I. Introduction

Unsteady flow problems in engineering practice are of significant importance because they can cause excessive pressures, noise, cavitation and vibration far beyond that indicated by steady flow analysis. In fact, the problems created by hydraulic transients may be so severe as to constitute actual or performance failure of a system.

1.1 Unsteady Flow Analysis

The analysis of unsteady flow in pipeline systems can be divided into two broad categories. The first, called "surge" or "rigid water column" theory treats the fluid as an inelastic substance wherein pressure changes propagate instantaneously throughout the system and elastic properties of the pipe walls are of no consequence. The equations describing this type of flow are generally ordinary differential equations which can be solved in closed form or with relatively straight-forward numerical techniques. Where applicable, this approach is the easiest to apply and should always be considered as a possibility to adequately approximate problems under consideration.

The second category of problems are classified under "elastic" or "water hammer" theory wherein the elasticity of both the fluid and the pipe walls is taken into account in the calculations. Pressure waves created by velocity changes depend on these elastic properties and they propagate throughout the pipeline system at speeds depending directly on these elastic properties. While the elastic theory more accurately reflects the behavior of the unsteady flow system, successful analysis hinges on the ability to solve two nonlinear partial differential equations. As a consequence, the analysis is more complex and difficult to manage than for inelastic theory. However, Streeter and Wylie [1] have demonstrated that with the assistance of a highspeed digital computer, the method of characteristics can be applied to solve the equations in a relatively general and easily understood manner. Their text represents a compilation of computer analysis techniques and is the most significant book in the area of water hammer analysis to be published in years.

Before computer analysis the general equations describing water hammer in pipeline systems were simplified in some manner to permit solution by arithmetic, graphical or algebraic means. Nonlinear terms were neglected, friction was included by lumping or approximating, or it was left out altogether. Matching of boundary conditions at pumps and turbines was, at best, difficult and understood by relatively few engineers. Today, modern analysis techniques, including numerical methods of solving partial differential equations, has brought within reach of most engineers the capability of solving accurately a wide range of water hammer problems. Although digital computers are needed, they need not be large. Undoubtedly in the near future certain simple types of unsteady flow problems will even be solved on programmable "pocket" calculators.

Accordingly, it is the purpose of this book to make available to the engineering profession the means of employing both rigid water column theory and elastic theory in solving problems related to the design of pipeline systems. Sophisticated and obscure points and developments will not be included. Methods of analysis other than recent computer-oriented methods will not be addressed. The emphasis will be placed on providing a readable self-study book with which the engineer can instruct himself on unsteady flow analysis with enough applications and computer programs included to give him a start in building his own library of programs.

Because the emphasis in this work is on water hammer analysis, the history of unsteady flow without elastic effects will not be included. Rather a brief history of water hammer analysis will be presented to give the reader a perspective on the evolution of this type of analysis over the last 100 years.

1.2 History of Water Hammer Analysis

While it is difficult to establish the beginnings of unsteady flow analysis in pipelines in general, it certainly dates back to early in the 19th Century. However, water hammer analysis history is more readily documented. Some of the earliest work, according to Wood [2], was when Wilhelm Weber in the 1850's measured the effects of pipe wall elasticity on wave propagation speed. He also developed the continuity and fluid dynamic equations which were the basis for later analytical studies. Wood [2] also states that in 1875 Marey published the results of his careful laboratory work which proved wave speed was constant for a given situation and depended on pipe elasticity. In 1878, Korteweg considered both pipe and fluid elasticity in developing an equation for wave speed and his equation is essentially that used today.

Again from Wood we learn that Jules Michaud first dealt experimentally with water hammer in 1878 by using air chambers and pressure relief valves in pipelines to reduce the effects of sudden gate or valve closures. In 1883, Grameka published an analysis showing the effects of friction but he was unable to solve the equations.

It is less difficult to identify the beginnings of water hammer analysis wherein fluid and pipe elasticity are important in the computation of water hammer pressure. According to Rouse and Ince [3], Nicolai Joukowsky in 1898 was clearly the first to show that the pressure rise in a water line was related to the change in flow velocity, the wave speed and the fluid density. However, Wood [2] states that in a less well known but equally important study, J. P. Frizell in 1897 conducted an analysis of the effect of water hammer pressures on speed regulation of a hydroelectric plant turbine in Ogden, Utah. Apparently, without knowledge of European work, he developed his own wave speed and pressure intensity equations for sudden valve closure. He also noted the effect of branched lines and wave reflection including the relationship between gate closure time and wave period.

At virtually the same time, Nicolai Joukowsky in Moscow published a report of his analytical and experimental studies of water hammer as it related to the Moscow municipal water system. Joukowsky was well-acquainted with previous work. He derived equations for wave speed and pressure increase and considered the problems of pressure wave propagation into smaller pipes, wave reflection from open pipes, the relationship between gate closure time and wave period, effects of air chambers, and the use of spring-controlled surge valves. Certainly, in retrospect, Frizell and Joukowsky would have to share the title of fathers of water hammer analysis.

The next giant to appear in the field of water hammer analysis was Lorenzo Allievi. According to Rouse et al. [3], in 1913, this Italian hydraulician created a mathematical and graphical treatment of water hammer problems which was the foundation for further developments in the field for the next 50 years. His contribution is too monumental to review in any detail.

The first part of the 20th Century was devoted to applying the work of Joukowsky and Allievi to water hammer problems. By a quirk of history, Frizell's significant contribution was largely ignored. Much of the work seemed directed to the problems associated with hydraulic turbines in hydroelectric plants. Most of the books available today [4, 5, 6, 7] are largely related to this application.

In the 1930's, friction was included in the analysis of water hammer problems and the First Symposium on Water Hammer was held in Chicago in 1933. Topics covered included high-head penstocks, compound pipes, surge tanks, centrifugal pump installations with air chambers and surge relief valves.

In 1937, the Second Water Hammer Symposium was held in New York with presentations by both American and European engineers. The leaders in the field were in attendance as papers were presented on air chambers, surge valves, water hammer in centrifugal pump lines and effects of friction on turbine governing.

During this period graphical techniques of analysis thrived under the work of Allievi, Angus, Bergeron, Schnyder, Wood, Knapp, Paynter, and Rich. In later years moves were made to more accurately incorporate frictional effects into the equations. Also more sophisticated boundary conditions were employed and more general forms of the basic equations were used in analysis.

The arrival of the 1960's and the advent of the high-speed digital computer heralded the beginning of a new era in water hammer analysis. The work of Streeter and Wylie [1] in showing the application of the computer to complete and comprehensive water hammer analysis opened the door to the engineering profession at large to consider water hammer analysis as part of normal design procedures without the need of hiring one of the few individuals previously capable of performing an analysis. In this way, as much as any other, Victor Streeter should rank along with Allievi and Joukowsky as one of the outstanding contributors in water hammer history.

Today the emphasis on unsteady flow analysis is almost entirely concentrated on computer applications. Since the appearance of Streeter and Wylie's [1] book only one other reference in recent years has appeared. In 1970 Tullis [8] published the proceedings of an institute on the control of flow in closed conduits held at Colorado State University. Interestingly enough, this year two other works in addition to this one are appearing in print. Wylie and Streeter [9] are offering a text on hydraulic transients in closed conduit systems and Martin [10] is also preparing a book on transient analysis. Hopefully, this trend to make available the latest techniques in unsteady flow analysis will continue in future years.

II. Fundamental Concepts

Before moving into the details of the unsteady flow analysis it is important to develop an understanding of the action of water hammer in a simple situation. Such an understanding will help determine when to apply elastic theory and understand the sequence of events occurring in later more complicated problems.

Because including friction in unsteady flow analysis is important and because it may be necessary to apply the analysis to a wide variety of problems including other liquids as well as water, a pipe friction formula must be used which is sufficiently versatile to encompass these needs. Accordingly, the Darcy-Weisbach formula

$$h_f = f \frac{L}{d} \frac{V^2}{2g} \qquad \text{(2-1)}$$

will be used in conjunction with the Moody Diagram which can be used to find the friction factor f. The reader would be well-advised to review the basis for this formula and its application by referring to any recent elementary fluid mechanics text.

2.1 Simplified Description of Water Hammer

To grasp a basic understanding of the action of a liquid pipe system under the action of water hammer waves, it is easiest to consider as simple a system as possible. The system we will examine is shown in Figure 2-1 as a horizontal, constant-diameter pipe leading from a reservoir to some unknown destination far downstream. A valve is placed a distance L from the reservoir. Friction in the line is assumed negligible to simplify the analysis; and because velocity heads are generally quite small in relation to water hammer pressures, the difference between the energy gradeline (EL) and the hydraulic gradeline (HGL) will be neglected.

Water hammer will be introduced into the system by suddenly closing the valve. The activity will occur both upstream and downstream of the valve but for our purposes, we will observe only what occurs upstream of the valve.

Upon sudden closure of the valve the velocity of water at the valve is forced suddenly to zero. As a consequence, the pressure head at the valve increases suddenly by an amount ΔH (see figure). The magnitude of ΔH is

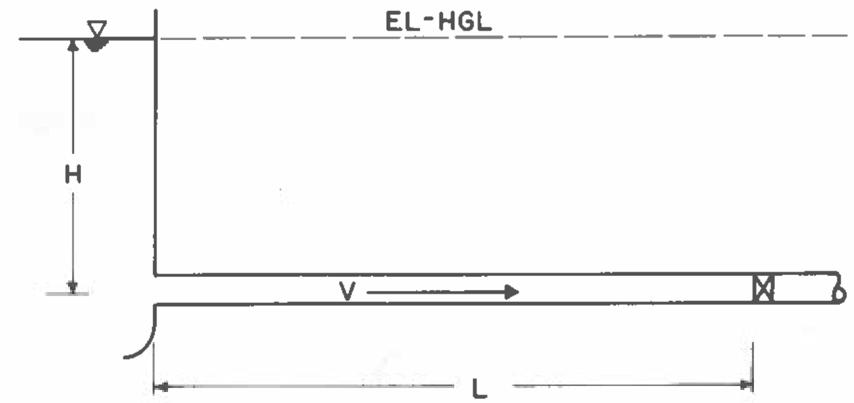


Figure 2-1. Steady state flow situation for simple water hammer.

just the amount of pressure head necessary to change the momentum of the liquid flowing at velocity V at the valve to zero.

The increase in pressure at the valve results in a swelling of the pipe and an increase in the density of the liquid. The amount of pipe stretching and liquid volume decrease depends on the pipe material and size and the liquid elasticity. Generally, for common pipe materials and liquids, the percentage change is less than 0.5 percent. The deformation has been greatly exaggerated in Figure 2-2 for purposes of illustration.

The pressure increase propagates upstream at a wave speed of **a**, which is determined by the elastic properties of the system and liquid and the system geometry. The wave speed will remain constant so long as the above remain constant. Traveling at a speed **a**, the wave will reach the reservoir in a time L/**a**. At this time the velocity in the pipe is everywhere zero, the pressure head is everywhere H + ΔH, the pipe is stretched and the fluid is compressed.

Under these conditions the liquid in the pipe is under a condition of non-equilibrium because the pressure head in the reservoir is only H. As a result, flow begins to occur toward the reservoir as the distended pipe ejects liquid in that direction. The reverse velocity is equal in magnitude to the initial steady velocity (as a result of neglecting friction) and the source of liquid for the reverse flow is the liquid previously stored in the stretched pipe walls as compressed liquid.

This process continues and at time 2 L/**a**, the pressure has returned to normal (but with reverse flow occurring) throughout the pipe. However, there is no source of liquid at the valve to supply the upstream flow hence the pressure head drops an additional ΔH to force the reverse velocity to zero. This drop in pressure causes the pipe to shrink and the liquid to expand.

