

In solid mechanics, it is straightforward to describe the motion of a particle or a rigid bo In contrast, the particles in a flowing fluid move in more complicated ways and it is not pra tical to track the motion of each particle. Thus, researchers invented a second way to descri motion.

The first way to describe motion (called the **Lagrangian approach**) involves selecti a body and then describing the motion of this body. The second way (called the **Euleri approach**) involves selecting a region in space and then describing the motion that is c curring at points in space. In addition, the Eulerian approach allows properties to be eval ated at spatial locations as a function of time. This is because the Eulerian approach us fields.

EXAMPLE. Consider falling particles (Fig. 4.7). The Lagrangian approach uses equatic that describe an individual particle. The Eulerian approach uses an equation for the locity field. Although the equations of the two approaches are different, they predict t same values of velocity. Note that the equation $v = \sqrt{2g|z|}$ in Fig. 4.7 was deriv by letting the kinetic energy of the particle equal the change in gravitational potent energy.

FIGURE 4.7
This figure shows small particles released from rest and falling under the action of gravity. Equations on the left side of the image show how motion is described using a Lagrangian approach. Equations on the right side show an Eulerian approach.



When the ideas in Fig. 4.7 are generalized, the independent variables of the Lagrang approach are initial position and time. The independent variable of the Eulerian appro are position in the field and time. Table 4.1 compares the Lagrangian and the Euler approaches.

TABLE 4.1 Comparison of the Lagrangian and the Eulerian Approaches

Feature	Lagrangian Approach	Eulerian Approach
Basic idea	Observe or describe the motion of a body of matter of fixed identity.	Observe or describe the motion of matter at spatial locations.
Solid mechanics (application)	Used in dynamics.	Used in elasticity. Can be used to model the f of materials.
Fluid mechanics (application)	Fluid mechanics uses Eulerian ideas (e.g., fluid particle, streakline, acceleration of a fluid particle). Equations in fluid mechanics are often derived from an Lagrangian viewpoint.	Nearly all mathematical equations in fluid mechanics are written using the Eulerian approach.
Independent variables	Initial position (x_0, y_0, z_0) and time (t).	Spatial location (x, y, z) and time (t).
Mathematical complexity	Simpler.	More complex; e.g., partial derivatives and nonlinear terms appear.
Field concept	Not used in the Lagrangian approach.	The field is an Eulerian concepts. When fields are used, the mathematics often includes the divergence, gradient, and curl.
Types of systems used	Closed systems, particles, rigid bodies, system-of-particles.	Control volumes.

Representing Velocity Using Components

When the velocity field is represented in Cartesian components the mathematical form is

$$\mathbf{V} = u(x, y, z, t)\mathbf{i} + u(x, y, z, t)\mathbf{j} + u(x, y, z, t)\mathbf{k} \quad (4.4)$$

where $u = u(x, y, z, t)$ is the x-component of the velocity vector in and $\mathbf{i}$ is a unit vector in the x direction. The coordinates (x, y, z) give the spatial location in the field and t is time. Similarly, the components v and w give the y- and z-components of the velocity vector.

Another way to represent a velocity is to use *normal and tangential components*. In this approach (Fig. 4.8), unit vectors are attached to the particle and move with the particle. The tangential unit vector $\mathbf{u}_t$ is tangent to the path of the particle and the normal unit vector $\mathbf{u}_n$ is normal to path and directed inward toward the center of curvature. The position coordinate s measures distance traveled along the path. The velocity of a fluid particle is represented as $\mathbf{V} = V(s, t)\mathbf{u}_t$ where V is the speed of the particle and t is time.

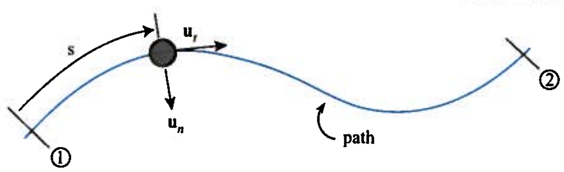


FIGURE 4.8 Describing motion of a fluid particle using normal and tangential components.

4.3 Describing Flow

Engineers use many words to describe flowing fluids. Speaking and understanding this language is seminal to professional practice. Thus, this section introduces concepts for describing flowing fluids. Because there are many ideas, a summary table is presented (see Table 4.4 on page 153).

Uniform and Nonuniform Flow

To introduce uniform flow, consider a velocity field of the form

$$\mathbf{V} = \mathbf{V}(s, t)$$

where s is distance traveled by a fluid particle along a path, and t is time (Fig. 4.9). This mat ematical representation is called *normal and tangential components*. This approach is use when the path of a particle is known.

In a **uniform flow**, the velocity is constant in magnitude and direction along a streamli at each instant in time. In uniform flow the streamlines must be rectilinear, which mea straight and parallel (see Fig. 4.10). Uniform flow can be described by an equation.

$$\left(\frac{\partial \mathbf{V}}{\partial s}\right)_t = \frac{\partial \mathbf{V}}{\partial s} = 0 \qquad \text{(uniform flow)} \qquad (4$$

Regarding notation in this text, we omit the variables that are held constant when writing p tial derivatives. For example, in Eq. (4.5), the leftmost terms show the formal way to writ partial derivative, and the middle term shows a simpler notation. The rationale for the simp notation is that variables that are held constant can be inferred from the context.

In **nonuniform flow**, the velocity changes along a streamline either in magnitude, dir tion, or both. It follows that any flow with streamline curvature is nonuniform. Also, any fl in which the speed of the flow is changing spatially is also nonuniform.

$$\frac{\partial \mathbf{V}}{\partial s} \neq 0 \qquad \text{(nonuniform flow)}$$

EXAMPLES. Nonuniform flow occurs in the converging duct in Fig. 4.11a because speed increases as the duct converges. Nonuniform flow occurs for the vortex in Fig. 4.: because the streamlines are curved.

FIGURE 4.9

Fluid particle moving along a pathline.

