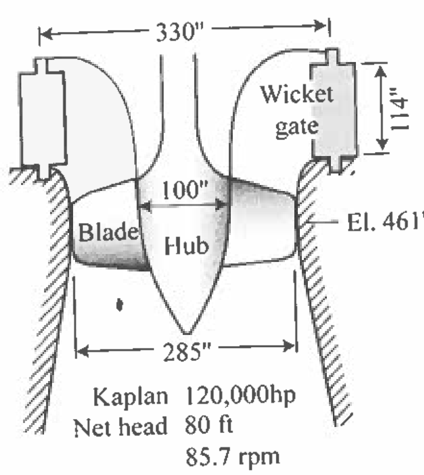

Fig. P-5.13

12. ♦♦♦ For the Kaplan turbine in Fig. E-5.6, draw the velocity diagrams at the trailing edge of the propeller turbine blade such that the tangential velocity $V_{u2} = 0$. Determine the blade angles β_2 for $r = 0.75$ ft, 1 ft, 1.5 ft and 2 ft.

Additional Problems

13. From the Kaplan turbine at Wanapum Dam in Fig. P-5.13, can you determine the blade angles?
14. Consider the following turbines compiled from Moody and Zowski (1984) in Table P-5.14, calculate N_s and compare with the characteristics described in this chapter.

Table P-5.14. Sample of turbine characteristics

Location	Type	Rotation (rpm)	Head (ft)	Power (hp)	Outside diameter (in.)
Paucartambo, Peru	Pelton	450	1,580	28,000	
Smith Mtn Dam	Francis	100	179	204,000	246
Garrison Dam	Francis	90	150	90,000	223
Hoover Dam	Francis	180	440	115,000	
Oxbow power plant	Francis	100	115	73,000	
R. Moses Niagara	Francis	120	300	210,000	
Cullingran, Scotland	Deriaz	300	180	30,500	
Priest Rapids	Kaplan	86	78	114,000	284-in. runner
Wanapum	Kaplan	86	80	120,000	285-in. runner
St-Lawrence Power	Propeller	95	81	79,000	240-in. runner
Pierre Benite, France	Bulb	83	26	27,000	240-in. runner
Ozark L&D	Tube	60	26	33,800	315-in. runner

6 Water Hammer

Water compressibility effects in closed conduits can be devastating and hydraulic structures like surge tanks are specifically designed to minimize the pressure fluctuations in pipe systems. Section 6.1 presents important knowledge on water compressibility. It is followed with the celerity of wave propagation in pipes in Section 6.2. The concept of hydraulic transients and water hammer is detailed in Section 6.3 with prevention measures such as surge tanks in Section 6.4.

6.1 Water Compressibility

The bulk modulus of elasticity E_w is a measure of fluid compressibility. It measures relative changes in volume and fluid density under pressure,

$$E_w = -\frac{\forall dp}{d\forall} = \frac{\rho\, dp}{d\rho}, \tag{6.1}$$

where dp is the increase in pressure required to decrease the volume $d\forall$ from the initial fluid volume $\forall$. From the definition of the mass density $\rho = m/\forall$, we note that the mass change $dm = \rho d\forall + \forall d\rho = 0$ such that $-d\forall/\forall = d\rho/\rho$. A typical elasticity value for water is $E_w = 2.1 \times 10^9$ Pa $\cong 3 \times 10^5$ psi. The bulk modulus of elasticity for water increases slightly with temperature and pressure, as shown in Table 6.1. Example 6.1 illustrates how to calculate the bulk modulus of elasticity of a fluid.

Table 6.1. Water modulus of elasticity E_w in GPa vs temperature and pressure

Pressure	E_w ($T° = 0$ °C)	E_w ($T° = 10$ °C)	E_w ($T° = 20$ °C)
$1 - 25 \times 10^5$ Pa	1.93	2.03	2.07
$25 - 50 \times 10^5$ Pa	1.96	2.06	2.13

♦ **Example 6.1:** Bulk modulus of elasticity of a fluid

A liquid has a volume of 2.0 ft^3 at a pressure of 100 psi. When compressed to 180 psi, the volume decreases to 1.8 ft^3. Find the bulk modulus of elasticity of this fluid.

Solution: $dp = 180 - 100 = 80$ psi $= 11{,}520$ psf and $d\forall = 1.8 - 2 = -0.2$ ft^3, and thus

$$E_{fluid} = -\frac{\forall dp}{d\forall} = -\frac{2 \times 11{,}520}{-0.2} = 115{,}200 \text{ psf}.$$

Note that 1 psi = 144 psf = 6.89 kPa.

6.2 Wave Celerity

We start with the celerity of a compressed water wave in an infinitely rigid pipe in Section 6.2.1, and in elastic pipes in Section 6.2.2. Celerity reduction methods are then discussed in Section 6.2.3.

6.2.1 Celerity in an Infinitely Rigid Pipe

The term celerity describes how fast pressure variations travel in water. It is equivalent to the speed of sound in air. Consider a pressure wave in a large body of water, or an infinitely rigid pipe shown in Figure 6.1. The wave is held stationary by moving with the wave at a constant celerity c, the mass flux entering the compressed area is $\dot{m} = \rho A(c + V)$ and the velocity decreases by $\Delta V = -V$ across the wave front.

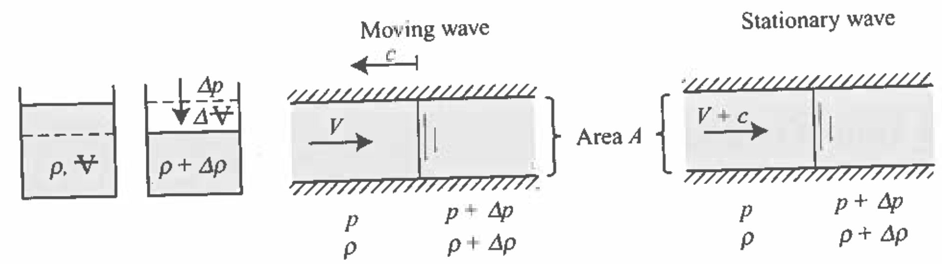

Figure 6.1 Compressible wave celerity in a rigid pipe

The force analysis in the flow direction includes an increase in pressure force from pA to $(p + dp)A$, which corresponds to the rate of change in linear flow momentum of the mass flux $\dot{m} = \rho A(V + c)$, where $c >> V$, or

$$pA - (p + dp)A = \dot{m}\Delta V = \rho A(c + V)(-V) \simeq -\rho AVc.$$

The pressure increase becomes

$$dp = \rho Vc. \tag{6.2}$$

Now, looking at the conservation of mass during a time interval dt, the mass entering the pipe $\rho AVdt$ equals the mass being compressed $d\rho Acdt$, which gives $\rho V = cd\rho$. This is combined with the modulus of elasticity $dp = E_w d\rho/\rho$ to eliminate V as

$$dp = \rho cV = \rho c\left(c\frac{d\rho}{\rho}\right) = \frac{E_w d\rho}{\rho},$$

or

$$c = \sqrt{\frac{E_w}{\rho}} = \sqrt{\frac{dp}{d\rho}}. \tag{6.3}$$

Thus, by application of the momentum and continuity equations, we have derived equations describing the increase in pressure dp (Eq. (6.2)) and the wave celerity c (Eq. (6.3)). From values of E_w in Table 6.1, the compression wave celerity in water is approximately $c = \sqrt{2.1 \times 10^9/1{,}000} \cong 1{,}450$ m/s $= 4{,}750$ ft/s.