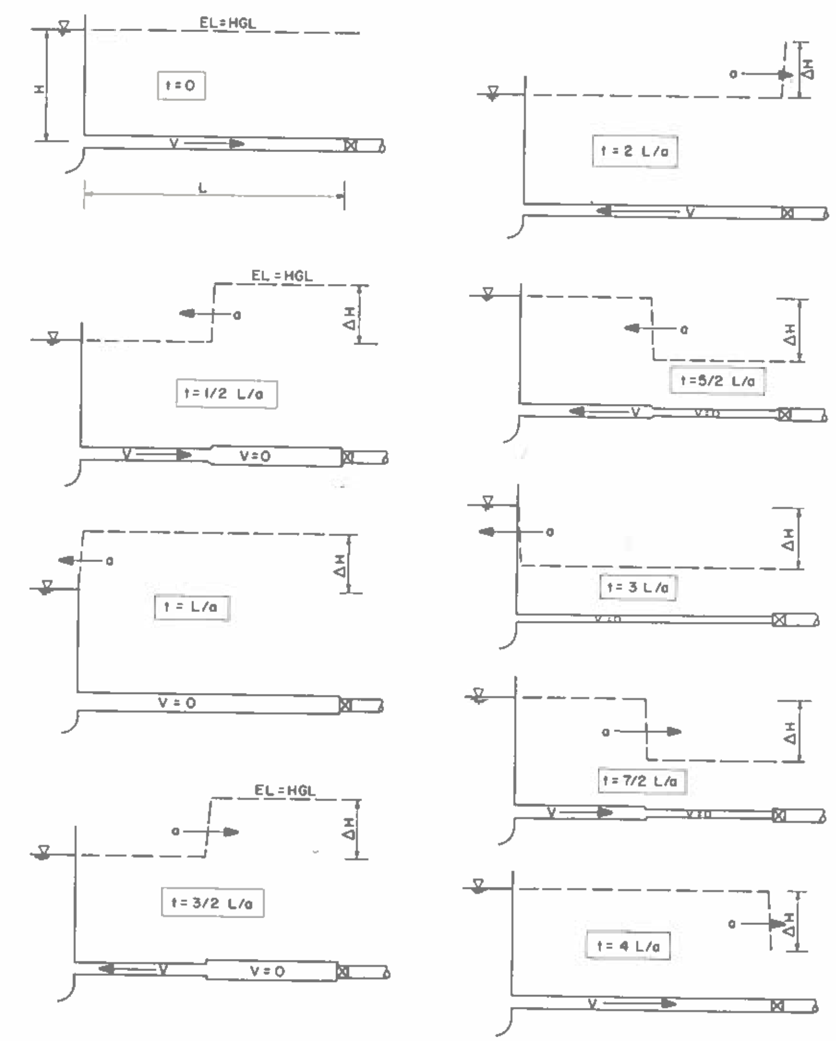


Figure 2-2. Pressure wave propagation in a simple pipe system.

At time 3 L/a this effect has propagated to the reservoir and the velocity of flow is everywhere zero. However, the pipe pressure head is ΔH below that of the reservoir. Consequently, the pipe sucks in liquid from the reservoir creating a velocity of flow equal to and in the same direction as the original steady flow. While this is occurring the pressure in the pipe is also returning to its original value.

After time 4 L/a this wave has reached the valve and at this instant the flow is identical to its original steady state configuration. This elapsed time constitutes one wave period. As time goes on, this cycle of events will continue without abatement (in the absence of friction).

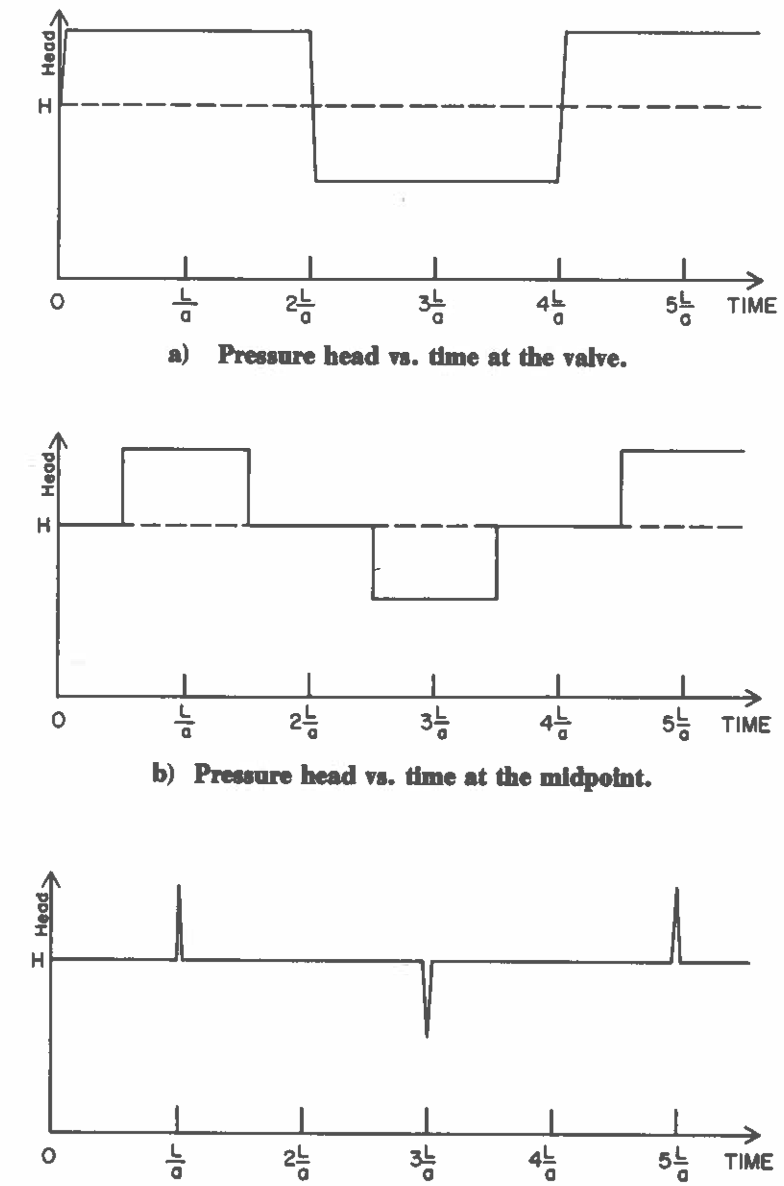
Some fundamental concepts can be gained from examining more closely what occurs in this system. For example, it is clear that the time parameter which best describes the sequence of events in a meaningful fashion is not time alone but the ratio L/a. It is informative to plot the pressure head at various points in the pipeline as a function of time as shown in Figure 2-3. Note the pressue head as the valve fluctuates between H ± ΔH whereas the pressure head at other locations also experiences periods of time when pressure head is H.

One basic point can be made from Figure 2-3b. Note that the pressure does not increase at a point until enough time has occurred for the wave to travel from the closed valve. Once the pressure head has increased, it remains there only long enough for "relief" to arrive back from the reservoir. This idea of "time of communication" or "message propagation time" is fundamental to a good understanding of the happenings in a system undergoing water hammer.

A second important point can be seen by examining Figure 2-3a more closely. Suppose that instead of closing the valve suddenly, we were to close it in 10 steps, each increasing the pressure head at the valve by ΔH/10. A further requirement would be that the complete closure of the valve would be accomplished before 2 L/a seconds had elapsed. It is clear that the pressure head at the valve would still build up to the full ΔH value because "relief" from the reservoir could not arrive before 2 L/a seconds. The point to be made is that a valve need not be closed suddenly to create the maximum water hammer pressure. Indeed, any closure time less than the time necessary for relief to return from a reservoir (a larger pipe may suffice) will result in full water hammer pressures. In fact, as we will see later, because of the nature of the way a valve shuts off flow in a pipeline by creating large head losses, it may be necessary to close the valve in a time much greater than 2 L/a to prevent high pressures from occurring.

2.2 Unsteady Flow in Piping Systems

Unsteady flow in piping systems is a common occurrence. Indeed, steady flow is so rare that one might question the advisability of devoting so much time to a study of its behavior. Virtually, all hydraulic design is based on steady flow analysis and to a significant degree, the unsteadiness



a) Pressure head vs. time at the valve.

b) Pressure head vs. time at the midpoint.

c) Pressure head vs. time just inside the pipe at the reservoir.

Figure 2-3. Pressure head vs. time at three locations along the pipe.

occurring in the pipeline systems is of little consequence because of its transient nature and its small magnitude of change. It is with those few cases wherein significant changes in velocity can cause large changes in pressure that we are concerned.

As discussed earlier, unsteady flows are divided into two categories, depending on the type of analysis required to accurately describe the flow behavior. Often it is not clear which type of analysis should be used because there is no distinct line of demarcation between the two areas of application. On the other hand, there are cases where it is obvious which type of approach should be used. For example, if a large storage tank 50 ft in diameter and 75 ft tall were to be drained through a 6-inch pipeline 1000 ft long, it would be foolish to use elastic theory in a traditional water hammer analysis. Yet, if during the draining process, there was the possibility of having to close the discharge valve suddenly, then significant water hammer could occur and elastic theory should be used.

2.3 The Unsteady Flow Equation

From earlier discussions it is clear that whenever changes in velocity in a pipe system are so slow that the elastic wave has time to propagate throughout the system many times during the period of change, then rigid water column theory can be applied. When elastic effects are ignored, the development of an appropriate equation is relatively easy. The resulting equation is referred to as the one-dimensional unsteady flow equation. Two- or three-dimensional equations have no practical application to pipeline flow so they will not be considered.

The unsteady flow equation, known widely in differential form as the Euler equation, is derived by applying Newton's Second Law to a small cylindrical fluid particle. Considering only the streamline direction,

$$\Sigma F_s = m\,a_s = m\frac{dv}{dt}$$

Substituting the force components and mass from Figure 2-4 into this equation results in

$$p\Delta A - \left(p + \frac{\partial p}{\partial s}\Delta s\right)\Delta A - W\sin\theta - \tau\Delta s\pi d = \frac{W}{g}\frac{dv}{dt} \quad \text{.......(2-2)}$$

After some manipulation, we end up with the one-dimensional Euler equation

$$-\frac{1}{\gamma}\frac{\partial p}{\partial s} - \frac{\partial z}{\partial s} - \frac{4\tau}{\gamma d} = \frac{1}{g}\frac{dv}{dt} \quad \text{..........................(2-3)}$$

Expanding the particle diameter to the size of the pipe cross-section and introducing the average velocity V gives a more useful equation

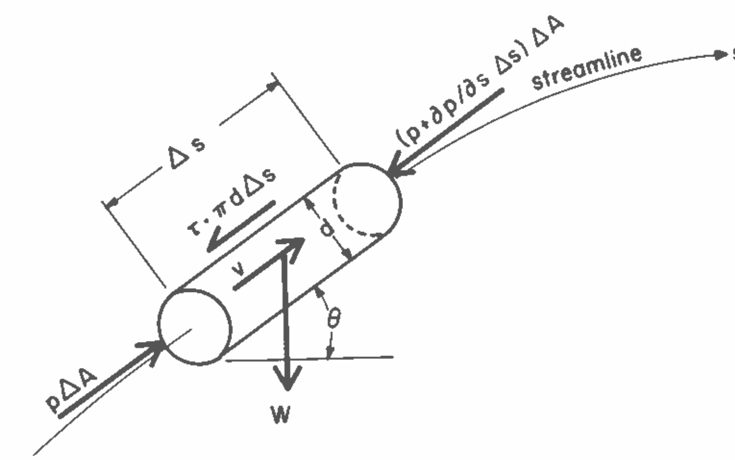


Figure 2-4. Definition sketch for unsteady flow equation derivation.

$$-\frac{1}{\gamma}\frac{\partial p}{\partial s} - \frac{\partial z}{\partial s} - \frac{4\tau_o}{\gamma D} = \frac{1}{g}\frac{dV}{dt} \quad \text{..........................(2-4)}$$

where D is the pipe diameter and τ_o is the shear stress at the wall.

Because the above form with the shear stress τ_o is not directly useful, we will substitute a reaction between τ_o and the Darcy-Weisbach friction factor f. The result of this substitution is

$$-\frac{1}{\gamma}\frac{\partial p}{\partial s} - \frac{\partial z}{\partial s} - \frac{f}{D}\frac{V^2}{2g} = \frac{1}{g}\frac{dV}{dt} \quad \text{..........................(2-5)}$$

Recognizing that z is a function only of s and represents the elevation above some datum of the pipe centerline, we can change the partial derivative to a total derivative. Finally, the equation has the form

$$-\frac{1}{\gamma}\frac{\partial p}{\partial s} - \frac{dz}{ds} - \frac{f}{D}\frac{V^2}{2g} = \frac{1}{g}\frac{dV}{dt} \quad \text{..........................(2-6)}$$

Familiarization with the application of the equation will be gained by examining several different flow situations.