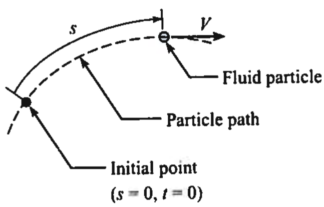


FIGURE 4.10

Uniform flow in a pipe.

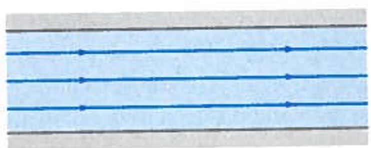
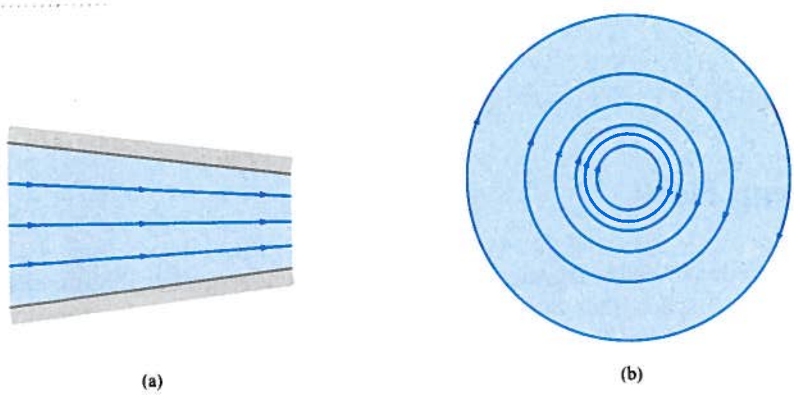


FIGURE 4.11

Flow patterns for nonuniform flow.
(a) Converging flow.
(b) Vortex flow.



Steady and Unsteady Flow

In general, a velocity field **V** depends of position **r** and time t: $\mathbf{V} = \mathbf{V}(\mathbf{r}, t)$. However, in many situations, the velocity is constant with time, so $\mathbf{V} = \mathbf{V}(\mathbf{r})$. This is called steady flow. **Steady flow** means that velocity at each location in space is constant with time. This idea can be written mathematically as:

$$\left.\frac{\partial \mathbf{V}}{\partial t}\right|_{\text{all points in velocity field}} = 0$$

In an **unsteady flow** the velocity is changing, at least at some points, in the velocity field. This idea can be represented with an equation.

$$\frac{\partial \mathbf{V}}{\partial t} \neq 0$$

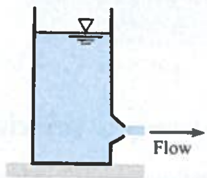
EXAMPLE. If the flow in a pipe changed with time due to a valve opening or closing, the flow would be unsteady; that is, the velocity at locations in the velocity field would be increasing or decreasing with time.

✔CHECKPOINT PROBLEM 4.2

As shown, water drains out of a small opening in a container. Which statement is true?

a. The flow in the container is steady.

b. The flow in the container is unsteady.



Laminar and Turbulent Flow

In a famous experiment, Osborne Reynolds showed that there are two different kinds of flow that can occur in a pipe.* The first type, called **laminar flow**, is a well-ordered state of flow in which adjacent fluid layers move smoothly with respect to each other. The flow occurs in layers or laminae. An example of laminar flow is the flow of thick syrup (Fig. 4.12a).

FIGURE 4.12

Examples of laminar and turbulent flow (a) the flow of maple syrup is laminar (Lauri Patterson/The Agency Collection/Getty Images) (b) the flow of steam out of a smokestack is turbulent (Photo by Donald Elger)

(a) (b)

*Reynolds experiment is described in Chapter 10.

The second type of flow identified by Reynolds is called **turbulent flow**, which is an un steady flow characterized by eddies of various sizes and intense cross-stream mixing. Turbu lent flow can be observed in the wake of a ship. Also, turbulent flow can be observed for smokestack (Fig. 4.12b). Notice that the mixing of the turbulent flow is apparent because th plume widens and disperses.

Laminar flow in a pipe (Fig. 4.13a) has a smooth parabolic velocity distribution. Turbulen flow (Fig. 4.13b) has a plug-shaped velocity distribution because eddies mix the flow, whic tends to keep the distribution uniform. In both laminar and turbulent flow, the no-slip cond tion applies.

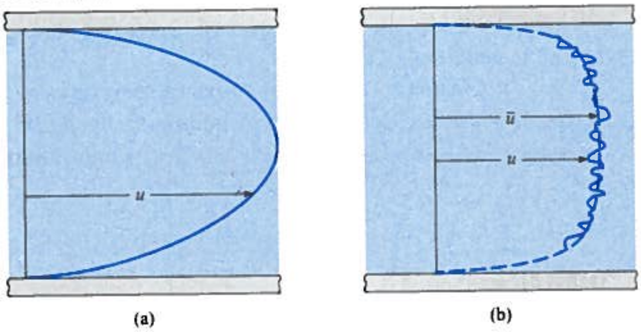


FIGURE 4.13
Laminar and turbulent flow in a straight pipe. (a) Laminar flow. (b) Turbulent flow. Both sketches assume fully developed flow.