6.2.2 Celerity in Elastic Pipes

In elastic pipes, the mass of water is not only stored by compression, but also in the expanded volume of the pipe under increased pressure. Table 6.2 lists values of the modulus of elasticity E_p for various pipe materials. Note that the values change at very high temperature, and the table lists commonly used values to solve hydraulic engineering problems.

Table 6.2. Modulus of elasticity E_p for various pipe materials

Pipe material	E_p(Pa)	E_p(psi)	E_p/E_w
Steel	1.9×10^{11}	28×10^6	90
Reinforced concrete	1.6×10^{11}	25×10^6	83
Cast iron	1.1×10^{11}	16×10^6	52
Copper	9.7×10^{10}	14×10^6	47
Glass	7.0×10^{10}	10×10^6	33
Concrete	3.0×10^{10}	4.3×10^6	14
PVC (polyvinyl chloride)	2.5×10^9	3.6×10^5	1.2
Water	2.1×10^9	3.0×10^5	1.0
HDPE (high-density polyethylene)	1×10^9	1.4×10^5	0.5
Lead	3.1×10^8	4.5×10^4	0.15

Figure 6.2 provides a schematic illustration for the analysis of wave propagation in expanding pipes. The main pipe parameters include the wall thickness e, the tensile force per unit length T_f, the pipe modulus of elasticity E_p, and the radial expansion dr under the pressure increase Δp.

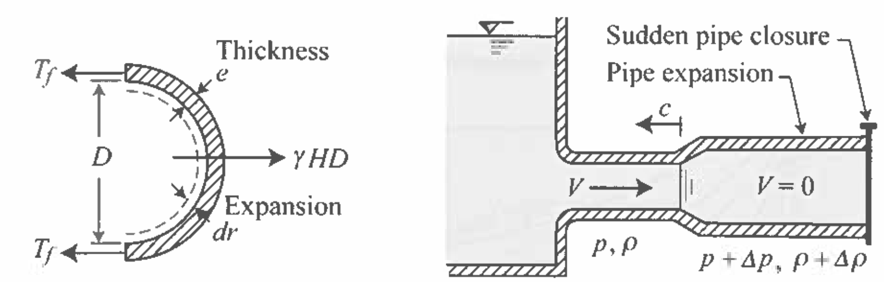

Figure 6.2 Wave propagation in an elastic pipe

Per unit pipe length, the pipe tension is

$$\sigma_p = \frac{T_f}{e} = \frac{pD}{2e} = \frac{\gamma HD}{2e},$$

and its increase is

$$d\sigma_p = \frac{dpD}{2e} = E_p\left(\frac{2dr}{D}\right)$$

or

$$dr = \frac{D^2 dp}{4E_p e}.$$

The expansion volume is given as

$$\frac{d\forall}{\forall} = \frac{4\pi D dr}{\pi D^2} = \frac{D dp}{E_p e} = -\frac{dp}{E_w},$$

which gives

$$\frac{E_w d\forall}{dp\forall} = \frac{E_w D}{E_p e}.$$

The volume change in the pipe therefore has two components: (1) the volume compressed for rigid pipes; and (2) the elastic pipe expansion volume. Therefore, when compared to the rigid-pipe analysis,

$$d\forall_{elastic} = \left[1 + \left(\frac{E_w D}{E_p e}\right)\right] d\forall_{rigid}.$$

Accordingly, the wave celerity in elastic pipes c' will be less than c for rigid pipes:

$$c' = \sqrt{\frac{E_w}{\rho}\left[1 + \left(\frac{E_w D}{E_p e}\right)\right]^{-1}},$$

or

$$\frac{c'}{c} = 1/\sqrt{\left[1 + \left(\frac{E_w D}{E_p e}\right)\right]}. \tag{6.4}$$

Obviously, the celerity in rigid pipes ($E_w/E_p = 0$) reduces to Eq. (6.3). The compressibility wave celerity in elastic pipes is shown in Figure 6.3.

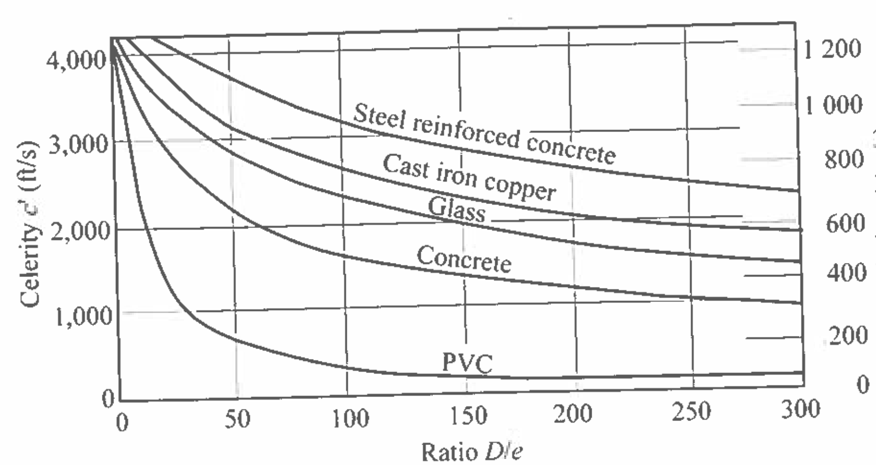

Figure 6.3 Wave celerity in elastic pipes

Example 6.2 indicates how to calculate the wave celerity in an elastic pipe.

♦ **Example 6.2:** Wave celerity in a pipe

Consider a 125-psi cast-iron pipe (Table 2.2) and calculate the wave-propagation celerity for a diameter $D = 42$ in. $= 1{,}068$ mm and thickness $e = 0.5(45.1 - 42.02)$ in. $= 39$ mm.