III. Rigid Water Column Theory

The unsteady flow equation can be used to solve a wide range of pipeline problems which fall within the domain of rigid water column theory. We will begin with some of the simple problems and proceed to more comprehensive ones.

3.1 Flow Establishment in a Horizontal Pipe

If the discharge in the pipeline shown in Figure 3-1 is controlled by the valve at the downstream end, the pressure in the pipe is everywhere equal to H_0 when the valve is closed. When the valve is suddenly opened, the pressure at the valve drops instantly to zero and the fluid begins to accelerate.

The equation describing this flow is obtained by integrating Equation 2-6 with respect to s.

$$-\int_L \frac{1}{\gamma}\frac{\partial p}{\partial s}\,ds - \int_L \frac{dz}{ds}\,ds - \int_L \frac{fV^2}{2gD}\,ds = \int_L \frac{1}{g}\frac{dV}{dt}\,ds$$

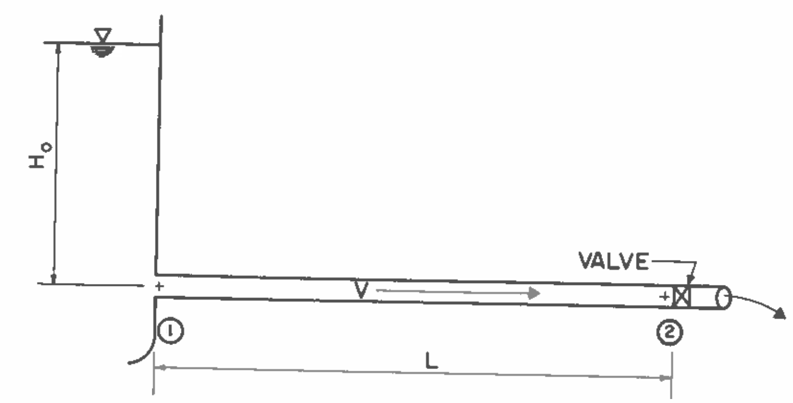


Figure 3-1. Simple system for applying rigid water column theory.

In a horizontal constant-diameter pipe, the integration is made quite easy because $(dz/ds) = 0$ and V is a function of time only. We also assume the f-value in unsteady flow is the same as for a steady flow at a velocity equal to the instantaneous value. The result is

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} - \frac{fL}{2gD} V^2 = \frac{L}{g}\frac{dV}{dt} \quad \text{(3-1)}$$

Because the pressure head p_1/γ = constant = H_o and because $p_2/\gamma = 0$ for $t > 0$, the equation is reduced to

$$H_o - \frac{fL}{2gD} V^2 = \frac{L}{g}\frac{dV}{dt} \quad \text{(3-2)}$$

Integration is performed by separating the variables to form

$$\int dt = \frac{L}{g} \int \frac{dV}{H_o - \frac{fL}{2gD} V^2} \quad \text{(3-3)}$$

The integration gives the following equation for the time necessary to accelerate the flow to a given velocity V.

$$t = \sqrt{\frac{LD}{2gfH_o}} \log \frac{\sqrt{\frac{2gD\,H_o}{fL}} + V}{\sqrt{\frac{2gD\,H_o}{fL}} - V} \quad \text{(3-4)}$$

Recognizing that $\sqrt{2g\,H_oD/fL} = V_o$, the steady state velocity, the equation for t becomes

$$t = \frac{LV_o}{2g\,H_o} \log \frac{V_o + V}{V_o - V} \quad \text{(3-5)}$$

It is important to note that as steady flow is approached, $V \rightarrow V_o$ and as a consequence $t \rightarrow \infty$. Of course this answer is unacceptable so we propose that when $V = 0.99\,V_o$, we have essentially steady flow. With this interpretation,

$$t_{99} = 2.65 \frac{L\,V_o}{g\,H_o} \quad \text{(3-6)}$$

Example 3-1

A horizontal pipe 24 inches in diameter and 10,000 feet long leaves a reservoir 100 feet below the surface and terminates in a valve. The steady state friction factor is 0.018 and it is assumed to remain constant during the acceleration process.

If the valve opens suddenly, calculate how long it will take for the velocity to reach 99 percent of its final values. Neglect minor losses.

Solution

(3-6) $$t_{99} = 2.65 \frac{L\,V_o}{g\,H_o} = \frac{2.65 \times 10{,}000 \times V_o}{32.2 \times 100}$$

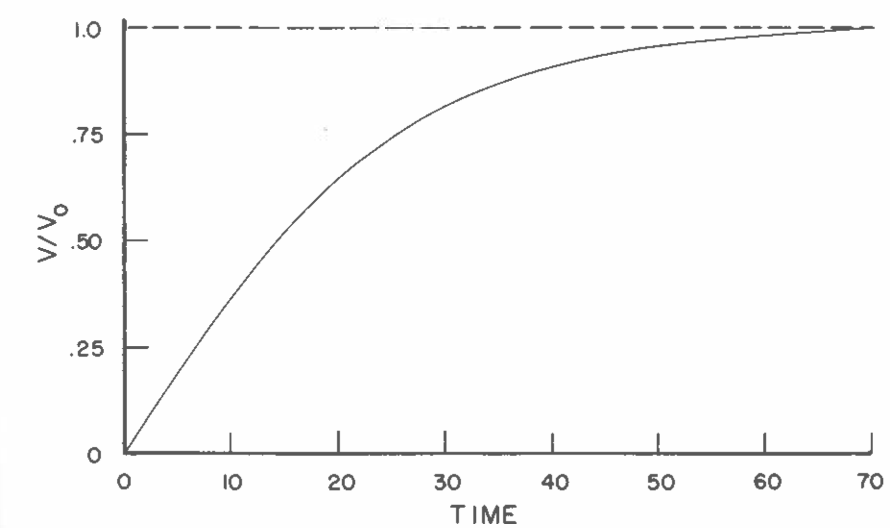
Solving for V_o, $h_f = f \frac{L}{D} \frac{V_o^2}{2g}$

$$100 = \frac{0.018 \times 10{,}000 \times V_o^2}{2 \times 64.4}, \quad V_o = 8.46 \text{ fps}$$

Substituting into (3-6), $t_{99} = \frac{2.65 \times 10{,}000 \times 8.46}{32.2 \times 100}$

$t_{99} = 70$ sec.

The following graph illustrates how the velocity approaches its steady state value with time.



3.2 Pressure Caused by Valve Closure in a Horizontal Pipe

Valve closure can cause some analysis problems beyond those of instantaneous valve openings. This possibility is apparent when one considers the rapid valve closure which caused the elastic water hammer problem discussed in an earlier chapter. The difficulty occurring in this problem is precipitated by the fact that the pressure just upstream of the valve is no longer zero, but is determined by loss characteristics of the flow through the valve.

Figure 3-1 can still be used to represent the problem. At t = 0 the velocity is V_0 and the EL - HGL is approximately a straight line between the reservoir surface and the pipe outlet (neglect minor losses) under steady flow conditions.

The differential equation representing this problem is the same as Equation 3-1

$$H_0 - \frac{p_2}{\gamma} - \frac{fL}{2gD} V^2 = \frac{L}{g}\frac{dV}{dt} \quad \text{(3-7)}$$

Unfortunately, there are two dependent variables so we need another equation.

The second equation devolves from an energy equation written across the valve

$$\frac{p_2}{\gamma} = K_L \frac{V^2}{2g} \quad \text{(3-8)}$$

where K_L is the valve loss coefficient. Substituting this equation into Equation 3-7 gives

$$H_0 - \left(K_L + \frac{fL}{D}\right)\frac{V^2}{2g} = \frac{L}{g}\frac{dV}{dt} \quad \text{(3-9)}$$

If K_L were a constant, integration would proceed as with the flow establishment case. However, K_L is a function of the amount the valve is open. Further complicating the problem is the fact that there is not an equation directly relating K_L to either time or velocity. Hence, the solution to the differential equation must be a numerical one.

The approach would be to write the equation in finite difference form. With a valve closing schedule specified, the value of K_L would be known at any time and would be assumed constant over each Δt time interval. One form of the equation would be

$$V(t+\Delta t) = V(t) + \frac{g\Delta t}{L}\left[H_0 - \left(K_L(t) + \frac{f(t)L}{D}\right)\frac{V^2(t)}{2g}\right]$$

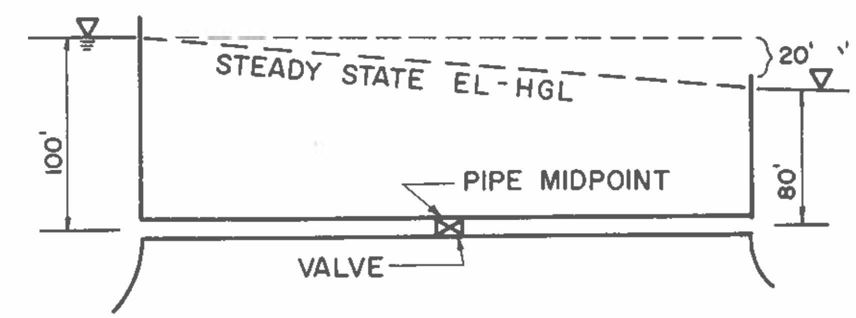
where each of the variables on the right hand side would be evaluated at time t.

That there is indeed a limit of applicability to this approach can be seen with Equation 3-7. As faster and faster valve closure times are used, dV/dt becomes quite large and, in the limit, goes to infinity. According to Equation 3-7, in the limit $p_2/\gamma \rightarrow \infty$ also. The point at which rigid water column theory fails to give acceptable results and a move to elastic theory is necessary is hard to establish, because it depends on the individual problem and the accuracy in analysis required.

Example 3-2

Water flows from one reservoir to another through the pipe at a velocity of 10 fps. The shutdown plan calls for a valve closure scheme which will cause the velocity to decrease linearly to zero in 100 seconds. The valve is located at the center of a 6440-ft long pipeline.

Estimate the maximum and minimum pressures which will occur in the system, locate them and give the time at which they will occur.



Solution

The general form of the unsteady flow equation applying to this situation is

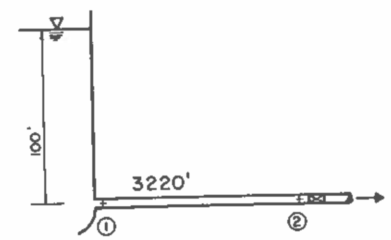
$$\frac{p_2}{\gamma} - \frac{p_1}{\gamma} + \frac{fL}{2g}\frac{V^2}{D} + \frac{L}{g}\frac{dV}{dt} = 0$$

Given that the velocity will decrease linearly with time,

$$\frac{dV}{dt} = \frac{-10}{100} = -0.10 \text{ ft/sec}^2$$

To solve the problem, we will consider it in two sections.

Upstream section



$$\frac{p_2}{\gamma} - \frac{p_1}{\gamma} + \frac{fL}{2gD}V^2 + \frac{L}{g}\frac{dV}{dt} = 0$$

$$\frac{p_2}{\gamma} = 100 - \frac{fL}{2gD}V^2 - \frac{3220}{g}(-0.10)$$

$$= 100 + 100\,(0.10) - \frac{fL}{2gD}V^2$$

$$\frac{p_2}{\gamma} = 110 - \frac{fL}{2gD}V^2$$

Because we are looking for extreme values of pressure, it is clear that $(p_2/\gamma)_{max} = 110$ ft when $V = 0$ which is at the instant of valve closure.

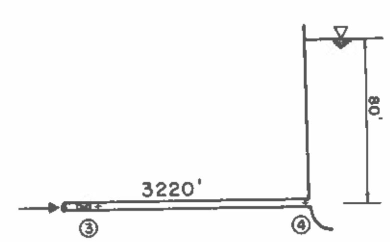
The minimum pressure occurs at steady flow where $p_2/\gamma = 90$ ft. An instant after the valve begins to close, p_2/γ jumps to 100 ft.