Time-Averaged Velocity

Turbulent flow is unsteady, so the standard approach is to represent the velocity as a tim averaged velocity $\bar{u}$ plus a fluctuating component u'. Thus, the velocity is expressed $u = \bar{u} + u'$ (see Fig. 4.13b). Thus, the fluctuating component is defined as the difference b tween the local velocity and the time-averaged velocity. A turbulent flow is designated "steady" if the time-averaged velocity is unchanging with time. For an interesting look at turb lent flows, see the film entitled *Turbulence* (3). Table 4.2 compares laminar and turbulent flow

TABLE 4.2 Comparison of the Laminar and Turbulent Flow

Feature	Laminar Flow	Turbulent Flow
Basic description	Smooth flow in layers (laminae).	The flow has many eddies of various sizes. The flow appears random, chaotic, and unsteady.
Velocity profile in a pipe	Parabolic; ratio of mean velocity to centerline velocity is 0.5 for fully developed flow.	Pluglike; ratio of mean velocity to centerline velocity is between 0.8 and 0.9.
Mixing of materials added to the flow	Low levels of mixing. Difficult to get a material to mix with a fluid in laminar flow.	High levels of mixing. Easy to get a material to mix; e.g., visualize cream mixing with coffee.
Variation with time	Can be steady or unsteady.	Always unsteady.
Dimensionality of flow	Can be 1D, 2D, or 3D.	Always 3D.
Availability of mathematical solutions	In principle, any laminar flow can be solved with an analytical or computer solution. There are many existing analytical solutions. Solutions are very close to what would be measured with an experiment.	There is no complete theory of turbulent flow. There are a limited number of semiempirical solution approaches. Many turbulent flows cannot be accurately predicted with computer models or analytical solutions. Engineers often rely on experiments to characterize turbulent flow.

Feature	Laminar Flow	Turbulent Flow
Practical importance	Although many problems of practical problems involve laminar flow, these problems are not nearly as common as problems that involve turbulent flow.	The majority of practical problems involve turbulent flow. Typically, the flow of air and water in piping systems is turbulent. Most flc of water in open channels are turbulent.
Occurrence (Reynolds number)	Occurs at lower values of Reynolds numbers. (Reynolds number is introduced in Chapter 8.)	Occurs at higher values of Reynolds number

One-Dimensional and Multidimensional Flows

The dimensionality of a flow field can be illustrated by example. Fig. 4.14a shows the velocity distribution for an axisymmetric flow in a circular duct. The flow is uniform, or fully developed, so the velocity does not change in the flow direction (z). The velocity depends on only one spatial dimension, namely the radius r, so the flow is one-dimensional or 1D. Fig. 4.14b shows the velocity distribution for uniform flow in a square duct. In this case the velocity depends on two dimensions, namely x and y, so the flow is two dimensional. Figure 4.14c also shows the velocity distribution for the flow in a square duct but the duct cross-sectional area is expanding in the flow direction so the velocity will be dependent on z as well as x and y. This flow is three-dimensional, or 3-D.

Another good example of three-dimensional flow is turbulence because the velocity components at any one time depend on the three coordinate directions. For example, the velocity component u at a given time depends on x, y, and z; that is, $u(x, y, z)$. Turbulent flow is unsteady, so the velocity components also depend on time.

Another definition frequently used in fluid mechanics is quasi-one-dimensional flow. By this definition it is assumed that there is only one component of velocity in the flow direction and that the velocity profiles are uniformly distributed; that is, constant velocity across the duct cross section.

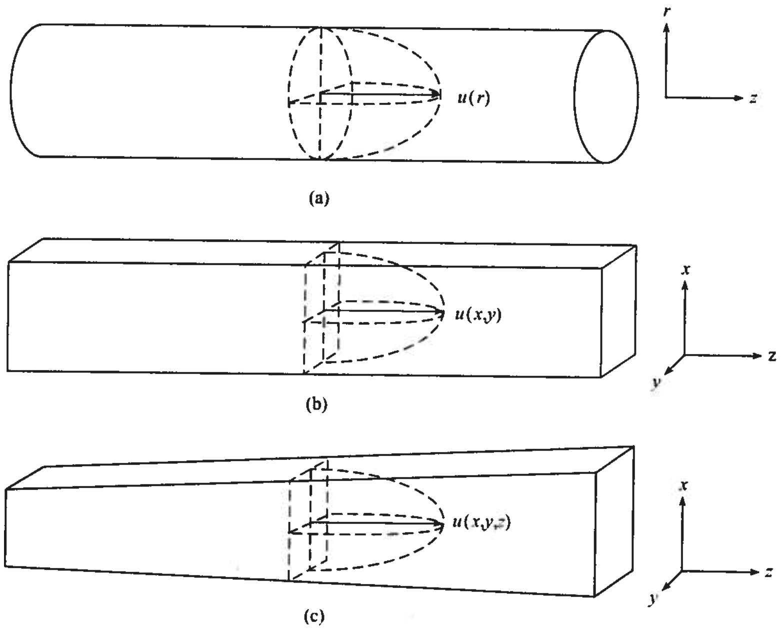


FIGURE 4.14

Flow dimensionality, (a) one-dimensional flo (b) two-dimensional flo and (c) three-dimension flow.

Viscous and Inviscid Flow

In a **viscous flow** the forces associated with viscous shear stresses are large enough to effect t dynamic motion of the particles that comprise the flow. For example, when a fluid flows ir pipe as shown in Fig. 4.13, this is a viscous flow. Indeed, both laminar and turbulent flows a types of viscous flows.

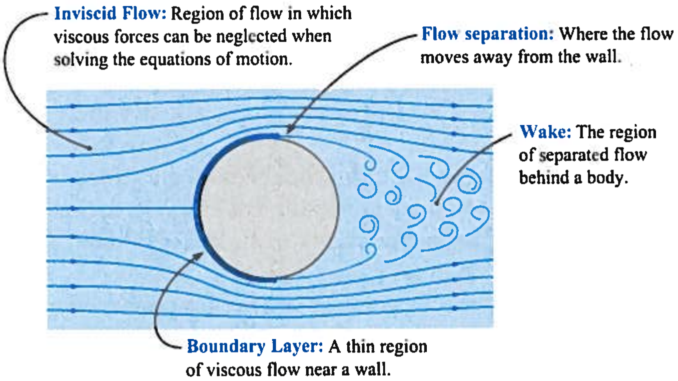
In a **inviscid flow** the forces associated with viscous shear stresses are small enough that they do not affect the dynamic motion of the particles that comprise the flow. Thus, inviscid flow, the viscous stresses can be neglected in the equations for motion.