Solution: Consider $\rho = 1{,}000\ \text{kg/m}^3$, $E_w = 2.1 \times 10^9$ Pa and $E_p = 1.1 \times 10^{11}$ Pa to get

$$c' = \sqrt{\frac{2.1 \times 10^9}{1{,}000}\left[1 + \left(\frac{2.1 \times 10^9 \times 1.068}{1.1 \times 10^{11} \times 39}\right)\right]^{-1}} = 1{,}174\ \text{m/s},$$

compared to $c = 1{,}450$ m/s.

Notice that this celerity reduction from the elasticity of the pipe results in significant decreases in pressure surge (Eq. (6.2)) from sudden valve closures.

6.2.3 Wave Celerity Reduction with Air

Let us consider water containing air bubbles. The total volume $\forall = \forall_w + \forall_g$ equals the sum of the water volume $\forall_w$ and the gas volume $\forall_g$, and the concentration of gas $C_g = \forall_g/\forall$. The density of the mixture is $\rho = \rho_w(\forall_w/\forall) + \rho_g(\forall_g/\forall)$. A pressure change brings about a volume change equal to $\Delta\forall = \Delta\forall_w + \Delta\forall_g$. The bulk modulus of elasticity of water $E_w = -\frac{\forall_w dp}{d\forall_w} \cong 2.1 \times 10^9$ Pa, compared to the high compressibility of gas $E_g = -\frac{\forall_g dp}{d\forall_g} \simeq 9{,}000$ psf (or 430 kPa). Combining these expressions yields the combined air–water bulk modulus $E = E_w/\{1 + C_g[-1 + (E_w/E_g)]\}$, and the celerity with air bubbles is $c' = \sqrt{E/\rho}$. Figure 6.4 illustrates the significant decrease in wave

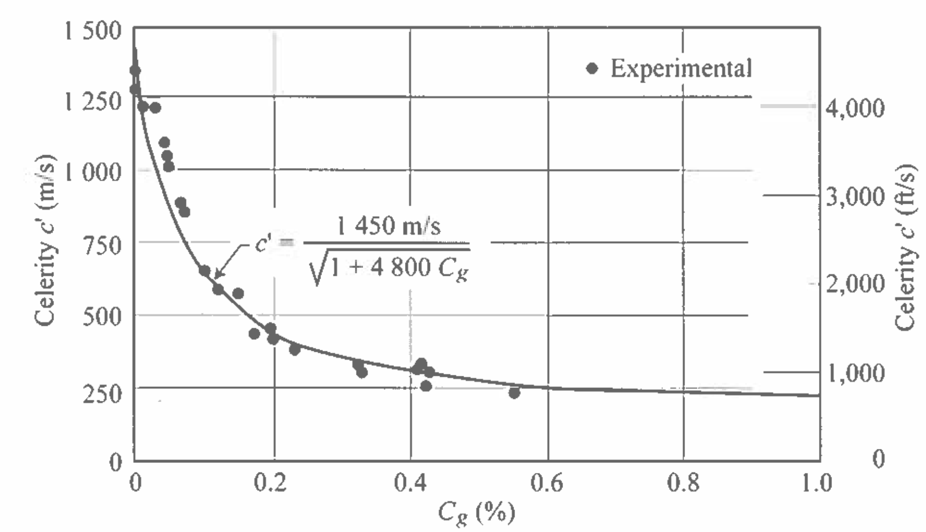

Figure 6.4 Wave celerity propagation in water with air bubbles

celerity as a function of C_g. The presence of air bubbles in water at low concentrations (e.g. $C_g > 0.2\%$) will significantly reduce the celerity of compression waves, as shown in Example 6.3.

Example 6.3: Wave celerity reduction with air
Calculate the wave-propagation speed in water containing 0.5% air ($C_g = 0.005$).

Solution: Consider $E_g = 9{,}000$ psf, $E_w = 3 \times 10^5$ psi $= 4.32 \times 10^7$ psf and the mass densities of air $\rho_g = 0.00238$ slug/ft^3 and water $\rho = 1.94$ slug/ft^3. The modulus is

$$E = \frac{E_w}{1 + C_g\left[-1 + (E_w/E_g)\right]} = \frac{4.32 \times 10^7}{1 + 0.005(4{,}800)} = 1.73 \times 10^6 \text{ psf}.$$

The celerity becomes $c' = \sqrt{E/\rho} = \sqrt{1.73 \times 10^6/1.94} = 943$ ft/s $= 288$ m/s. Air significantly reduces the pressure surge $dp = \rho V c'$ from sudden valve closures in pipes.

6.3 Hydraulic Transients

This section deals with the maximum pressure generated by a sudden valve closure (Section 6.3.1) and a gradual closure (Section 6.3.2), followed by the timescale for valve opening (Section 6.3.3) and emptying large tanks (Section 6.3.4).

6.3.1 Sudden Valve Closure

In a pipe, the time of the valve closure t_C is compared to the travel time $t_T = 2L/c'$ for the wave to propagate back and forth between the valve and the reservoir. A sudden closure occurs when $t_C < 2L/c'$. The pressure increase is $dp = \rho V c'$ and the corresponding pressure-head increase is

$$\Delta H = \frac{\Delta p}{\gamma} = \frac{\rho V c'}{\gamma} = \frac{V c'}{g}.$$

Figure 6.5 sketches the main propagation features of a water compressibility wave in a pipe after a sudden valve closure. At (a), the water enters the pipe and the pressure increase propagates to the reservoir at celerity c'. At (b), the water comes out of the pipe as pressure returns to hydrostatic condition and the wave propagates back to the valve at the same celerity. Upon return (c), and reaching the valve, the water is still pulled out of the pipe and a zone of pipe contraction develops as the change in pressure now becomes negative. This zone of lower pressure propagates from the valve to the reservoir. Upon reaching the reservoir in (d), water reenters the pipe to reestablish hydrostatic pressure conditions and the wave propagates to the valve. At the end of this double sweep, we reach the starting point from which the cycle is repeated. This series of "coup de bélier" or hydraulic ram is called a water hammer, leading to vibrations in short pipes.