In summary,

$$(p_2/\gamma)_{max} = 110 \text{ ft at } t = 100 \text{ sec}$$

$$(p_2/\gamma)_{min} = 90 \text{ ft just before valve starts to close}$$

Downstream section



$$\frac{p_4}{\gamma} - \frac{p_3}{\gamma} + \frac{fL}{2gD}V^2 + \frac{L}{g}\frac{dV}{dt} = 0$$

$$\frac{p_3}{\gamma} = 80 + \frac{fL}{2gD}V^2 + \frac{3220}{g}(-0.10)$$

$$= 80 - 10 + \frac{fL}{2gD}V^2$$

$$\frac{p_3}{\gamma} = 70 + \frac{fL}{2gD}V^2$$

Under steady flow conditions, $p_3/\gamma = 90$ ft. At the instant the valve begins to move, p_3/γ drops to 80 ft.

At the instant of valve closure, $p_3/\gamma = 70$ ft.

In summary,

$(p_3/\gamma)_{max}$ = 90 ft at steady flow just before valve begins to close
$(p_3/\gamma)_{min}$ = 70 ft at t = 100 sec

Both sections

$(p/\gamma)_{max}$ = 110 ft at upstream side of valve at t = 100 sec
$(p/\gamma)_{min}$ = 70 ft at downstream side of valve at t = 100 sec

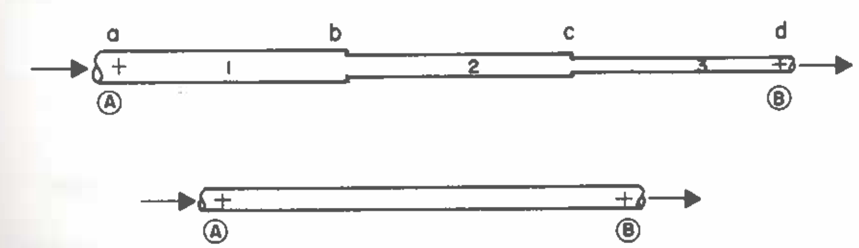
3.3 Unsteady Flow in Series Pipes

Engineers are generally confronted with piping systems which are more complex than single constant-diameter pipes. This section illustrates how to reduce a series pipe to a dynamically equivalent single pipe so that the previous analysis techniques for single pipes can be used.

The concept of equivalent pipes is familiar to engineers. It is possible to replace any minor loss or any given pipe with another pipe of any convenient diameter. The only concern is that both the actual system and the equivalent system have the same frictional losses at the given flow rate.

In applying the equivalent pipe idea to unsteady flow problems, the concept of equivalence must be extended to include dynamic behavior as well as friction. That is, the inertial effects of the actual and equivalent systems must be similar.

Using the three-part series pipe below as a general model, we will determine the relationships necessary to size the equivalent single diameter pipe shown.



Equivalent Pipe

The first requirement is that the frictional head loss between A and B be the same in each case. The equation is

$$H_{f_1} + H_{f_2} + H_{f_3} = H_{f_{eq.}}$$

$$K\frac{f_1L_1}{D_1^5}Q^2 + K\frac{f_2L_2}{D_2^5}Q^2 + K\frac{f_3L_3}{D_3^5}Q^2 = K\frac{f_{eq}L_{eq}}{D_{eq}^5}Q^2 \quad \ldots\ldots\ldots\ldots (3\text{-}10)$$

where K is a collection of numerical values including pi and g. Because the discharge in each section of the series pipe is the same at any instant, an expression for friction loss equivalent follows from the above.

$$\frac{f_{eq}L_{eq}}{D_{eq}^5} = \frac{f_1L_1}{D_1^5} + \frac{f_2L_2}{D_2^5} + \frac{f_3L_3}{D_3^5}$$

In more general terms for any number of pipes in series,

$$\left[\frac{fL}{D^5}\right]_{eq} = \sum_{i=1}^{N}\left[\frac{f_iL_i}{D_i^5}\right] \quad \ldots\ldots\ldots\ldots (3\text{-}11)$$

where N is the number of pipes in series.

Now considering the dynamic behavior of the flow in the series pipe, we begin by writing the unsteady flow equations for each section.

$$\frac{p_a}{\gamma} - \frac{p_b}{\gamma} - H_{f_1} = \frac{L_1}{g}\frac{dV_1}{dt} = \frac{L_1}{g}\frac{1}{A_1}\frac{dQ}{dt}$$

$$\frac{p_b}{\gamma} - \frac{p_c}{\gamma} - H_{f_2} = \frac{L_2}{g}\frac{dV_2}{dt} = \frac{L_2}{g}\frac{1}{A_2}\frac{dQ}{dt}$$

$$\frac{p_c}{\gamma} - \frac{p_d}{\gamma} - H_{f_3} = \frac{L_3}{g}\frac{dV_3}{dt} = \frac{L_3}{g}\frac{1}{A_3}\frac{dQ}{dt}$$

Adding the three equations together,

$$\frac{p_a}{\gamma} - \frac{p_d}{\gamma} - (H_{f_1} + H_{f_2} + H_{f_3}) = \left(\frac{L_1}{gA_1} + \frac{L_2}{gA_2} + \frac{L_3}{gA_3}\right)\frac{dQ}{dt}$$

Now, let us write the unsteady flow equation for the equivalent pipe.

$$\frac{p_A}{\gamma} - \frac{p_B}{\gamma} - H_{f_{eq}} = \frac{L_{eq}}{g}\frac{dV_{eq}}{dt} = \frac{L_{eq}}{g}\frac{1}{A_{eq}}\frac{dQ}{dt}$$

Noting that $p_a = p_A$ and $p_d = p_B$, and that friction loss equivalence gives

$$H_{f_{eq}} = H_{f_1} + H_{f_2} + H_{f_3}$$

we conclude

$$\frac{L_{eq}}{A_{eq}} = \frac{L_1}{A_1} + \frac{L_2}{A_2} + \frac{L_3}{A_3}$$

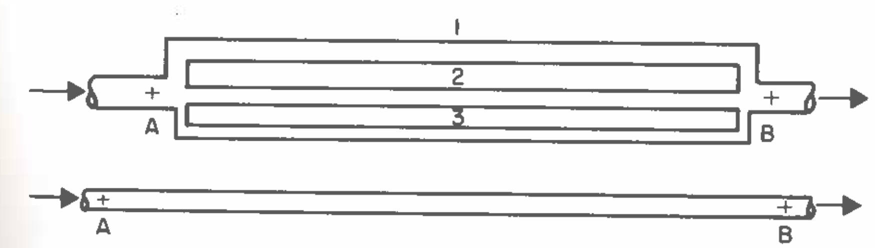
In more general form,

$$\left[\frac{L}{D^2}\right]_{eq} = \sum_{i=1}^{N}\left[\frac{L_i}{D_i^2}\right] \quad \ldots\ldots\ldots\ldots (3\text{-}12)$$

where N is the number of pipes in series. With Equations 3-11 and 3-12 and an arbitrarily picked f-value for the equivalent pipe, it is possible to solve for the length and diameter of the equivalent pipe. Once the equivalent pipe configuration is found, Equation 3-1 can be used to solve for pressures.

3.4 Unsteady Flow in Parallel Pipes

The development of an equivalent pipe for a parallel pipe system is similar to that previously done for series pipes. Again, we will use three pipes and generalize the results to any number of pipes.



Equivalent Pipe

Following the criterion of frictional head loss equivalence,

$$h_{f_1} = h_{f_2} = h_{f_3} = h_{f_{eq}}$$

This relationship leads to the equations

$$K\frac{f_1L_1}{D_1^5}Q_1^2 = K\frac{f_2L_2}{D_2^5}Q_2^2 = K\frac{f_3L_3}{D_3^5}Q_3^2 = K\frac{f_{eq}L_{eq}}{D_{eq}^5}Q_{eq}^2 \quad \ldots\ldots (3\text{-}13)$$

where

$$Q_1 + Q_2 + Q_3 = Q_{eq}$$

Substituting Equation 3-13 into the above expression for continuity gives

$$\left[\frac{f_{eq}\,L_{eq}}{D_{eq}^5}\right]^{1/2}\left[\frac{D_1^5}{f_1 L_1}\right]^{1/2} Q_{eq} + \left[\frac{f_{eq}\,L_{eq}}{D_{eq}^5}\right]^{1/2}\left[\frac{D_2^5}{f_2 L_2}\right]^{1/2} Q_{eq}$$

$$+\left[\frac{f_{eq}\,L_{eq}}{D_{eq}^5}\right]^{1/2}\left[\frac{D_3^5}{f_3 L_3}\right]^{1/2} Q_{eq} = Q_{eq} \quad \text{.........(3-14)}$$

Dividing out Q_{eq} and regrouping,

$$\left[\frac{D_{eq}^5}{f_{eq}\,L_{eq}}\right]^{1/2} = \sum_{i=1}^{N}\left[\frac{D_i^5}{f_i\,L_i}\right]^{1/2} \quad \text{...........................(3-15)}$$

Now, addressing the dynamic behavior of the parallel system, we write a dynamic equation for each pipe

$$\frac{p_A}{\gamma} - \frac{p_B}{\gamma} - H_{f_1} = \frac{L_1}{g}\frac{dV_1}{dt} = \frac{L_1}{g}\frac{1}{A_1}\frac{dQ_1}{dt}$$

$$\frac{p_A}{\gamma} - \frac{p_B}{\gamma} - H_{f_2} = \frac{L_2}{g}\frac{dV_2}{dt} = \frac{L_2}{g}\frac{1}{A_2}\frac{dQ_2}{dt}$$

$$\frac{p_A}{\gamma} - \frac{p_B}{\gamma} - H_{f_3} = \frac{L_3}{g}\frac{dV_3}{dt} = \frac{L_3}{g}\frac{1}{A_3}\frac{dQ_3}{dt}$$

$$\frac{p_A}{\gamma} - \frac{p_B}{\gamma} - H_{f_{eq}} = \frac{L_{eq}}{g}\frac{dV_{eq}}{dt} = \frac{L_{eq}}{g}\frac{1}{A_{eq}}\frac{dQ_{eq}}{dt}$$

Because of friction loss equivalence, the left hand side of the above equations are equal, giving

$$\frac{L_1}{A_1}\frac{dQ_1}{dt} = \frac{L_2}{A_2}\frac{dQ_2}{dt} = \frac{L_3}{A_3}\frac{dQ_3}{dt} = \frac{L_{eq}}{A_{eq}}\frac{dQ_{eq}}{dt}$$

Writing the equation of continuity in differential form,

$$dQ_1 + dQ_2 + dQ_3 = dQ_{eq}$$

and substituting expressions for the dynamic equation gives

$$\frac{L_{eq}}{A_{eq}}\frac{A_1}{L_1}\,dQ_{eq} + \frac{L_{eq}}{A_{eq}}\frac{A_2}{L_2}\,dQ_{eq} + \frac{L_{eq}}{A_{eq}}\frac{A_3}{L_3}\,dQ_{eq} = dQ_{eq}$$

Dividing out dQ_{eq},

$$\frac{A_{eq}}{L_{eq}} = \frac{A_1}{L_1} + \frac{A_2}{L_2} + \frac{A_3}{L_3}$$

Writing the areas in terms of diameter and generalizing

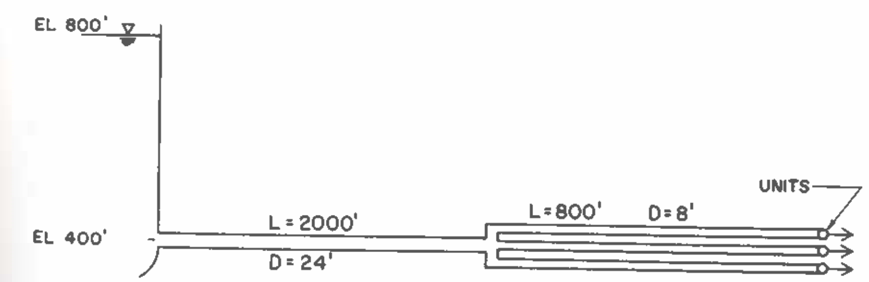
$$\frac{D_{eq}^2}{L_{eq}} = \sum_{i=1}^{N}\frac{D_i^2}{L_i} \quad \text{.................................(3-16)}$$

An example problem is included to illustrate the use of the equivalent pipe concept for complex pipe systems.

Example 3-3

A three-unit pumped storage facility is operating in the generating mode. During emergency shut-down, the wicket gates on the turbines are closed in such a manner that the velocities in the penstocks at the turbines decrease linearly from 60 fps to zero in 30 seconds.