Boundary Layer, Wake, and Potential Flow Regions

To idealize many complex flows, engineers use ideas that can be illustrated by flow over sphere (Fig. 4.15). As shown, the flow is divided into three regions: an inviscid flow region wake, and a boundary layer.

FIGURE 4.15

Flow pattern around a sphere when the Reynolds number is high. The sketch shows the regions of flow.

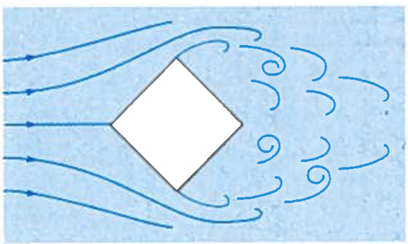


Flow Separation

Flow separation (Fig. 4.15) occurs when the fluid particles adjacent to a body deviate from t contours of the body. Fig. 4.16 shows flow separation behind a square rod. Notice that the fl separates from the shoulders of the rod and that the wake region is large. In both Figs. 4.15 a 4.16 the flow follows the contours of the body on the upstream sides of the objects. The regi in which a flow follows the body contour is called **attached flow.**

FIGURE 4.16

Flow pattern past a square rod illustrating separation at the edges.



When flow separates (Fig. 4.16), the drag force on the body is usually large. Thus, design strive to reduce or eliminate flow separation when designing products such as automobiles a airplanes. In addition, flow separation can lead to structural failure because the wake is u steady due to vortex shedding, and this creates oscillatory forces. These forces cause structu vibrations, which can lead to failure when the structure's natural frequency is closely matched to the vortex shedding frequency. In a famous example, vortex shedding associated with flow separation caused the Tacoma Narrows Bridge near Seattle, Washington, to oscillate wildly and to fail catastrophically.

Fig. 4.17 shows flow separation for an airfoil (an airfoil is a body with the cross sectional shape of a wing). Flow separation occurs when the airfoil is rotated to an angle of attack that is too high. Flow separation in this context causes an airplane to stall, which means that the lifting force drops dramatically and the wings can no longer keep the airplane in level flight. Stall is to be avoided.

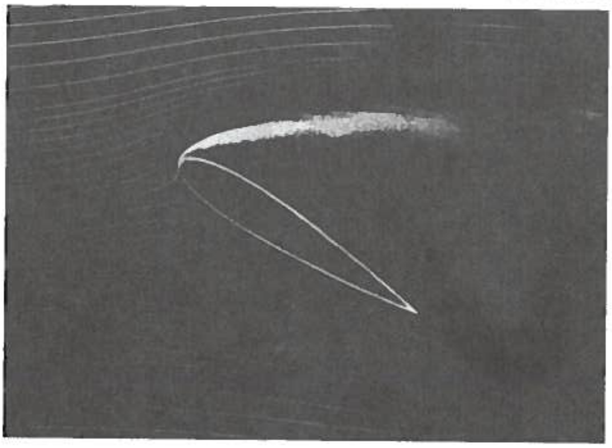


FIGURE 4.17

Smoke traces showin separation on an airf section at a large ang attack. (Courtesy Edu Development Center, Waltham, MA)

Flow separation can occurs inside pipes. For example, flow passing through an orifice in a pipe will separate (see Fig. 13.14 in Section 13.2). In this case, the zone of separated flow is usually called a recirculating zone. Separating flow within a pipe is usually undesirable because it causes energy losses, low pressure zones that can lead to cavitation and vibrations.

Summary *Attached flow* means that flow is moving parallel to walls of a body. *Flow separation*, which occurs in both internal and external flows, means the flow moves away from the wall. Flow separation is related to phenomenon of engineering interest such as drag, structural vibrations, and cavitation.

4.4 Acceleration

Predicting forces is important to the designer. Because forces are related to acceleration, this section describes what acceleration means in the context of a flowing fluid.

Definition of Acceleration

Acceleration is a property of a fluid particle that characterizes the change in speed of the particle and the change in the direction of travel at an instant in time. The mathematical definition of acceleration is:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} \tag{4.6}$$

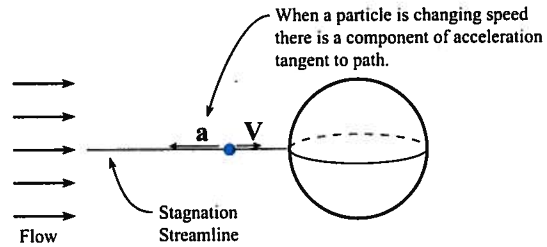
where $\mathbf{V}$ is the velocity of the particle and t is time.

Physical Interpretation of Acceleration

Acceleration occurs when *a fluid particle is changing its speed, changing its direction of trav*
or both.

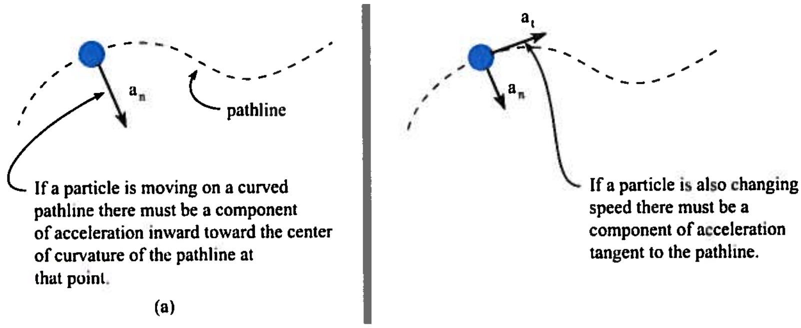
EXAMPLE. As a particle moves along the straight streamline in Fig 4.18, it is slowing dow
Because the particle is changing speed, it is accelerating (actually decelerating in this cas
Anytime a particle is changing speed, there must be a component of the acceleration vect
tangent to the path. This component of acceleration is called the *tangential component*
acceleration.

FIGURE 4.18
This figure shows flow over a sphere. The blue sphere is a fluid particle that is moving along the stagnation streamline.