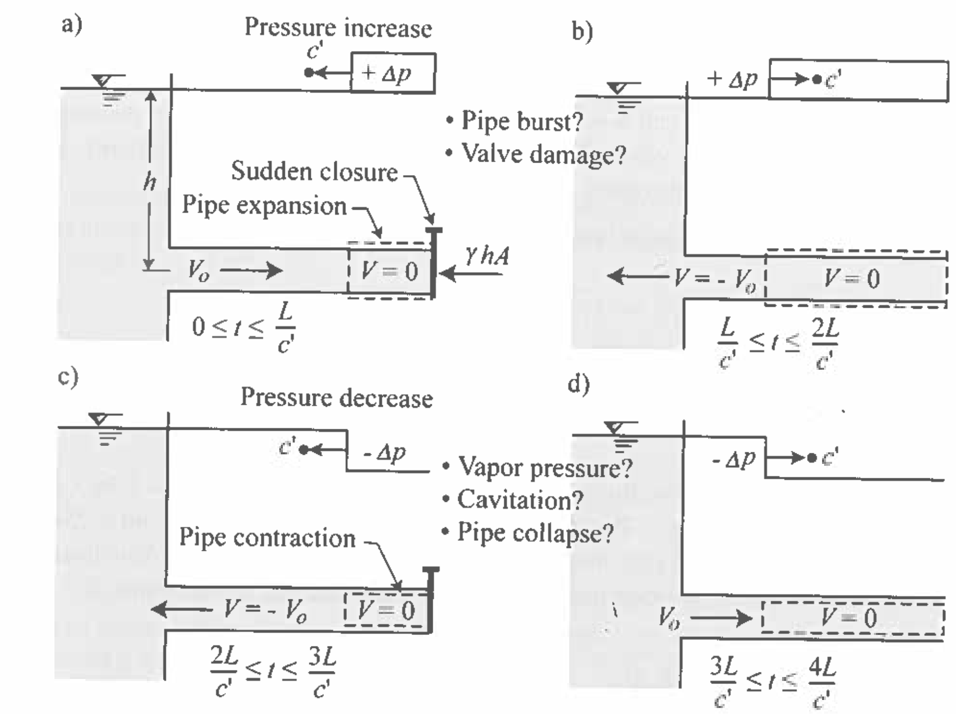

Figure 6.5 Wave propagation in a pipe after a sudden valve closure

Of course, the very high pressures induced by the product of high flow velocities and high wave celerities can lead to pipe bursting. Also, during the phase of pipe contraction, the pressure may become close enough to the vapor pressure to cause cavitation. At all times, the minimum negative pressure head can only be -10 m and cannot be below the absolute zero pressure, as shown from experiments in Figure 6.6. Example 6.4 determines the maximum pressure from a sudden pipe closure.

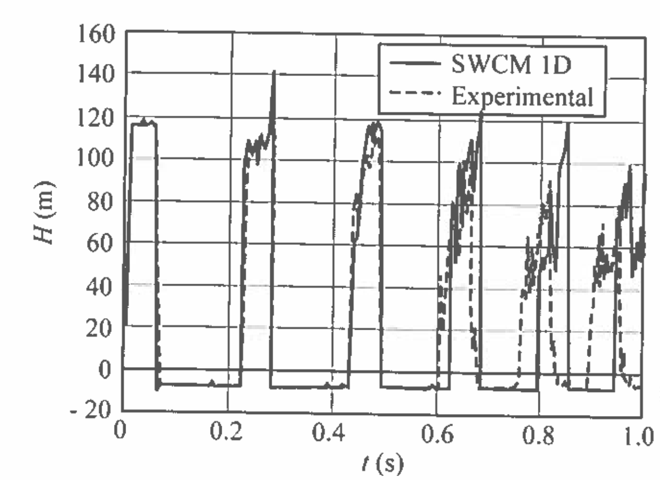

Figure 6.6 Pressure-head measurements from a water hammer in a pipe (Pezzinga and Santoro 2017)

♦♦ **Example 6.4:** Sudden valve closure in a pipe

A 3,000-ft-long, 36-in.-diameter pipe conveys water at a velocity of 4 ft/s. If the initial pressure at the downstream end is $p_0 = 40$ psi, what maximum pressure will develop at the downstream end when a valve is closed in 1 second? Would a 125- or 250-psi cast-iron pipe hold the pressure increase?

Solution: As a first approximation, consider the celerity in a rigid pipe

$$c = \sqrt{E_w/\rho} = \sqrt{(300{,}000 \times 144)/1.94} = 4{,}720 \text{ ft/s}.$$

Next, the travel time for the wave to propagate back in forth in the pipe is

$$t_T = 2L/c = 2 \times 3{,}000/4{,}720 = 1.27 \text{ s}.$$

Since the closure time $t_C < t_T$, the maximum pressure increase $\Delta p = \rho Vc = 1.94 \times 4 \times 4{,}720 = 36{,}630$ psf $= 254$ psi. The maximum pressure $p_{max} = p + \Delta p = 40 + 254 = 294$ psi. (Note that a 125-psi cast-iron pipe would burst open from this sudden closure.)

From Table 2.2, a 250-psi cast-iron pipe is 2 in. thick. The celerity becomes

$$c' = c/\sqrt{\left[1 + \left(\frac{E_w D}{E_p e}\right)\right]} = 4{,}720/\sqrt{\left[1 + \left(\frac{3 \times 36}{160 \times 2}\right)\right]} = 4{,}080 \text{ ft/s}$$

and

$$p_{max} = p_0 + \Delta p = 40 + \rho V c' = 40 + (1.94 \times 4 \times 4{,}080/144) = 260 \text{ psi}.$$

This pipe would burst open from a sudden closure. Options include a slower flow velocity and/or a gradual closure ($t_C > 2L/c' = 2 \times 3{,}000/4{,}080 = 1.5$ s).

6.3.2 Gradual Valve Closure

In the case of a gradual pipe closure without friction losses with ($t_C > 2L/c'$), the attention turns to the force required to slow down a large mass of water moving in the pipe. In general, converting the momentum flux $\rho QV = \rho AV^2$ into pressure $pA = \rho gHA$ gives $V = C\sqrt{H}$. Figure 6.7 shows a valve fully open under the initial pressure head H_0. In this case, the conversion of potential to kinetic energy results in $V_0 = \sqrt{2gH_0}$, or $V_0 = C\sqrt{H_0}$. A gradual valve closure is assumed $V = V_0\left(1 - \frac{t}{t_C}\right)$.

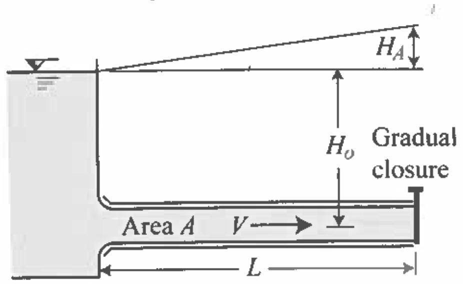

Figure 6.7 Gradual valve closure

Under a slow closure, the head at the valve is expected to increase to $H_0 + H_A$ such that the exit velocity would reach $V = C\sqrt{H_0 + H_A}$. This gives the main approximation for the velocity as a function of time:

$$\frac{V}{V_0} = \left(1 - \frac{t}{t_C}\right)\sqrt{\frac{H_0 + H_A}{H_0}}$$

and its derivative

$$\frac{dV}{dt} = \frac{-V_0}{t_C}\sqrt{\frac{H_0 + H_A}{H_0}}.$$

The maximum pressure-head increase H_A is obtained from the force difference at both ends of the pipe $F = \gamma H_A A$, which decelerates the fluid mass $M = \rho AL$ at an acceleration $a = F/M = -gH_A/L$, or