Compute the maximum pressure head at the wicket gates during shut-down. Assume f-values are the same for all pipes.



Solution

First, the three parallel pipes are replaced by a single equivalent pipe.

$$\text{(3-16)} \qquad \frac{D_{eq}^2}{L_{eq}} = 3\left(\frac{8^2}{800}\right) = 0.240$$

$$(3\text{-}15) \qquad \left[\frac{D_{eq}^5}{L_{eq}}\right]^{1/2} = 3\left[\frac{8^5}{800}\right]^{1/2} = 19.20$$

Solving the above two equations simultaneously gives

D_{eq} = 11.54 ft and L_{eq} = 555 ft

Now, the pipe system has been reduced to the series pipe shown below.

Next, using the series pipe equivalence relationships we reduce the series pipe to a single constant-diameter pipe.

$$(3\text{-}12) \qquad \frac{L_{eq}}{D_{eq}^2} = \frac{2000}{24^2} + \frac{555}{11.54^2} = 7.64$$

$$(3\text{-}11) \qquad \frac{L_{eq}}{D_{eq}^5} = \frac{2000}{24^5} + \frac{555}{11.54^5} = 0.00296$$

Solving these two equations simultaneously gives

D_{eq} = 13.7 ft and L_{eq} = 1437 ft

Computing the velocity in the equivalent pipe,

$$V_{eq} = \frac{Q}{A_{eq}} = 61.3 \text{ fps}$$

Now, that we have a single constant diameter pipe, we can use the unsteady flow equation to solve for the maximum pressure.

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} - \frac{fL\,V^2}{2g\,D} = \frac{L}{g}\frac{dV}{dt}$$

$$400 - \frac{p_2}{\gamma} - \frac{fL\,V^2}{2g\,D} = \left(\frac{1437}{32.2}\right)\frac{0-61.3}{30}$$

$$\frac{p_2}{\gamma} = 400 + 91.2 - \frac{fL\,V^2}{2g\,D}$$

The above equation for p_2/γ is good during the deceleration process. It is clear from this equation that the highest pressure will result when $V = 0$, hence,

$(p_2/\gamma)_{max}$ = 491 ft (213 psi) at the time valve is completely closed

3.5 Accounting For Significant Minor Losses

If minor losses occur in the pipe system to the extent that they have a noticeable effect on the results, then they must be incorporated into the analysis. For a single pipe this can be done in two ways. In the first method the pipe is broken into two pieces (as in Example 3.2) and each portion is set up separately with the two solutions coupled at the minor loss location via an energy equation. The second method includes the minor loss in the differential equation along with the pipe friction term. This can be done by absorbing it into the pipe friction term by increasing the friction factor or simply adding it in as a separate term.

Assuming that the minor loss can be represented as $h_L = K_L (V^2/2g)$, these two latter techniques result in the following modifications of Equation 3-1, respectively

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} - \frac{f'L}{2gD}V^2 = \frac{L}{g}\frac{dV}{dt} \quad \text{..................(3-17)}$$

where

$$f' = f + K_L\frac{D}{L} \quad \text{................................(3-18)}$$

and

$$\frac{p_1}{\gamma} - \frac{p_2}{\gamma} - \left(f\frac{L}{D} + K_L\right)\frac{V^2}{2g} = \frac{L}{g}\frac{dV}{dt} \quad \text{...........(3-19)}$$

It is important not to use the traditional equivalent length method to represent the minor loss. This technique adds length to the pipe and the subsequent increase in liquid mass will distort the true dynamic behavior of the system.

If the pipe system is complex, the previous techniques must be applied to the individual components of the system. The second method discussed earlier is recommended with the technique of computing an f' to distribute the minor loss along the entire pipe. After this has been accomplished, the analysis proceeds as before.

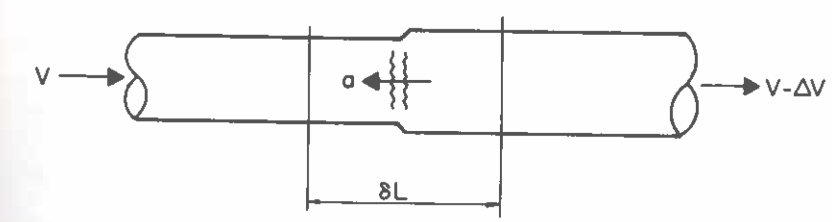
IV. Elastic Theory

For situations in which the velocity changes suddenly and the pipeline is relatively long, the elastic properties of the pipe and liquid enter into the analysis. In Chapter II we saw how a pipeline behaves under the action of a sudden closed valve. The suddenly closed valve caused an increase in pressure head ΔH to occur, which propagated at a speed **a**. It remains now to develop means to calculate ΔH and **a** and broaden the range of applications from that of the simple example in Chapter II.

The previously derived and integrated unsteady flow equations cannot be used because they have not included elastic effects. We will employ the impulse-momentum equation and the conservation of mass principle to develop an appropriate set of equations for an impulsive change in velocity.

4.1 The ΔH Equation

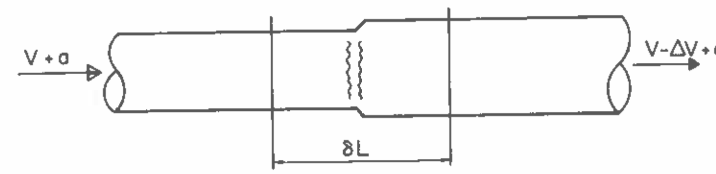
The impulse-momentum equation will be used to develop an equation for ΔH. We know that a change in velocity ΔV will cause a pressure head change ΔH to propagate upstream at some speed **a**. To begin, we will use a piece of pipe δL long, where δL is arbitrarily small but not differentially small as dL would be. The pressure wave and the pipe bulge (which is caused by the pressure head change ΔH) propagate upstream at a speed **a**. The wave speed in this work is defined as the speed relative to the observer at rest with respect to the pipe rather than the speed relative to the flowing water. In the case of relatively rigid pipes, either approach used gives essentially the same results. Because this is an unsteady flow situation, the impulse momentum equation for steady flow cannot be used. However, in



Unsteady Case

this case it is possible to use a translating coordinate system to transform the unsteady flow into a "steady" flow.

If we move our reference system to the left at a speed **a** we have, for all appearances, a steady flow.



Steady Flow

From basic fluid mechanics we have available the one-dimensional impulse-momentum equation

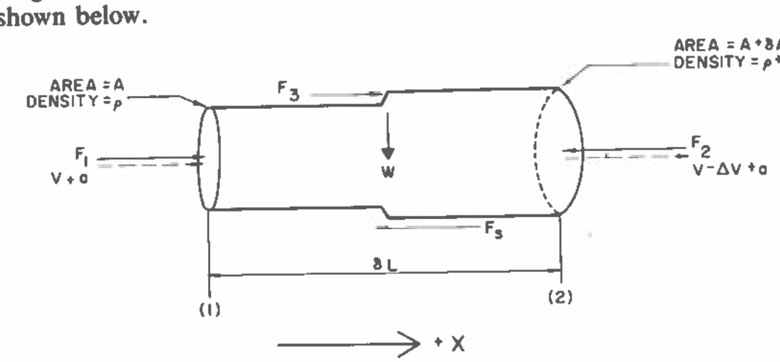
$$\Sigma \vec{F}_{ext} = (\Sigma Q \rho \vec{V})_{out} - (\Sigma Q \rho \vec{V})_{in} \quad \text{.......... (4-1)}$$

where Q is the discharge, ρ is the liquid density and $\Sigma \vec{F}_{ext}$ is the sum of the external forces acting. The momentum correction factor for nonuniform velocity profiles has been assigned the value of 1.0.

Considering only the component of this vector equation parallel to the pipe and noting that momentum enters and leaves the section of pipe δL long at only one section each, we can write

$$(\Sigma \vec{F}_{ext})_x = Q\rho (V_{out} - V_{in}) \quad \text{.......... (4-2)}$$

To apply the impulse-momentum equation we must specify a control volume and take into account all forces acting on the fluid in the control volume at a particular instant and at that same instant evaluate the momentum fluxes into and out of the control volume. We will choose a control volume coinciding with the inside of the pipe walls over the length δL and including the flow cross-section at each end of the pipe section δL long. This control volume, the fluid in it and the external forces acting are shown below.



The side shear force caused by friction will be neglected because its size is limited by a very small δL. Also, because we are considering only relatively rigid pipe (steel, concrete, etc.), the pipe bulge will be very small and F_3 will also be negligible.

Application of Equation 4-2 gives

$$F_1 - F_2 = Q\rho (V - \Delta V + a - V - a) = Q\rho (-\Delta V)$$

where $Q\rho = (V + a) A\rho$

If the pressure at (1) were p_o then the pressure at (2) would be $p_o + \Delta p$.

$$p_o A - (p_o + \Delta p)(A + \delta A) = (V + a) A\rho (-\Delta V)$$

Expanding this equation and recognizing that $\Delta p = \gamma \Delta H$ and δA is very small compared to H_o, ΔH, A and γ, we can neglect the small terms with the result

$$-\Delta H \gamma A = (V + a) A\rho (-\Delta V)$$

In slightly different form, this equation can be written

$$\Delta H = \frac{\rho}{\gamma} \Delta V (V + a)$$

or

$$\Delta H = \frac{a \Delta V}{g} \left(1 + \frac{V}{a}\right) \quad \text{.......... (4-3)}$$

In most cases involving rigid pipes (even PVC with a wave speed of about 1200 fps), the value of V/**a** is less than 0.01. Accordingly, Equation 4-3 is generally used (and is always used in this text) as

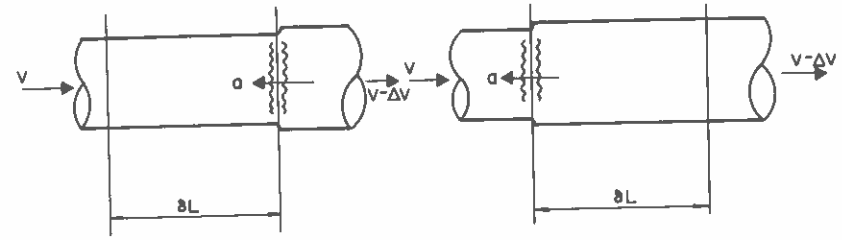
$$\boxed{\Delta H = \frac{a}{g} \Delta V} \quad \text{.......... (4-4)}$$

It is clear from Equation 4-4 that ΔH depends on **a** and cannot be determined until a value of **a** is established.

4.2 The a Equation

To develop an equation for **a** we will consider conservation of mass into the section of pipe δL long, which was used in the previous section to find an equation for ΔH. The procedure used will be to examine the mass flow into and out of the portion of pipe δL long over the time period required for the wave to pass through that portion of the pipe. The net inflow of mass will be equated to the increased mass storage in δL to yield an equation for **a**.

To begin, we will look at the situation when the wave first reaches the δL section and then at the time the wave has just passed through the section δt later.



time = t   time = t + δt

It is clear that δL and δt are related via the wave speed by

$$\delta L = a\delta t$$

Net Mass Inflow

During the time period δt an amount of liquid has accumulated in the section of pipe given by the amount

$$\delta M = \text{Mass accumulated} = VA\rho\delta t - (V-\Delta V)(\rho+\delta\rho)(A+\delta A)\delta t$$

Expanding parentheses and neglecting small terms gives

$$\delta M = A\rho\Delta V\delta t$$

or writing in terms of wave speed and δL,

$$\delta M = A\rho\Delta V\frac{\delta L}{a} \quad \text{(4-5)}$$

This amount of extra liquid is accumulated in section δL by being compressed slightly and by stretching the pipe slightly to provide storage room.

Change in Liquid Volume

Because the pressure has increased during the passage of the wave, the volume of the liquid in the section will compress slightly to a higher density. The equation describing this relationship is that defining the bulk modulus of elasticity which can be found in any text on fluid mechanics.

$$K = -\frac{dp}{\frac{d\forall}{\forall}} \quad \text{(4-6)}$$

where K = the bulk modulus of elasticity of the liquid and p, ∀ are the pressure and volume, respectively. Recognizing that dp ≈ Δp (K is relatively constant over a wide range of pressure), Equation 4-6 becomes

$$\delta\forall = -\Delta p\frac{\delta LA}{K} \quad \text{(4-7)}$$

where δ∀ is the change in volume of the liquid in the pipe section δL long as the result of a pressure change of Δp.