EXAMPLE. As a particle moves along a curved streamline (see Fig 4.19), the particle mu
have a component of acceleration directed inward as shown. This component is called t
normal component of the acceleration vector. In addition, if the particle is changing spee
the tangential component will also be present.

FIGURE 4.19
This figure shows a particle moving on a curved streamline.



Summary Acceleration is a property of a fluid particle. The tangential component of t
acceleration vector is associated with a change in speed. The normal component is asso
ated with a change in direction. The normal component will be nonzero anytime a parti
is moving on a curved streamline because the particle is continually changing its directi
of travel.

Describing Acceleration Mathematically

Because the velocity of a flowing fluid is described with a velocity field (i.e., an Euleri
approach), the mathematical representation of acceleration is different from what is present
in courses like physics and dynamics. This subsection develops the ideas.

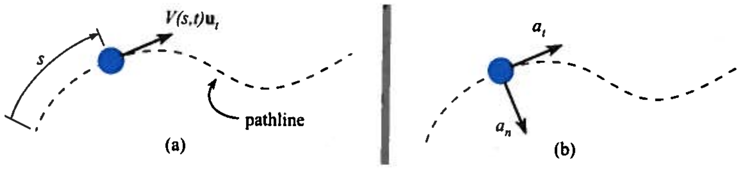
To begin, picture a fluid particle on a streamline as shown in Fig. 4.20. Write the velocity using normal-tangential components:

$$\mathbf{V} = V(s, t)\mathbf{u}_t$$

In this equation, the speed of the particle V is a function of position s and time t. The direction of travel of the particle is given by the unit vector $\mathbf{u}_t$ which, by definition, is tangent to the streamline.

FIGURE 4.20

Particle moving on a pathline. (a) Velocity. (b) Acceleration.



Using the definition of acceleration,

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \left(\frac{dV}{dt}\right)\mathbf{u}_t + V\left(\frac{d\mathbf{u}_t}{dt}\right) \qquad (4.7)$$

To evaluate the derivative of speed in Eq. (4.7), the chain rule for a function of two variables can be used.

$$\frac{dV(s,t)}{dt} = \left(\frac{\partial V}{\partial s}\right)\left(\frac{ds}{dt}\right) + \frac{\partial V}{\partial t} \qquad (4.8)$$

In a time dt, the fluid particle moves a distance ds, so the derivative ds/dt corresponds to the speed V of the particle, and Eq. (4.8) becomes

$$\frac{dV}{dt} = V\left(\frac{\partial V}{\partial s}\right) + \frac{\partial V}{\partial t} \qquad (4.9)$$

In Eq. (4.7), the derivative of the unit vector $d\mathbf{u}_t/dt$ is nonzero because the direction of the unit vector changes with time as the particle moves along the pathline. The derivative is

$$\frac{d\mathbf{u}_t}{dt} = \frac{V}{r}\mathbf{u}_n \qquad (4.10)$$

where $\mathbf{u}_t$ is the unit vector perpendicular to the pathline and pointing inward toward the center of curvature (2).

Substituting Eqs. (4.9) and (4.10) into Eq. (4.7) gives the acceleration of the fluid particle:

$$\mathbf{a} = \left(V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t}\right)\mathbf{u}_t + \left(\frac{V^2}{r}\right)\mathbf{u}_n \qquad (4.11)$$

The interpretation of this equation is as follows. The acceleration on the left side is acceleration of the fluid particle. The terms on the right side represent a way to evaluate this acceleration by using the velocity, the velocity gradient, and the velocity change with time.

Eq. (4.11) shows that the magnitude of the normal component of acceleration is V^2/r The direction of this acceleration component is normal to the streamline and inward towar the center of curvature of the streamline. This term is sometimes called the **centripeta acceleration**, where the centripetal means *center seeking*.

Convective and Local Acceleration

In Eq. (4.11), the term $\partial V/\partial t$ means the time rate of change of speed while holding posi tion (x, y, z) constant. Time derivative terms in Eulerian formulation for acceleration ar called **local acceleration** because position is held constant. All other terms are called **con vective acceleration** because they typically involve variables associated with flui motion.

EXAMPLE. The concepts of Eq. (4.11) can be illustrated by use of the cartoon in Fig. 4.21 The carriage represents the fluid particle, and the track, the pathline. A direct way to mea sure the acceleration is to ride on the carriage and read the acceleration off an accelerom eter. This gives the acceleration on the left side of Eq. (4.11). The Eulerian approach is t record data so terms on the right side of Eq. (4.11) can be calculated. One would measur the carriage velocity at two locations separated by a distance Δs and calculate the convec tive term using

$$V\frac{\partial V}{\partial s} \approx V\frac{\Delta V}{\Delta s}$$

Next, one would measure V and r and then calculate V^2/r. The local acceleration, for thi example, would be zero. When one did the calculations on the right side of Eq. (4.11), th calculated value would match the value recorded by the accelerometer.

FIGURE 4.21
Measuring convective acceleration by two different approaches. (Sketch by Chad Crowe)

Summary The physics of acceleration are described by considering changing speed ar changing direction of a fluid particle. Local and convective acceleration are labels for the mat ematical terms that appear in the Eulerian formulation of acceleration.

When a velocity field is specified, this denotes an Eulerian approach, and one can calcula the acceleration by substituting into an appropriate formula. Example 4.1 illustrates th method.

EXAMPLE 4.1

Calculating Acceleration when a Velocity Field is Specified

Problem Statement

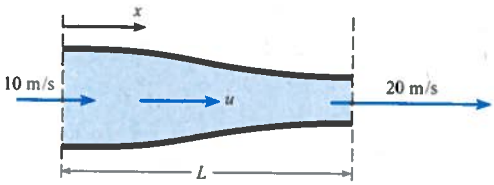
A nozzle is designed such that the velocity in the nozzle varies as

$$u(x) = \frac{u_0}{1.0 - 0.5x/L}$$

where the velocity u_0 is the entrance velocity and L is the nozzle length. The entrance velocity is 10 m/s, and the length is 0.5 m. The velocity is uniform across each section. Find the acceleration at the station halfway through the nozzle ($x/L = 0.5$).