$$a = \frac{dV}{dt} = \frac{-gH_A}{L} = \frac{-V_0}{t_C}\sqrt{\frac{H_0 + H_A}{H_0}},$$

or

$$\frac{H_A}{H_0} = \frac{LV_0}{gH_0 t_C}\sqrt{1 + \frac{H_A}{H_0}}.$$

The solution to this quadratic equation is known as Allievi's formula, in memory of the Italian Lorenzo Allievi. It defines the maximum head increase H_A (or pressure increase $\Delta p = \gamma H_A$) from

$$\frac{H_A}{H_0} = \frac{\Delta p}{p_0} = \frac{N}{2} + \sqrt{\frac{N^2}{4} + N}, \qquad (6.5)$$

where $N = (LV_0/gH_0 t_C)^2 = (\rho L V_0/p_0 t_C)^2$, given the fluid mass density ρ; the initial pressure p_0, velocity V_0 and head H_0; the pipe length L; and the pipe closure time t_C. The maximum head becomes $H_0 + H_A$, and the total pressure is $p = p_0 \pm \Delta p$. Example 6.5 shows how to calculate the pressure increase in a pipe after a gradual valve closure.

♦ **Example 6.5:** Gradual valve closure in a pipe

A 2-km-long cast-iron pipe conveys a discharge of 27 m³/s in a 5-m-diameter pipe with an exit in air through a 1-m-diameter valve. The initial pressure head is 60 m and the pipe is 5-cm thick. If the valve closes in 5 seconds, what pressure surge would develop?

Solution:

Step (1): The main conditions for this pipe are $D = 5$ m, $A = 19.6$ m². Assume $f = 0.015$, and find the steady friction losses: $V_0 = 4Q/\pi D^2 = 4 \times 2.7/\pi\, 5^2 = 1.37$ m/s,

$$\Delta H = \frac{fL}{D}\frac{V^2}{2g} = \frac{0.015 \times 2{,}000}{5}\frac{1.37^2}{2 \times 9.81} = 0.6 \text{ m}$$

and friction is negligible;

or $\Delta p = \gamma H_o = 9{,}810 \times 60 = 589$ kPa (or 85.4 psi).

Step (2): Is it a sudden or gradual closure?

The bulk modulus for water is $E_w = 2.1 \times 10^9$ Pa and, for the cast-iron pipe, $E_p = 1.1 \times 10^{11}$ Pa. The wave celerity in the pipe is

$$c' = \sqrt{\frac{E_w}{\rho}\left[\frac{1}{1+\left(\frac{E_w D}{E_p e}\right)}\right]} = \sqrt{\frac{2.1\times10^9}{1{,}000}\left[\frac{1}{1+\left(\frac{2.1\times10^9\times5}{1.1\times10^{11}\times0.05}\right)}\right]} = 850\ \text{m/s}.$$

The back-and-forth travel time for the wave is $t_T = 2L/c' = 2 \times 2{,}000/850 = 4.7$ s, which is less than the closure time $t_C = 5$ s, resulting in a gradual valve closure.

Step (3): Calculate the pressure increase.

The pressure increase from Allievi's formula is estimated as

$$N = \left(\frac{\rho L V_0}{p_0 t_C}\right)^2 = \left(\frac{L V_0}{g H_0 t_C}\right)^2 = \left(\frac{2{,}000\times1.37}{9.81\times60\times5}\right)^2 = 0.867.$$

Then,

$$\frac{H_A}{H_0} = \frac{N}{2} + \sqrt{\frac{N^2}{4}+N} = \frac{0.867}{2} + \sqrt{\frac{0.867^2}{4}+0.867} = 1.46.$$

This corresponds to a pressure head $H_A = 1.46 H_0 = 1.46 \times 60 = 88$ m, and a corresponding maximum pressure $p = p_o + \Delta p = p_0 + \gamma H_0 = 589 + (9.81 \times 88) = 589 + 863 = 1{,}452$ kPa (or 210 psi).

For comparison, the sudden closure formula would yield a pressure increase $\Delta p = \rho V c' = 1{,}000 \times 1.37 \times 850 = 1{,}173$ kPa (or 170 psi) and a total pressure $p = p_0 + \Delta p = 589 + 1{,}173 = 1{,}762$ kPa (or 255 psi) which would burst a 250-psi pipe.

With reference to Figure 6.5, each high-pressure pulse for a sudden pipe closure would reoccur every period $T = 4L/c' = 4 \times 2{,}000/850 = 9.4$ s.

The benefits of a gradual closure to prevent pipe bursting can be significant.

6.3.3 Valve Opening

Opening valves in long pipes with friction can require a long time to reach steady flow conditions. After a valve is suddenly opened in Figure 6.8, the total head H accelerates the flow. As the velocity increases, the pressure head is reduced by friction f until steady flow conditions are reached.

The steady velocity V_0 is given from

$$H = \frac{fL}{D}\frac{V_0^2}{2g},$$

or

$$\frac{gH}{L} = \frac{fV_0^2}{2D}$$

and

$$\frac{2D}{f} = \frac{LV_0^2}{gH}.$$

The equation of motion,

$$\gamma A\left(H - \frac{fL}{D}\frac{V^2}{2g}\right) = m\frac{dV}{dt} = \frac{\gamma A L}{g}\frac{dV}{dt},$$

reduces to

$$\frac{dV}{dt} = \frac{gH}{L} - \frac{fV^2}{2D} = \frac{f}{2D}\left(V_0^2 - V^2\right).$$

By solving for dt and integrating gives

$$\int_0^t dt = \frac{2D}{f}\int_0^V \frac{dV}{V_0^2 - V^2} = \frac{LV_0^2}{gH}\int_0^V \frac{dV}{V_0^2 - V^2}$$

or

$$t = \frac{LV_0}{2gH}\ln\left[\frac{1+(V/V_0)}{1-(V/V_0)}\right]. \qquad (6.6)$$

We can also use the hyperbolic tangent function $\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$ since we also know that $\ln[(1+x)/(1-x)] = 2\tanh^{-1}(x)$. Therefore, we get

$$t = \frac{LV_0}{gH}\tanh^{-1}\left(\frac{V}{V_0}\right),$$

or

$$\frac{V}{V_0} = \tanh\left(\frac{gHt}{LV_0}\right). \qquad (6.7)$$

From Figure 6.8, we notice that $t \approx 2LV_0/gH$, with more details in Example 6.6.