Change in Pipe Volume

Because the increased pressure stretches the pipe, there is more room made available to store the net mass inflow of liquid. When the pipe stretches circumferentially it may also stretch longitudinally so both contributions to change in pipe volume should be evaluated.

Before proceeding, it is important to recognize that there is an interconnection or relationship between pipe wall strains in two perpendicular directions. If a material is strained in one direction an amount ϵ_1 then a strain ϵ_2 will occur in the perpendicular direction (provided the material is free to strain without developing stress in that direction) such that $\epsilon_2 = \mu\epsilon_1$ where μ is Poisson's ratio. If there is a restriction to strain in either direction caused either by restraint or applied stress, the relationship is more complicated. In any case, Timoshenko [11] gives, for thin-walled pipes,

$$\sigma_1 = \frac{\epsilon_1 + \mu\epsilon_2}{1-\mu^2}E \text{ or } \epsilon_1 = \frac{1}{E}(\sigma_1 - \mu\sigma_2) \quad \text{(4-8a)}$$

$$\sigma_2 = \frac{\epsilon_2 + \mu\epsilon_1}{1-\mu^2}E \text{ or } \epsilon_2 = \frac{1}{E}(\sigma_2 - \mu\sigma_1) \quad \text{(4-8b)}$$

where σ_1 and ϵ_1 are the stress and strain, respectively, in the direction parallel to the pipe axis and σ_2 and ϵ_2 are the values in the circumferential direction. E is the modulus of elasticity of the pipe wall material.

In the case of water hammer waves, there is generally a resident stress and strain already occurring in the pipe before wave passage. Hence, we will write the above equations in incremental form

$$\Delta\sigma_1 = \frac{\Delta\epsilon_1 + \mu\Delta\epsilon_2}{1-\mu^2}E \text{ or } \Delta\epsilon_1 = \frac{1}{E}(\Delta\sigma_1 - \mu\Delta\sigma_2) \quad \text{(4-9a)}$$

$$\Delta\sigma_2 = \frac{\Delta\epsilon_2 + \mu\Delta\epsilon_1}{1-\mu^2}E \text{ or } \Delta\epsilon_2 = \frac{1}{E}(\Delta\sigma_2 - \mu\Delta\sigma_1) \quad \text{(4-9b)}$$

The change in volume caused by circumferential stretching is

$$\delta\forall_c = \pi D\frac{\delta D}{2}\delta L$$

where $\pi\delta D = \pi D\Delta\epsilon_2$

Combining the two equations gives

$$\delta \mathcal{V}_c = \frac{1}{2} \pi D^2 \, \delta L \, \Delta\epsilon_2 \quad \text{(4-10)}$$

The change in volume caused by longitudinal stretching is

$$\delta \mathcal{V}_L = \frac{\pi}{4} D^2 \, \delta L \, \Delta\epsilon_1 \quad \text{(4-11)}$$

Adding the two equations together gives the total volume change

$$\delta \mathcal{V} = \frac{\pi}{2} D^2 \, \delta L \left(\frac{\Delta\epsilon_1}{2} + \Delta\epsilon_2 \right) \quad \text{(4-12)}$$

Change in circumferential stress in the pipe wall caused by Δp is

$$\Delta\sigma_2 = \frac{\Delta p D}{2 e}$$

where e is the pipe wall thickness.

So Equation 4-9b becomes

$$\frac{\Delta p D}{2e} = \frac{\Delta\epsilon_2 + \mu \Delta\epsilon_1}{1 - \mu^2} E \quad \text{(4-13)}$$

Unfortunately, the longitudinal pipe restraint condition determines $\Delta\sigma_1$. For example, if the pipe were anchored at some point and free to stretch longitudinally (much like a long slender pressure vessel), the longitudinal stress would be

$$\Delta\sigma_1 = \frac{\Delta p D}{4e}$$

under static conditions. However, the dynamic conditions of a water hammer situation will cause the pipe to stretch axially in a dynamic fashion wherein the hardware inertia is important. That is to say, any valves, fittings, etc. in addition to the weight of the pipe itself must be displaced by the pressure changes. The pipe may even be partially restrained by supports. In order to determine the value of $\Delta\sigma_1$, we would have to solve a rather complex coupled set of equations relating the fluid dynamics to the hardware dynamics.

Rather than attempt this task, it is suggested that the dynamics of the pipe be ignored and the above equation for $\Delta\sigma_1$ be used. Because this type of restraint is rare and because restraint does not have an excessive impact on wave speeds in typical pipelines, we will not be greatly concerned with precisely fixing this type of restraint.

On the other hand, if the pipe were rigidly anchored to prevent axial strain then $\Delta\sigma_1 = \mu\Delta\sigma_2$ because $\Delta\epsilon_1 = 0$. If, however, the pipe had functioning expansion joints throughout its length, then $\Delta\sigma_1 = 0$ and $\Delta\epsilon_1$ is of no interest. Following Streeter and Wylie [1], we will identify the above as case (a), (b), and (c) restraints, respectively. In a practical sense, the pipe restraint lies between these values somewhere.

Because a buried pipeline might be expected to be restrained from strain by soil friction and anchor blocks, we will pursue case (b) restraint to develop an equation for wave speed.

Case (b) restraint — For this restraint, $\Delta\epsilon_1 = 0$ and Equation 4-9a becomes

$$\Delta\sigma_1 = \frac{\mu \Delta\epsilon_2}{1 - \mu^2} E = \mu \Delta\sigma_2$$

and Equation 4-13 becomes

$$\frac{\Delta p D}{2e} = \frac{\Delta\epsilon_2}{1 - \mu^2} E$$

Substituting this equation into Equation 4-12 gives

$$\delta \mathcal{V} = \frac{\pi}{2} D^2 \, \delta L \left(\frac{1 - \mu^2}{E} \right) \left(\frac{\Delta p D}{2e} \right) \quad \text{(4-14)}$$

Now considering conservation of mass, we already have Equation 4-5 expressing the amount of mass which has accumulated in the δL pipe section in δt seconds. We can write a different expression for the mass change in the δL pipe section after wave passage. The mass change in the section is

$$\delta M = (\rho + \delta\rho)(A \delta L + \delta \mathcal{V}) - \rho A \delta L$$

Equating this expression with Equation 4-5, expanding and dropping small terms gives

$$\delta\rho A \delta L + \rho \delta \mathcal{V} = A \rho \Delta V \frac{\delta L}{a} \quad \text{(4-15)}$$

To arrange this equation in more useable form, note that for a mass of a given substance, an increase in pressure causes a decrease in volume and an increase in density.

$$\rho \mathcal{V} = \text{constant}$$

$$\mathcal{V} \delta\rho + \rho \delta \mathcal{V} = 0$$

$$\delta\rho = -\frac{\delta \mathcal{V}}{\mathcal{V}} \rho$$

Substituting Equation 4-6 into this equation gives

$$\delta\rho = \rho \left(\frac{\Delta p}{K} \right)$$

Replacing Δp with $\gamma \Delta H$ in the above equation, substituting it and Equation 4-14 into Equation 4-15,

$$\gamma \Delta H \left[\frac{1}{K} + \left(\frac{1 - \mu^2}{E} \right) \frac{D}{e} \right] = \frac{\Delta V}{a} \quad \text{(4-16)}$$

Combining this equation with Equation 4-4 gives

$$a^2\rho\left[\frac{1}{K}+\frac{D}{e}\left(\frac{1-\mu^2}{E}\right)\right] = 1$$

or in a more conventional form for wave speed,

$$a = \frac{[K/\rho]^{1/2}}{\left[1+\frac{K}{E}\frac{D}{e}(1-\mu^2)\right]^{1/2}} \quad \text{(case b)} \qquad (4\text{-}17)$$

It is now possible for us to compute wave speed and pressure increase in simple situations where Equation 4-4 can be used.

Streeter and Wylie [1] have shown that the equation for wave speed can be more conveniently expressed as

$$\boxed{a = \frac{[K/\rho]^{1/2}}{\left[1+\frac{K}{E}\frac{D}{e}(C)\right]^{1/2}}} \qquad (4\text{-}18)$$

where

$C = 5/4\text{-}\mu$ for case (a) restraint .... (4-19a)

$C = 1\text{-}\mu^2$ for case (b) restraint .... (4-19b)

$C = 1.0$ for case (c) restraint .... (4-19c)

Recall that this set of equations applies only to thin-walled pipes where D/e is generally greater than about 40 (see Art. 4.3).

To assist in calculating wave speeds in pipes constructed of common materials, the following table of E-values and μ-values is included. The value of K for water can generally be taken as approximately 300,000 psi.

In the limit the pipe can become completely rigid without causing the wave speed to become infinite. This limiting value is obtained by passing E to ∞ in Equation 4-18. With the nominal value of K = 300,000 psi, the resulting wave speed is approximately 4720 fps. This number has no practical value in design because it is far too high to serve as even an approximate wave speed for preliminary design. With even a limited amount of experience, the designer can make far better estimates for wave speed in the pipe he is working with.

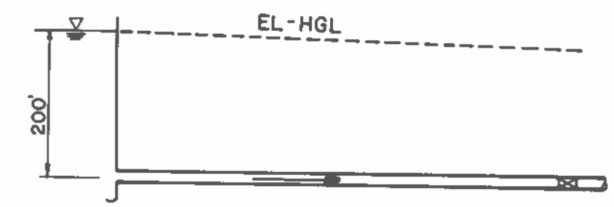
Table 4-1. Moduli of elasticity and Poisson's ratio for common pipe materials.

Steel	$E = 30 \times 10^6$ psi	$\mu \approx 0.30$
Ductile Cast Iron	$E = 24 \times 10^6$ psi	$\mu \approx 0.28$
Copper	$E = 16 \times 10^6$ psi	$\mu \approx 0.30$
Brass	$E = 15 \times 10^6$ psi	$\mu \approx 0.34$
Aluminum	$E = 10.5 \times 10^6$ psi	$\mu \approx 0.33$
PVC	$E \approx 4 \times 10^5$ psi	$\mu \approx 0.45$
Fiberglass reinforced plastic (FRP)	$E_2 = 4.0 \times 10^6$ psi $E_1 = 1.3 \times 10^6$ psi	$\mu_2 = 0.27 - 0.30$ $\mu_1 = 0.20 - 0.24$
Asbestos Cement	$E \approx 3.4 \times 10^6$ psi	$\mu \approx 0.30$
Concrete	$E = 57{,}000\sqrt{f'_c}$	$\mu \approx 0.30$

where f'_c = 28 day strength.

Example 4-1

As an illustration of the elastic deformations and pressure head changes caused by a water hammer situation and the effect of restraint on wave speed, the following problem is analyzed.