Define the Situation

A velocity distribution is specified in a nozzle.



Assumptions: Flow field is quasi one-dimensional (negligible velocity normal to nozzle centerline).

State the Goal

Calculate the acceleration at nozzle midpoint.

Generate Ideas and Make a Plan

1. Select the pathline along the centerline of the nozzle.
2. Evaluate the terms in Eq. (4.11).

Take Action (Execute the Plan)

The distance along the pathline is x, so s in Eq. (4.11) becomes x and V becomes u. The pathline is straight, so $r \to \infty$.

1. Term-by-term analysis:
   - Convective acceleration

$$\frac{\partial u}{\partial x} = -\frac{u_0}{(1 - 0.5x/L)^2} \times \left(-\frac{0.5}{L}\right) = \frac{1}{L}\frac{0.5u_0}{(1 - 0.5x/L)^2}$$

$$u\frac{\partial u}{\partial x} = 0.5\frac{u_0^2}{L}\frac{1}{(1 - 0.5x/L)^3}$$

Evaluation at $x/L = 0.5$:

$$u\frac{\partial u}{\partial x} = 0.5 \times \frac{10^2}{0.5} \times \frac{1}{0.75^3} = 237\ \text{m/s}^2$$

   - Local acceleration

$$\frac{\partial u}{\partial t} = 0$$

   - Centripetal acceleration (also a convective acceleration

$$\frac{u^2}{r} = 0$$

2. Combine the terms

$$a_x = 237\ \text{m/s}^2 + 0 = \boxed{237\ \text{m/s}^2}$$

$$a_n\ (\text{normal to pathline}) = \boxed{0}$$

Review the Solution and the Process

Knowledge. Because a_x is positive, the direction of the acceleration is positive; that is, the velocity increases in the x-direction, as expected. Even though the flow is steady, the fluid particles still accelerate.

4.5 Applying Euler's Equation to Understand Pressure Variation

Euler's equation, the topic of this section, is used by engineers to understand pressure variation.

Derivation of Euler's Equation

Euler's equation is derived by applying $\Sigma \mathbf{F} = m\mathbf{a}$ to a fluid particle. The derivation is similar to the derivation of the hydrostatic differential equation (Chapter 3).

To begin, select a fluid particle (Fig. 4.22a) and orient the particle in an arbitrary directic ℓ and at an angle α with respect to the horizontal plane (Fig. 4.22b). Assume that viscous forc are zero. Assume the particle is in a flow and that the particle is accelerating. Now, apply Nev ton's second law in the ℓ-direction:

$$\sum F_\ell = ma_\ell$$

$$F_{\text{pressure}} + F_{\text{gravity}} = ma_\ell \quad \text{(4.1}$$

The mass of the particle is

$$m = \rho \Delta A \Delta \ell$$

The net force due to pressure in the ℓ-direction is

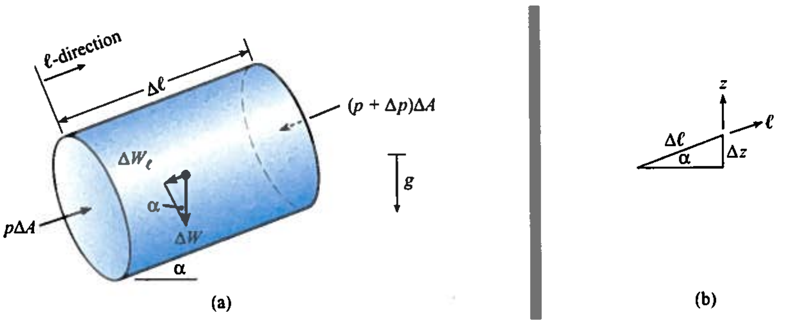
$$F_{\text{pressure}} = p\Delta A - (p + \Delta p)\Delta A = -\Delta p \Delta A$$

The force due to gravity is

$$F_{\text{gravity}} = -\Delta W_\ell = -\Delta W \sin\alpha \quad \text{(4.1}$$

FIGURE 4.22

(a) Forces acting on a fluid particle, and (b) sketch showing the geometry.



From Fig. 4.22b note that $\sin\alpha = \Delta z/\Delta \ell$, so Eq. (4.13) becomes

$$F_{\text{gravity}} = -\Delta W \frac{\Delta z}{\Delta \ell}$$

The weight of the particle is $\Delta W = \gamma \Delta \ell \Delta A$. Substituting the mass of the particle and the forc on the particle into Eq. (4.12) yields

$$-\Delta p \Delta A - \gamma \Delta \ell \Delta A \frac{\Delta z}{\Delta \ell} = \rho \Delta \ell \Delta A a_\ell$$

Dividing through by the volume of the particle $\Delta A \Delta \ell$ results in

$$-\frac{\Delta p}{\Delta \ell} - \gamma \frac{\Delta z}{\Delta \ell} = \rho a_\ell$$

Taking the limit as $\Delta \ell$ approaches zero (reduce the particle to an infinitesimal size) leads to

$$-\frac{\partial p}{\partial \ell} - \gamma \frac{\partial z}{\partial \ell} = \rho a_\ell \quad \text{(4.}$$

Assume a constant density flow, so γ is constant and Eq. (4.14) reduces to

$$-\frac{\partial}{\partial \ell}(p + \gamma z) = \rho a_\ell \quad \text{(4.}$$

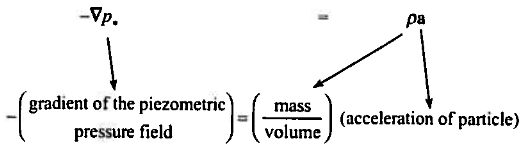
Equation (4.15) is a scalar form of *Euler's equation*. Because this equation is true in any scalar direction, one can write this in an equivalent vector form:

$$-\nabla p_z = \rho \mathbf{a} \tag{4.16}$$

where ∇p_z is the gradient of the piezometric pressure, and $\mathbf{a}$ is the acceleration of the fluid particle.