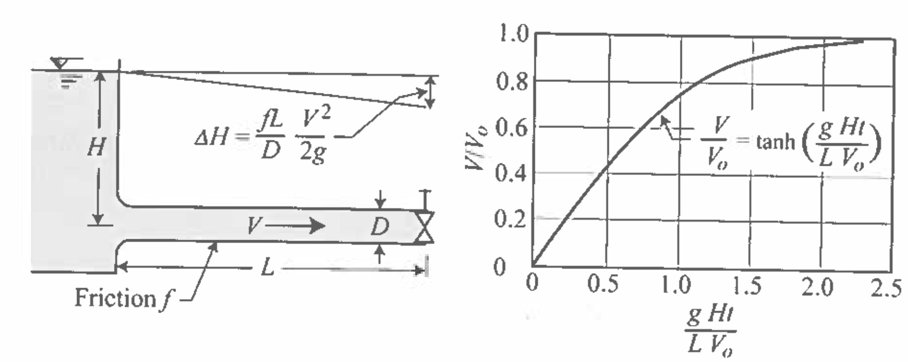

Figure 6.8 Definition sketch for a valve opening

Example 6.6: Valve opening

Consider a pipe $L = 10{,}000$ ft, $D = 8$ ft, $f = 0.03$ and $H = 60$ ft. How long does it take after opening the valve to reach 90% of the steady-state flow velocity $V = 0.9\ V_0$?

Solution: The equilibrium velocity is $V_0 = \sqrt{\frac{2gHD}{fL}} = \sqrt{\frac{2\times 32.2\times 60\times 8}{0.03\times 10{,}000}} = 10.2$ ft/s and after substituting $V = 0.9V_0$ we get

$$t_{0.9} = \frac{LV_0}{2gH}\ln\left(\frac{1.9}{0.1}\right) = \frac{10{,}000\times 10.2}{2\times 32.2\times 60}\ln(19) = 78\ \text{s}.$$

6.3.4 Emptying Large Tanks

To empty a circular tank of diameter D through an orifice of diameter d, as shown in Figure 6.9, we consider the exit flow velocity through the orifice is $V_0 = C_c\sqrt{2gh}$ where C_c is the orifice coefficient. The outflowing jet forms a contracted vein of fluid called "vena contracta," a term coined by Evangelista Torricelli in the seventeenth century. Torricelli was a student of Galileo.

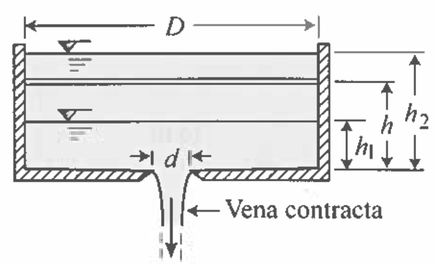

Figure 6.9 Orifice flow

The second condition with $Q = (\pi d^2/4)V_0$ stems from continuity which implies that the volumetric change is $Qdt = A_0V_0dt = (\pi d^2/4)C_c\sqrt{2gh}dt = -(\pi D^2/4)dh$. Separating t and h gives

$$dt = -\frac{D^2 dh}{d^2 C_c\sqrt{2gh}},$$

which is integrated from h_2 to h_1 as

$$t_{0-1} = \frac{2D^2}{C_c d^2\sqrt{2g}}\left(\sqrt{h_2} - \sqrt{h_1}\right). \tag{6.8}$$

Orifice flow through a large tank is analyzed in Example 6.7.

Example 6.7: Draining a tank

A 28-ft-diameter tank contains 18 ft of water. What is the time to drain the tank through a 1-inch orifice ($\alpha = 90°$) at the bottom of the tank, as shown in Fig. E-6.7.

Solution: The area of the tank is $A_T = \pi(28)^2/4 = 616$ ft^2 and the area of the orifice is $A_0 = \pi/(4\times 12^2) = 0.00545$ ft^2, and $C_c = 0.61$ for $\alpha = 90°$ from Fig. E-6.7.

The time needed to empty the tank is

$$t_{0-1} = \frac{2D^2}{C_c d^2\sqrt{2g}}\left(\sqrt{h_o} - \sqrt{h_1}\right) = \frac{2\times 28^2\times 12^2}{0.61\sqrt{2\times 32.2}}\left(\sqrt{18} - 0\right) = 195{,}000\ \text{s}$$
$$= 2.26\ \text{days}$$

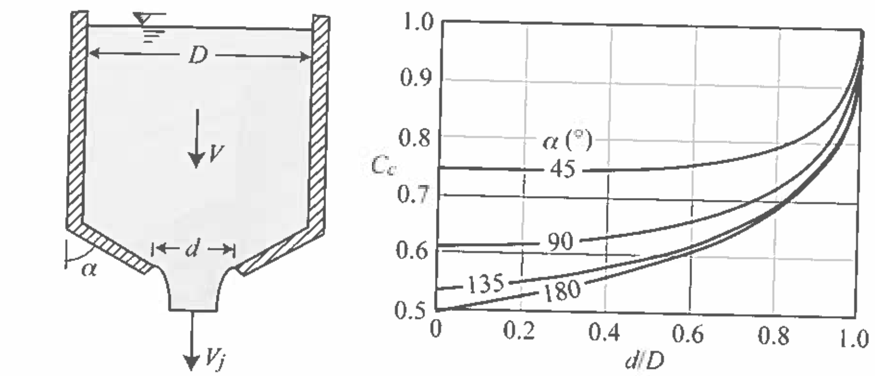

Fig. E-6.7 Vena contracta coefficient

The reader will notice that the tank would empty faster when $\alpha < 90°$.

6.4 Surge Tanks

Surge tanks are covered in Section 6.4.1 and pressure reduction in Section 6.4.2.

6.4.1 Surge Tanks

Surge tanks are a major part of hydropower projects and serve to reduce pressure surges when turbines or pump operations are stopped. A surge tank is typically a large vertical pipe connected to the main pipe. We are concerned with the flow oscillations between the large reservoir and the surge tank. As sketched in Figure 6.10, the water level in the tank rises to a level S above the initial level.