Flow in the 24-inch pipeline above occurs at a velocity of 6 fps. The pipeline is fabricated of steel and has a wall thickness of 0.25 inches.

a) Calculate the wave speed for all three cases of restraint.

$$\text{Case (a)} \quad a = \frac{4720}{\sqrt{1+\frac{3\times10^5}{3\times10^7}\,\frac{24}{.25}\,(5/4-0.30)}} = 3413 \text{ fps}$$

$$\text{Case (b)} \quad a = \frac{4720}{\sqrt{1+0.96\,(1-0.30^2)}} = 3448 \text{ fps}$$

$$\text{Case (c)} \quad a = \frac{4720}{\sqrt{1+0.96\,(1.0)}} = 3371 \text{ fps}$$

In a practical sense, the differences are negligible.

b) Find the head increase resulting from sudden valve closure for all three cases of restraint.

$$\text{Case (a)} \quad \Delta H = \frac{3413}{32.2} \times 6 = 636 \text{ ft}$$

$$\text{Case (b)} \quad \Delta H = \frac{3448}{32.2} \times 6 = 642 \text{ ft}$$

$$\text{Case (c)} \quad \Delta H = \frac{3371}{32.2} \times 6 = 628 \text{ ft}$$

The variation in head increase among the three cases is about 2 percent.

c) Compute the axial and circumferential pipe wall stresses before and after valve closure for all three cases of restraint.

$$\text{Case (a) Before} \quad \sigma_2 = \frac{200 \times 62.4 \times 24}{144 \times 2 \times .25} = 4160 \text{ psi}, \ \sigma_1 = \tfrac{1}{2}\,\sigma_2 = 2080 \text{ psi}$$

$$\Delta\sigma_2 = \frac{636 \times 62.4 \times 24}{144 \times 2 \times .25} = 13{,}230 \text{ psi}, \Delta\sigma_1 = 6615 \text{ psi}$$

$$\text{After} \quad \sigma_2 = \sigma_2 + \Delta\sigma_2 = 17{,}390 \text{ psi}, \ \sigma_1 = 8695 \text{ psi}$$

$$\text{Case (b) Before} \quad \sigma_2 = \text{same as above}, \ \sigma_1 = \mu\sigma_2 = 1250 \text{ psi}$$

$$\Delta\sigma_2 = \text{same as above}, \ \Delta\sigma_1 = \mu\Delta\sigma_2 = 3970 \text{ psi}$$

$$\text{After} \quad \sigma_2 = \text{same as above} = 17{,}390 \text{ psi}, \ \sigma_1 = 5220 \text{ psi}$$

$$\text{Case (c) Before} \quad \sigma_2 = \text{same as above}, \ \sigma_1 = 0$$

$$\Delta\sigma_2 = \text{same as above}, \ \Delta\sigma_1 = 0$$

$$\text{After} \quad \sigma_2 = \text{same as above} = 17{,}390 \text{ psi}, \ \sigma_1 = 0$$

d) Calculate the percent increase in diameter of the pipe caused by sudden valve closure.

$$100\,\frac{\delta D}{D} = 100\,\Delta\epsilon_2 = \frac{100}{E}(\Delta\sigma_2 - \mu\Delta\sigma_1)$$

$$\text{Case (a)} \quad \%\text{ change} = \frac{100}{30 \times 10^6}\,(13{,}230 - 0.3 \times 6615) = 0.037\%$$

$$\text{Case (b)} \quad \%\text{ change} = \frac{100}{30 \times 10^6}\,(13{,}230 - 0.3 \times 3970) = 0.040\%$$

$$\text{Case (c)} \quad \%\text{ change} = \frac{100}{30 \times 10^6}\,(13{,}230 - 0.3 \times 0) = 0.044\%$$

This result substantiates many of our previous assumptions used in neglecting small terms.

4.3 Wave Speeds in Other Types of Conduits

The simplest case of thin-walled pipes has been used previously to derive equations for wave speed. It is obvious that many hydraulic conduits are constructed of thick-walled pipe and often using two or more materials (reinforced concrete). Also tunnels may be carved in rock, lined with steel and back-filled with concrete. It is necessary to be able to calculate wave speeds in all these cases.

A concise summary of the calculation of the wave speeds for these cases is given by Halliwell [12]. The most obvious extension of the previous example of thin-walled pipes is to thick-walled pipes. In a thick-walled pipe, the wall thickness is so great that stress varies noticeably between the inner and outer surfaces and this affects the expression for wave speed. An analysis reveals that we may continue to use the same basic form for the wave speed equation, but we must find a different value for the C in Equation 4-18.

Thick-walled Pipes

Summarizing the results for thick-walled pipe for the same restraint conditions as before with D as the inside diameter,

Case (a) — For pipes free to stress and strain both laterally and longitudinally (anchored at only one point)

$$C = \frac{1}{1 + \frac{e}{D}} \left[(5/4 - \mu) + 2\frac{e}{D}(1 + \mu)\left(1 + \frac{e}{D}\right)\right] \quad \text{(4-20a)}$$

A limiting process shows that as $e/D \rightarrow 0$, this equation degenerates to Equation 4-19a.

Case (b) — For pipes anchored against longitudinal strain,

$$C = \frac{1}{1 + \frac{e}{D}} \left[(1 - \mu^2) + 2\frac{e}{D}(1 + \mu)\left(1 + \frac{e}{D}\right)\right] \quad \text{(4-20b)}$$

Case (c) — For pipes with functioning expansion joints throughout their length.

$$C = \frac{1}{1 + \frac{e}{D}} \left[1 + 2\frac{e}{D}(1 + \mu)\left(1 + \frac{e}{D}\right)\right] \quad \text{(4-20c)}$$

As in case (a), both cases (b) and (c) degenerate to the thin-walled pipe values when $e/D \rightarrow 0$.

The question arises as to when the more complex thick-walled formulas should be applied. For deciding, it is helpful to examine the plot of these equations in Figure 4-1. To assist in making this decision,

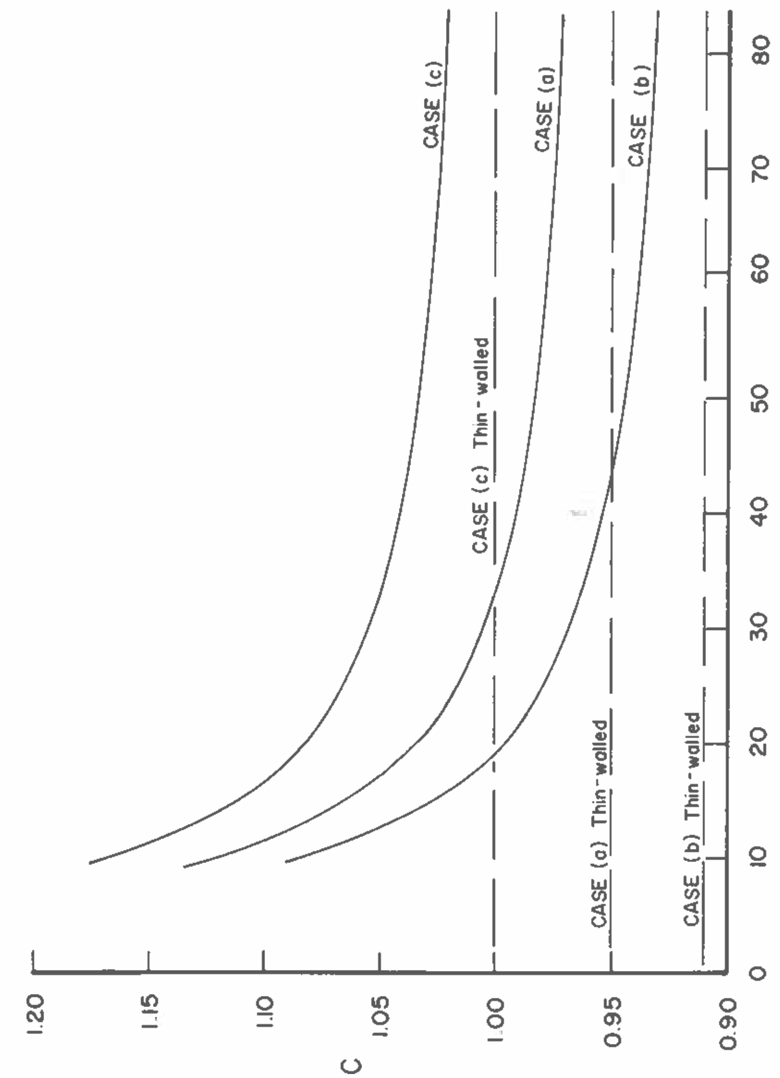


Figure 4-1. The effect of wall thickness on C-value for μ = 0.30.

consider the uncertainties of pipe restraint and its effect on wave speed. Figure 4-1 shows that uncertainty with respect to the type of restraint occurring can cause differences of about 10 percent between C-values at the two extremes of restraint. If we accept a similar error in deciding whether to use thin-walled or thick-walled formulas, then a D/e value of 20 is an appropriate dividing line. If, however, we decide to remove as much uncertainty as possible, then the thick-walled formulas should always be used. The additional computation required is negligible. In a practical sense though, because of the relative size of terms in the denominator of Equation 4-18, using thick-walled formulas beyond D/e values of 40 generally makes no significant improvement in the value of the wave speed except in cases where softer pipes such as PVC are used. It should also be noted that using the thin-walled formulas leads to higher (more conservative) wave speeds. To see the effect, consider the following example.

Example 4-2

A steel pipe 10 inches in diameter is used to convey water between two reservoirs. The inside diameter of the pipe is 9.522 inches and the wall thickness is 0.239 inches.

Compute the C-values and wave speeds using both thin and thick-walled formulas and compare results.

	Wave Speed-fps		
Restraint	Thin-walled	Thick-walled	Error (D/e = 40)
Case (a)	4023	3999	0.6%
Case (b)	4047	4021	0.6%
Case (c)	3994	3971	0.6%

Circular Tunnels

Wave speed formulas which apply to circular tunnels can be found by taking the thick-walled pipe formulas and letting the thickness go to infinity. The portion of Equation 4-18 that we will use is

$$(D/e)\ C = D/e\ \frac{1}{1+\frac{e}{D}}\left[(5/4-\mu)+2\frac{e}{D}(1+\mu)\left(1+\frac{e}{D}\right)\right]$$

$$\lim_{e\to\infty}(D/e)\ C = \frac{D}{e}\times\frac{D}{e}\left[2\frac{e}{D}(1+\mu)\frac{e}{D}\right] = 2(1+\mu)$$

The resulting equation of wave speed is

$$a = \frac{[K/e]^{1/2}}{\left[1 + \frac{2K}{E}(1 + \mu)\right]^{1/2}} \quad \text{.................(4-21)}$$

For tunnels which are concrete-lined or steel-lined with concrete backfill, the elastic analysis is more difficult. Refer to Halliwell [12] for the rather lengthy equations which must be used to find the (D/e)C value for Equation 4-18.

Reinforced Concrete Pipe

For reinforced concrete pipe, the transformed section technique can be used to convert the pipe into an equivalent homogeneous pipe. Then analysis can proceed according to the rules for homogeneous pipes. However, it should be emphasized that the manner in which the pipe is constructed should be investigated thoroughly. In most cases some of the concrete section should not be expected to carry load. This is true particularly if the load to be carried is tensile.

The transformed section technique replaces the steel with concrete (or vice versa) using the relationship

$$A_{steel} = \frac{E_{conc.}}{E_{steel}} A_{conc.} \quad \text{.....................(4-22)}$$

The technique of replacing concrete with steel is recommended because the resulting steel pipe is thin-walled and the computation of the wave speed is straight forward. If the reverse technique is used, the cross-section of concrete is too bulky to handle properly.

In working with reinforced concrete pipe that is not prestressed, the concrete is generally assumed to take no load in tension. The reinforcing steel is transformed into a thin-walled steel pipe having the same area in the axial and circumferential directions as did the reinforcing steel in the same respective directions. The wave speed is then calculated in the normal manner using the equivalent thin-walled steel pipe.

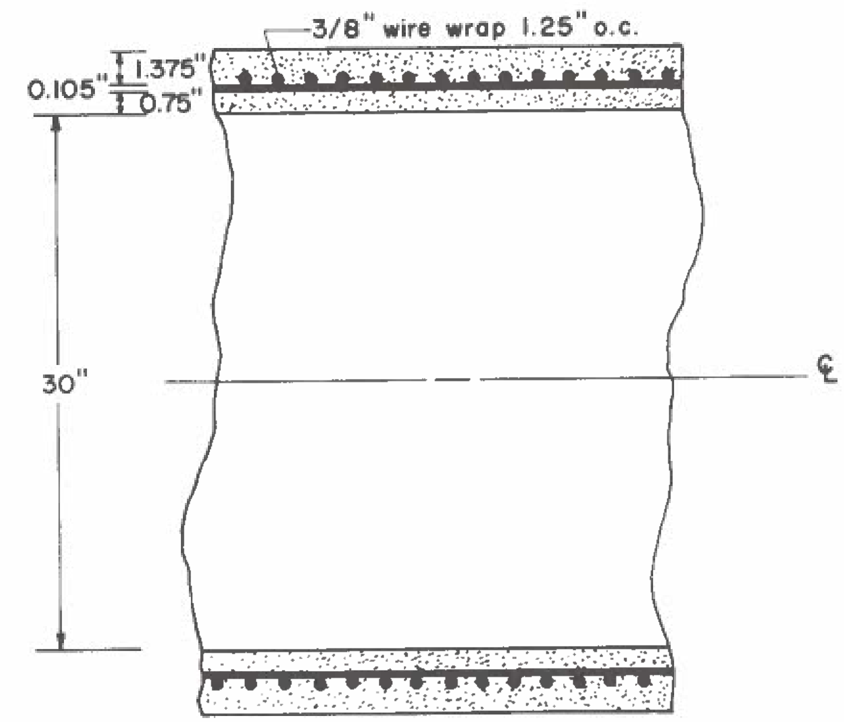
If the reinforced concrete pipe is pre-tensioned or post-tensioned, the area of the concrete placed in compression must be included in the transformed section. This pre-stressing makes the pipe much stronger, but it also results in higher wave speeds which give higher water hammer pressures. An example problem is worked out to illustrate the application of the transformed section technique.