Physical Interpretation of Euler's Equation

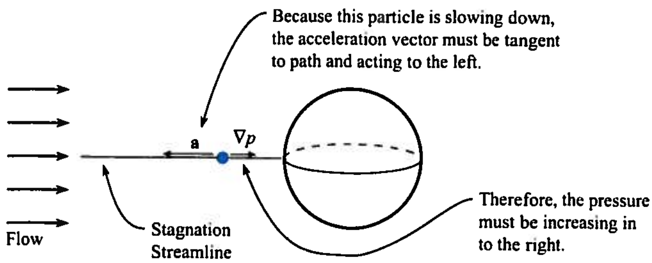
Euler's equation shows that the pressure gradient is colinear with the acceleration vector and opposite in direction.



Thus, by using knowledge of acceleration, one can make inferences about the pressure variation. Three important cases are presented next. At this point, we recommend the film entitled *Pressure Fields and Fluid Acceleration* (4) because this film illustrates fundamental concepts using laboratory experiments.

Case 1: Pressure Variation Due to Changing Speed of a Particle
When a fluid particle is speeding up or slowing down as it moves along a streamline, then pressure will vary in a direction tangent to the streamline. For example Fig. 4.23 shows a fluid particle moving along a stagnation streamline. Because the particle is slowing down, the acceleration vector points to the left. Therefore the pressure gradient must point to the right. Thus, the pressure is increasing along the streamline, and the direction of increasing pressure is to the right. *Summary.* When a particle is changing speed, then pressure will vary in a direction that is tangent to the streamline.

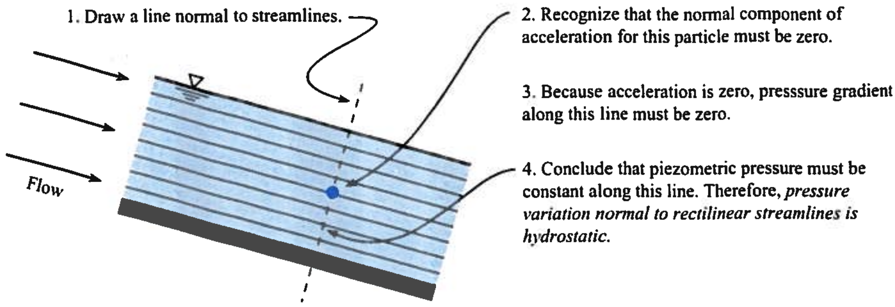
FIGURE 4.23
This figure shows flow over a sphere. The blue object is a fluid particle moving along the stagnation streamline.



Case 2: Pressure Variation Normal to Rectilinear Streamlines
When streamlines are straight and parallel (Fig. 4.24), then *piezometric pressure will be constant along a line that is normal to the streamlines.* To prove this fact, draw a line that is normal to the streamlines (see Fig. 4.24). Then recognize that

$$a_n = \frac{V^2}{r} = \frac{V^2}{\infty} = 0$$

FIGURE 4.24
Flow with rectilinear streamlines. The numbered steps give the logic to show that pressure variation normal to rectilinear streamlines is hydrostatic.

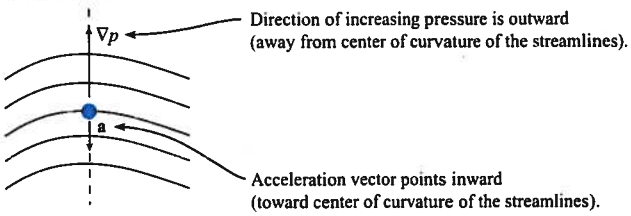


Because $a_n = 0$, Euler's equation shows that the pressure gradient must be zero: $\partial(p + \gamma$ $\partial n = 0$. Thus, conclude that piezometric pressure $(p + \gamma z)$ is constant along any line that normal to the streamlines. *Summary.* Pressure variation normal to rectilinear streamlines hydrostatic.

Case 3: Pressure Variation Normal to Curved Streamlines

When streamlines are curved (Fig. 4.25), then *piezometric pressure will increase along a l that is normal to the streamlines.* The direction of increasing pressure will be outward from t center of curvature of the streamlines. Fig. 4.25 shows why pressure will vary. A fluid parti on a curved streamline must have a component of acceleration inward. Therefore, the gradi of the pressure will point outward. Because the gradient points in the direction of increasi pressure, we conclude that pressure will increase along the line drawn normal to the strea lines. *Summary.* When streamlines are curved, then pressure increases outward from the ce ter of curvature* of the streamlines.

FIGURE 4.25
Flow with curved streamlines. Assume that the fluid particle has constant speed. Thus, the acceleration vector points inward towards the center of curvature.



Calculations Involving Euler's Equation

In most cases, calculations involving Euler's equation are beyond the scope of this book. Hc ever, when a fluid is accelerating as a rigid body, then Euler's equation can be applied i simple way. Examples 4.2 and 4.3 show how to do this.

*Each streamline has a center of curvature at each point along the streamline. There is not a single center of curva of a group of streamlines.

EXAMPLE 4.2

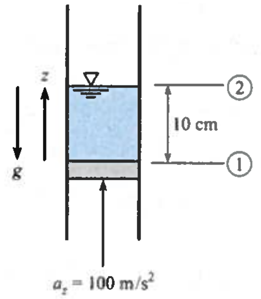
Applying Euler's equation to a Column of Fluid being Accelerated Upward

Problem Statement

A column water in a vertical tube is being accelerated by a piston in the vertical direction at 100 m/s². The depth of the water column is 10 cm. Find the gage pressure on the piston. The water density is 10^3 kg/m³.

Define the Situation

A column of water is being accelerated by a piston.