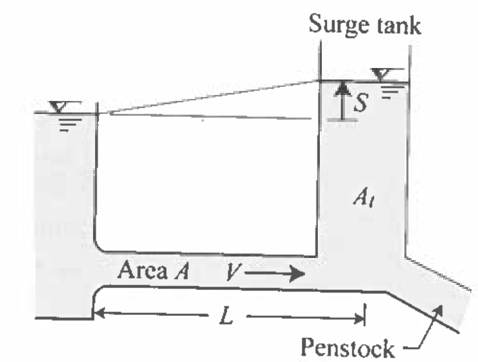

Figure 6.10 Surge tank

The fluid motion is analyzed for the case of negligible friction losses between the reservoir and the surge tank. The flow discharge Q_0 through the pipe is given by the pipe cross section A and steady flow velocity V_0 before the closure. The flow will enter the surge tank of cross-sectional area A_t and the velocity in the tank corresponds to the change in water level S over time dS/dt; therefore, the continuity relationship is written as $\frac{dS}{dt} = \frac{AV}{A_t}$. The momentum equation $F = m\frac{dV}{dt}$ is applied to the fluid mass in the pipe $(\rho AL)\frac{dV}{dt} = -\rho g A S$, which gives

$$\frac{dV}{dt} = \frac{-gS}{L} = \frac{dV}{dS}\frac{dS}{dt} = \frac{AV}{A_t}\frac{dV}{dS}.$$

Integrating $VdV = \frac{-gA_tSdS}{AL}$, with $V = V_0$ and $S = 0$ at $t = t_o$ gives $V^2 = \frac{-gA_tS^2}{AL} + V_0^2$. The maximum surge S_{max} is obtained when $V = 0$,

$$S_{max} = V_0\sqrt{\frac{AL}{A_t g}},$$

and

$$T = 2\pi\sqrt{\frac{A_t L}{Ag}}; \qquad (6.9)$$

Chapter 7 will provide more details on the derivation of the oscillation period T. Both S_{max} and T depend on the tank area A_t, the pipe length L, the pipe area A and the initial flow velocity V_0. Typical surge tank calculations are shown in Example 6.8; and Case Study 6.1 provides some details of a surge tank.

♦ Example 6.8: Surge tank

A 10-ft-diameter pipe carries 843 cfs over a length of 3,000 ft before reaching a surge tank with a 314-ft^2 cross-sectional area. During a sudden turbine shutdown, what will be the magnitude and time to reach the maximum water level S_{max} in the surge tank?

Solution: The pipe area is $A = 0.25\pi D^2 = 25\pi$ and the velocity $V_0 = Q/A = 843/25\pi = 10.7$ ft/s. The maximum surge height is

$$S_{max} = V_0\sqrt{\frac{AL}{A_t g}} = 10.73\sqrt{\frac{25\pi \times 3{,}000}{314 \times 32.2}} = 51.8 \text{ ft}$$

at a quarter of the flow oscillation period, or

$$t_{max} = \frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{A_t L}{Ag}} = \frac{\pi}{2}\sqrt{\frac{314 \times 3{,}000}{25\pi \times 32.2}} = 30 \text{ s.}$$

♦ Case Study 6.1: The Edmonston surge tank, USA

This case study supplements the information presented in Case Study 4.1 from the California Department of Water Resources (Deukmejian et al. 1985). The A.D. Edmonston Pumping Plant has a surge tank located at the top of the two pipelines, as shown in Fig. CS-6.1. At the top, the water enters a cylindrical surge tank which is 62 ft high and 50 ft in diameter. The normal water level in the surge tank is 3,165 ft with a maximum at 3,180 ft. The two main pipes are 8,400-ft long and each conveys 2,205 cfs. The pipe diameters are 12.5 ft for the lower half and 14 ft for the upper half. Each pipe contains 1.2×10^6 ft^3 of water. The closure time for the 4-ft-diameter ball valve at the base of the pipelines is 80% in 10 s and the remainder in the next 20 s. The reader can check that this time of closure is shorter than: (1) the time to fill or drain the surge tank;

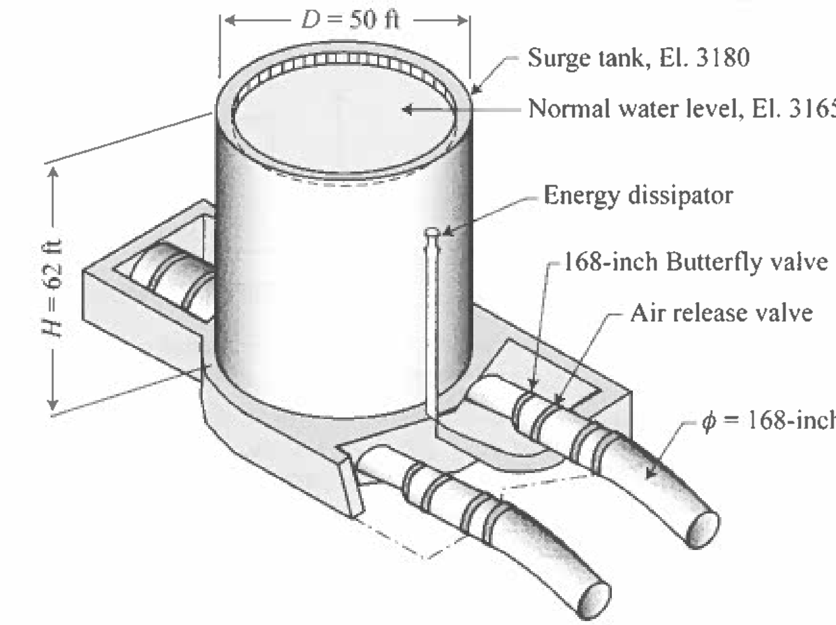

Fig. CS-6.1 Edmonston surge tank

(2) the period of oscillations in the pipeline and surge tank; and the time to drain the pipeline through the ball valve at the base.

6.4.2 Pressure Reduction

Other devices to reduce excessive pressure in pipes include compressed air chambers, and relief valves in Figure 6.11. They are usually placed beside pumps to prevent the pressure surge caused by power outages. These devices are connected to pipe systems to reduce the pressure surge when a pump suddenly stops operating.

Additional Resources

Benjamin Wylie from Michigan advanced the analysis of fluid transients (Wylie and Streeter 1978). Additional resources on water hammer and surge tanks include Parmakian (1963), Rich (1984a and b), Ghidaoui et al. (2005), Chaudhry (2014) and Guo et al. (2017). Recent reviews on hydraulic transients and negative pressure include Chaudhry (2020), Ferras et al. (2020) and Karney (2020).

EXERCISES

These exercises review the essential concepts from this chapter.

1. What does the modulus of elasticity measure?
2. What is a wave celerity?
3. Is the wave celerity of water greater in the ocean or in a pipe? Why?
4. Why is the wave celerity in pipes important?
5. Why should we care about air bubbles in pipes?