Example 4-3

A reinforced concrete pipe, 30 inches inside diameter, is prestressed using 3/8-inch diameter wrapping wire placed 1.25 inches o.c. The pipe is constructed by first fabricating the thin steel cylinder, then centrifugally placing a 0.75-inch cement lining inside the pipe.

After curing, the prestressing is accomplished by stressing the wire as it is wrapped around the outside of the steel cylinder. This process places the cement lining in compression. The ends of the wrapping wire are welded to the thin steel cylinder to maintain the prestressing. A cover of concrete is applied over the wire wrap to give a one-inch protective coating.

Assuming that the 28-day strength of the concrete is 6,000 psi, we will compute the wave speed.



$$E_{conc.} = 57{,}000\sqrt{6000} = 4.4 \times 10^6 \text{ psi}$$

$$\text{Area of steel wire} = 0.7854 \times \frac{0.375^2}{1.25} = 0.0884 \text{ sq. in/in.}$$

In this example we will replace the cement lining (prestressed in compression) by an equivalent area of steel.

$$A_{st} = \frac{4.4 \times 10^6}{30 \times 10^6} 0.75 = 0.110 \text{ sq. in/in.}$$

Now the thickness of the equivalent steel pipe is

$$e_{eq} = 0.105 + 0.0884 + 0.110 = 0.303 \text{ in.}$$

The wave speed is computed using Equation 4-18, case (b) restraint.

$$a = \frac{4720}{\sqrt{1 + \dfrac{3 \times 10^5}{30 \times 10^6}\dfrac{31.5}{0.303}(1 - 0.3^2)}} = 3384 \text{ fps}$$

If the effect of the cement lining is neglected, the wave speed is 2994 fps. The engineer must make the judgment as to the proper wave speed to use or he must analyze the system under both conditions to determine the most extreme behavior.

Effect of Air Entrainment on Wave Speed

When free air occurs in a substantial portion of a pipeline, either as small bubbles or in larger discrete lumps, the wave speed in the pipe is decreased. As a consequence, the pressure extremes and the wave propagation patterns are altered.

The wave speed of the air-liquid mixture is computed as previously done for a homogeneous liquid, but with the use of an average density for the mixture. This approach implicitly assumes the mixture is evenly distributed throughout a significant portion of the pipe. The elasticity of the liquid-air mixture is dramatically affected by a small amount of entrained air so the elasticity of both substances must be included.

Application of the momentum equation and conservation of mass leads to the following equation for wave speed from Tullis, Streeter and Wylie [13],

$$a = \frac{\sqrt{K_\ell/\rho_{ave}}}{\sqrt{1 + \dfrac{K_\ell}{E}\dfrac{D}{e}C + (\text{void fraction})\dfrac{K_\ell}{K_a}}} \quad \text{........(4-23)}$$

where the subscripts ℓ and a refer to properties of liquid and air, respectively, and ρ_{ave} is the average density of the mixture. The value of K_a is dependent on the thermodynamic process, e.g., for an isothermal process, K_a equals the absolute pressure. The void fraction is the volume of air per unit volume of mixture at a given pressure.

Unfortunately, the wave speed depends on the pressure because the void fraction is a function of pressure. Hence, an accurate analysis must

consider this effect as water hammer progresses through the system. Because a simple, reduced, constant value of wave speed should not be used in the standard analysis, further discussion of the analysis of water hammer with entrained air will be deferred to Chapter VI.

4.4 Basic Differential Equations of Unsteady Flow

To this point we have shown that for a known given impulsive change in velocity at a section in a pipeline, we can compute the pressure head change ΔH which will result. It now remains to expand this capability so that we may find the velocity and pressure head at any pipe section at any time as the result of boundary conditions imposed at sections either within or at the extremities of the system.

To accomplish this we will employ the previously derived unsteady flow equation (Euler equation) and develop a second equation based on conservation of mass (continuity).

Equation 2-4 is recopied here for convenience.

$$-\frac{1}{\gamma}\frac{\partial p}{\partial s} - \frac{\partial z}{\partial s} - \frac{4\tau_o}{\gamma D} = \frac{1}{g}\frac{dV}{dt} \quad \text{........................(4-24)}$$

As before, the pipe wall friction will be difficult to evaluate directly. Because we will be working with circular cylindrical pipes, we use the equation

$$\tau_o = \frac{1}{8}\rho f V|V|$$

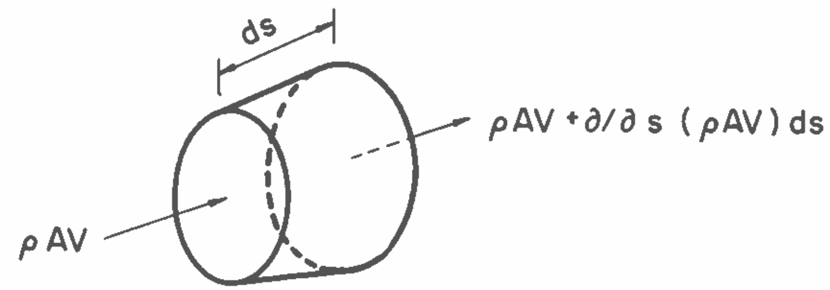
for relating wall shear to the Darcy-Weisbach friction factor. The peculiar form of the velocity representation is introduced in the above equation to point out how the proper shear force direction is preserved when the velocity reverses. If we had used V^2 instead, then the negative velocity sign occurring on flow reversal would be lost in the squaring process.

Substituting this equation into Equation 4-24 gives

$$\frac{dV}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{\partial z}{\partial s} + \frac{f}{2D}V|V| = 0 \quad \text{..............(4-25)}$$

It is appropriate to point out at this time that both V and p are functions of time t and location s along the pipe. The term $\partial z/\partial s$ is the slope of the pipe and can be written as the total derivative dz/ds. Equation 4-25 is an equation with two dependent variables V (s,t) and p (s,t) hence, we need a second equation relating the same dependent variables.

To obtain this equation the conservation of mass is applied to a control volume coinciding with the interior of the pipe and of length ds.



Conservation of mass gives

$$\rho AV - \left[\rho AV + \frac{\partial}{\partial s}(\rho AV)\,ds\right] = \frac{\partial}{\partial t}(\rho A ds) \qquad \text{(4-26)}$$

$$-\frac{\partial}{\partial s}(\rho AV)ds = \frac{\partial}{\partial t}(\rho A\,ds)$$

At this point we employ a rather unusual form of the control volume concept in that we permit the ends of the control volume to move longitudinally with the pipe as it stretches. This device is employed because the pipe stretching affects the storage volume available and the connection between pipe elasticity and the volume available for the fluid is identical to that developed in the previous section.

Expanding the parentheses of Equation 4-26 gives

$$-\left(\rho A\frac{\partial V}{\partial s}ds + \rho V\frac{\partial A}{\partial s}ds + AV\frac{\partial \rho}{\partial s}ds\right)$$

$$= \rho A\frac{\partial}{\partial t}(ds) + \rho ds\frac{\partial A}{\partial t} + A\,ds\frac{\partial \rho}{\partial t}$$

Regrouping and dividing by $\rho A ds$,

$$\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial s}\right) + \frac{1}{A}\left(\frac{\partial A}{\partial t} + V\frac{\partial A}{\partial s}\right) + \frac{1}{ds}\frac{\partial}{\partial t}(ds) + \frac{\partial V}{\partial s} = 0 \qquad \text{(4-27)}$$

Recognizing that

$$\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial s} = \frac{d\rho}{dt}$$

and

$$\frac{\partial A}{\partial t} + V\frac{\partial A}{\partial s} = \frac{dA}{dt}$$

Equation 4-27 becomes

$$\frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{A}\frac{dA}{dt} + \frac{\partial V}{\partial s} + \frac{1}{ds}\frac{d}{dt}(ds) = 0 \qquad \text{(4-28)}$$

because ds, fixed to the pipe walls, varies only with time.

From the previous section,

$$K = -\frac{dp}{\frac{d\forall}{\forall}} = \frac{dp}{\frac{d\rho}{\rho}}$$

so

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{1}{K}\frac{dp}{dt} \qquad \text{(4-29)}$$

To develop a useful expression for dA/dt in terms of pressure, the elastic pipe deformations must be considered. For stretching of the cross-sectional area, Equation 4-10 is used to give

$$dA = \tfrac{1}{2}\pi D^2\, d\epsilon_2 = \tfrac{1}{2}\pi\frac{D^2}{E}(d\sigma_2 - \mu\, d\sigma_1)$$

$$\frac{1}{A}dA = \frac{2}{E}(d\sigma_2 - \mu\, d\sigma_1)$$

In evaluating the stresses we will again use case (b) restraint

$$d\sigma_2 = dp\frac{D}{2e}$$

$$d\sigma_1 = \mu\, d\sigma_2$$

so

$$d\sigma_2 - \mu\, d\sigma_1 = (1-\mu^2)\,d\sigma_2 = (1-\mu^2)\frac{D}{2e}dp$$

Finally,

$$\frac{1}{A}\frac{dA}{dt} = (1-\mu^2)\frac{D}{eE}\frac{dp}{dt} \qquad \text{(4-30)}$$

Considering longitudinal expansion,

$$d(ds) = d\epsilon_1\, ds$$

which for case (b) restraint is zero. So,

$$\frac{1}{ds}\frac{d}{dt}(ds) = 0 \qquad \text{(4-31)}$$

Combining all of these terms into Equation 4-28 results in

$$\frac{1}{K}\frac{dp}{dt} + (1-\mu^2)\frac{D}{eE}\frac{dp}{dt} + \frac{\partial V}{\partial s} = 0$$

$$\frac{dp}{dt}\left[\frac{1}{K} + (1-\mu^2)\frac{D}{eE}\right] + \frac{\partial V}{\partial s} = 0$$

From Equation 4-17 it is clear that the term in brackets is $1/(a^2\rho)$. This statement is true for case (a) and case (c) pipe restraint as well. Substituting this term into the previous equation gives

$$\frac{1}{\rho}\frac{dp}{dt} + a^2\frac{\partial V}{\partial s} = 0 \quad \text{.......................... (4-32)}$$

We now have the necessary set of two simultaneous independent partial differential equations which will enable us to solve for p (s,t) and V (s,t).

$$\boxed{\frac{dV}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{dz}{ds} + \frac{f}{2D}V|V| = 0} \quad \text{........... (4-33)}$$

$$\boxed{a^2\frac{\partial V}{\partial s} + \frac{1}{\rho}\frac{dp}{dt} = 0} \quad \text{.......................... (4-34)}$$

V. Solution by Method of Characteristics

The history of water hammer analysis is marked by various clever and practical schemes for solving the Euler and Continuity Equations 4-33 and 4-34. The methods generally reflect the level of numerical analysis capability of their time as well as the ingenuity of the practitioners. It remains, then, in the age of electronic digital computers, to bring their full power to bear in solving more exactly and inexpensively these equations as they apply to a wide range of problems.

At present the most general and exact technique for solving this set of equations is the *method of characteristics*. Fortunately this technique is also very compatible with numerical solution by digital computer. For these reasons, this work will address only this solution approach and refer the reader to other works such as Streeter and Wylie [1], Parmakian [6] and Rich [4] for details on other analysis techniques.

5.1 Approximate Method of Characteristics

The Approximate Equations

In view of the likelihood that many engineers today are unfamiliar with the method of characteristics as a solution technique, it seems appropriate to introduce the method using approximate versions of Equations 4-33 and 4-34. The approximate equations are obtained by neglecting the spacial variation of V and p whenever both spacial and time-varying terms appear in the same equation, because in general, spacial variation is much less than time-varying variation.

In accordance with this approach, Equation 4-33 becomes

$$\frac{\partial V}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{dz}{ds} + \frac{f}{2D}V|V| = 0 \quad \text{............... (5-1)}$$

and Equation 4-34 becomes