Assumptions:

- Acceleration is constant.
- Viscous effects are unimportant.
- Water is incompressible.

Properties: $\rho = 10^3$ kg/m³

State the Goal

Find: The gage pressure on the piston.

Generate Ideas and Make a Plan

1. Apply Euler's equation, Eq. (4.15), in the z-direction.
2. Integrate between locations 1 and 2.
3. Set pressure equal to zero (gage pressure) at section 2.
4. Calculate the pressure on the piston.

Take Action (Execute the Plan)

1. Because the acceleration is constant, there is no dependence on time, so the partial derivative in Euler's equation can be replaced by an ordinary derivative. Eule equation becomes:

$$\frac{d}{dz}(p + \gamma z) = -\rho a_z$$

2. Integration between sections 1 and 2:

$$\int_1^2 d(p + \gamma z) = \int_1^2 (-\rho a_z) dz$$

$$(p_2 + \gamma z_2) - (p_1 + \gamma z_1) = -\rho a_z (z_2 - z_1)$$

3. Algebra:

$$p_1 = (\gamma + \rho a_z)\Delta z = \rho(g + a_z)\Delta z$$

4. Evaluation of pressure:

$$p_1 = 10^3 \text{ kg/m}^3 \times (9.81 + 100) \text{ m/s}^2 \times 0.1 \text{ m}$$

$$p_1 = \boxed{10.9 \times 10^3 \text{ Pa} = 10.9 \text{ kPa, gage}}$$

EXAMPLE 4.3

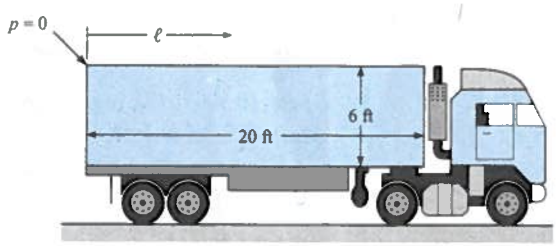
Applying Euler's Equation to Gasoline in a Decelerating Tanker

Problem Statement

The tank on a trailer truck is filled completely with gasoline, which has a specific weight of 42 lbf/ft³ (6.60 kN/m³). The truck is decelerating at a rate of 10 ft/s² (3.05 m/s²).

a. If the tank on the trailer is 20 ft (6.1 m) long and if the pressure at the top rear end of the tank is atmospheric, what is the pressure at the top front?

b. If the tank is 6 ft (1.83 m) high, what is the maximum pressure in the tank?



Define the Situation

Situation: Decelerating tank of gasoline with pressure equa zero gage at top rear end.

Assumptions:

1. Deceleration is constant.
2. Gasoline is incompressible.

Properties: $\gamma = 42\ \text{lbf/ft}^3\ (6.60\ \text{kN/m}^3)$

State the Goal

Find:

1. Pressure (psfg and kPa, gage) at top front of tank.
2. Maximum pressure (psfg and kPa, gage) in tank.

Make a Plan

1. Apply Euler's equation, Eq. (4.15), along top of tank. Elevation, z, is constant.
2. Evaluate pressure at top front.
3. Maximum pressure will be at front bottom. Apply Euler's equation from top to bottom at front of tank.
4. Using result from step 2, evaluate pressure at front bottom.

Take Action (Execute the Plan)

1. Euler's equation along the top of the tank

$$\frac{dp}{d\ell} = -\rho a_\ell$$

Integration from back (1) to front (2)

$$p_2 - p_1 = -\rho a_\ell \Delta\ell = -\frac{\gamma}{g} a_\ell \Delta\ell$$

2. Evaluation of p_2 with $p_1 = 0$

$$p_2 = -\left(\frac{42\ \text{lbf/ft}^3}{32.2\ \text{ft/s}^2}\right) \times (-10\ \text{ft/s}^2) \times 20\ \text{ft}$$

$$= \boxed{261\ \text{psfg}}$$

In SI units

$$p_2 = -\left(\frac{6.60\ \text{kN/m}^3}{9.81\ \text{m/s}^2}\right) \times (-3.05\ \text{m/s}^2) \times 6.1\ \text{m}$$

$$= \boxed{12.5\ \text{(kPa gage)}}$$

3. Euler's equation in vertical direction

$$\frac{d}{dz}(p + \gamma z) = -\rho a_z$$

4. For vertical direction, $a_z = 0$. Integration from top of tank (2) to bottom (3):

$$p_2 + \gamma z_2 = p_3 + \gamma z_3$$

$$p_3 = p_2 + \gamma(z_2 - z_3)$$

$$p_3 = 261\ \text{lbf/ft}^2 + 42\ \text{lbf/ft}^3 \times 6\ \text{ft} = \boxed{513\ \text{psfg}}$$

In SI units

$$p_3 = 12.5\ \text{kN/m}^2 + 6.6\ \text{kN/m}^3 \times 1.83\ \text{m}$$

$$p_3 = \boxed{24.6\ \text{kPa gage}}$$

4.6 Applying the Bernoulli Equation along a Streamline

Because the Bernoulli equation is used frequently in fluid mechanics, this section introduc this topic.

Derivation of the Bernoulli Equation

Select a particle on a streamline (Fig. 4.26). The position coordinate s gives the particle's po tion. The unit vector $\mathbf{u}_t$ is tangent to the streamline, and the unit vector $\mathbf{u}_n$ is normal to t streamline. Assume steady flow so the velocity of the particle depends on position only. That $V = V(s)$.

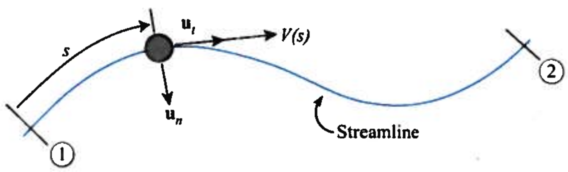


FIGURE 4.26 Sketch used for the derivation of the Bernoulli equation.