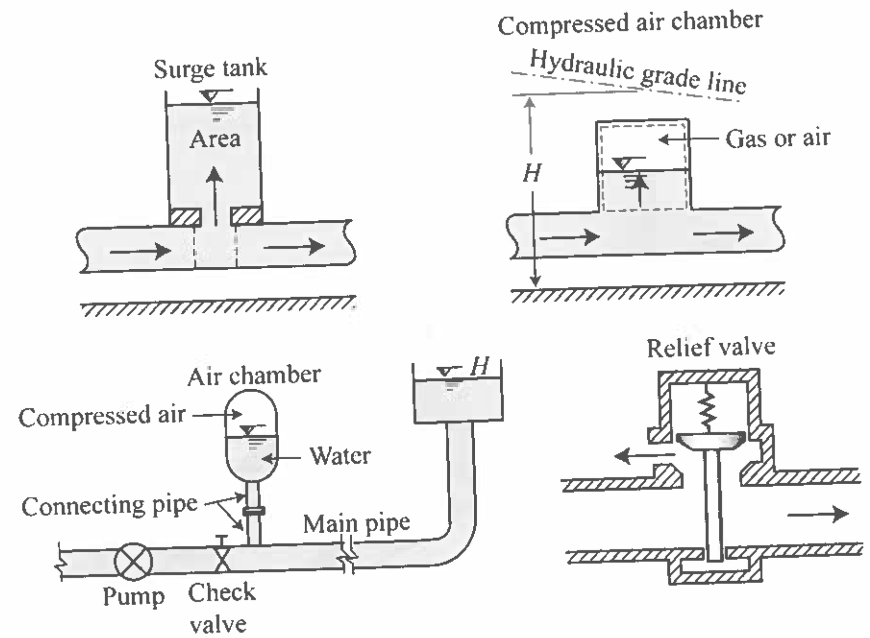

Figure 6.11 Pressure reduction in pipes

6. Why do we use twice the pipe length in the analysis of water hammer?
7. What is the lowest pressure generated from a sudden valve closure?
8. Why is the time of closure of valves important?
9. Are the water compressibility effects included in Allievi's formula?
10. Is the pipe friction loss included in Allievi's formula?
11. What is a vena contracta? Does it increase or decrease the flow rate of a given opening?
12. What does a relief valve do?
13. What is the purpose of a surge tank?
14. True or false?
    (a) The modulus of elasticity of water does not change significantly with temperature or pressure.
    (b) The wave celerity in a pipe is approximately one mile per hour.
    (c) Water is more elastic than steel.
    (d) The pipe expansion under pressure increases with pipe thickness.
    (e) Thicker pipes contribute to higher pressures from sudden closures.
    (f) Lower wave celerities reduce the pressure increase from a sudden valve closure.
    (g) Compared with steel, PVC pipes reduce the pressure from sudden valve closures.
    (h) The time required to establish the flow in a valve opening is approximately $2LV/gH$.
    (i) Compressed air is an effective way to reduce the pipe pressure fluctuations.
    (j) The present worth analysis distributes the cost of infrastructure over long periods of time.

SEARCHING THE WEB

Find photos of the following features, study them carefully and write down your observations.

1. ♦ Burst pipes.
2. ♦ Surge tanks.

PROBLEMS

Hydraulic Transients and Surge Tanks

1. ♦♦ A cast-iron 18-in.-diameter pipeline carries water at 70 °F over 1,000 ft from a reservoir to a powerhouse. The flow velocity is 5 ft/s and the initial pressure is 46 psi. If the cast-iron pipe is 1-in. thick, consider the following questions.
    (a) The wave celerity in this pipe.
    (b) The added pressure generated by a sudden closure.
    (c) Would the sudden closure cause cavitation or burst the 175-psi pipe open?
    (d) How long would the closure time have to be to reduce the maximum pressure?
    (e) What is the increased pressure if the time of closure is 1 second?
    (f) A 10-ft-diameter surge tank is built halfway in the line. Find the maximum surge height?
    (g) What is the period of oscillations in the surge tank?
2. ♦♦ The City of Thornton considers a water pipeline. From a 2015 newspaper article, you extract the main statements: (a) current population 138,000; (b) demographic expansion up to 242,000 in 10 years; (c) pipeline delivery of 14,000 ac. · ft of water per year; (d) 60-mile length; (e) 48-inch pipe diameter; (f) preliminary cost estimate \$400 million; and (g) on line in 2025. A second newspaper article in 2017 indicates that 1 ac. · ft of water meets the demand of three–four urban households and the value of water recently increased from \$6,500 to \$16,700 per ac. · ft. Consider the following questions.
    (a) What is the cost of the pipeline per linear foot?
    (b) If the population starts at 138,000 residents, what is the annual population increase?
    (c) What is the continuous equivalent discharge: (i) in cfs for 14,000 ac. · ft per year; and (ii) in gallons per household per day?
    (d) Assuming $p_0 = 40$ psi, what is the pressure head in the pipe?
    (e) How long would it take to establish 95% of the equilibrium flow velocity after a sudden opening?
    (f) Assuming that friction is negligible, what is the period of oscillations in this pipe?
    (g) Assuming a rigid pipe with $f = 0.02$ and $p_0 = 40$ psi, what would be the pressure generated from a sudden closure?
    (h) What if a valve closure requires 10 seconds?
    (i) Based on the value of water from the second article, do you think this project has a high or low benefit to cost ratio?

Cost Analysis (see also Appendix A)

3. ♦♦ The cost to build a hydropower plant is $50,000,000. The annual energy generation is equal to 52,000 MWh (megawatt-hours) and the value of electricity is constant at $70 per MWh. The period of contract for energy supply is 20 years and the annual interest rate is 5%.
   (a) Is this contract lucrative? [Hint: use present worth analysis.]
   (b) What should be the minimum value of electricity to break even in this contract?
   (c) Suppose that value of electricity is fixed, but you can sell the hydropower plant at the end of the contract. What should be the minimum sale value?
4. ♦♦ Consider the 7.5-m-diameter penstock in Case Study 5.1. You could increase your revenue by $250,000 per year over a period of 30 years by enlarging the penstock diameter to 9.5 m. Consider interest rates of 0% and 5% and compare the results.
   (a) What is the incremental construction cost?
   (b) Is this contract lucrative over 30 years if the interest rate is 0%?
   (c) Is this expansion project valuable if the interest rate is 5%?
   (d) Is this contract profitable if the interest rate is 4% but you lose the first year of revenue because of the project construction?

7 Pipe Flow Oscillations

This brief chapter supplements Chapter 6 and is normally not covered in undergraduate courses. The material helps graduate and honors students bridge the gap between spring–mass systems covered in engineering mechanics and flow oscillations in pipes. This more advanced treatment focuses on fluid oscillations in pipes without friction in Section 7.1, with laminar friction covered in Section 7.2, turbulent friction in Section 7.3, and oscillations between reservoirs considered in Section 7.4.

7.1 Oscillations without Friction

We first review spring–mass systems in Section 7.1.1, with applications to water in Section 7.1.2.

7.1.1 Spring–Mass Oscillations

Let's consider the free vibrations of a spring–mass system in Figure 7.1.

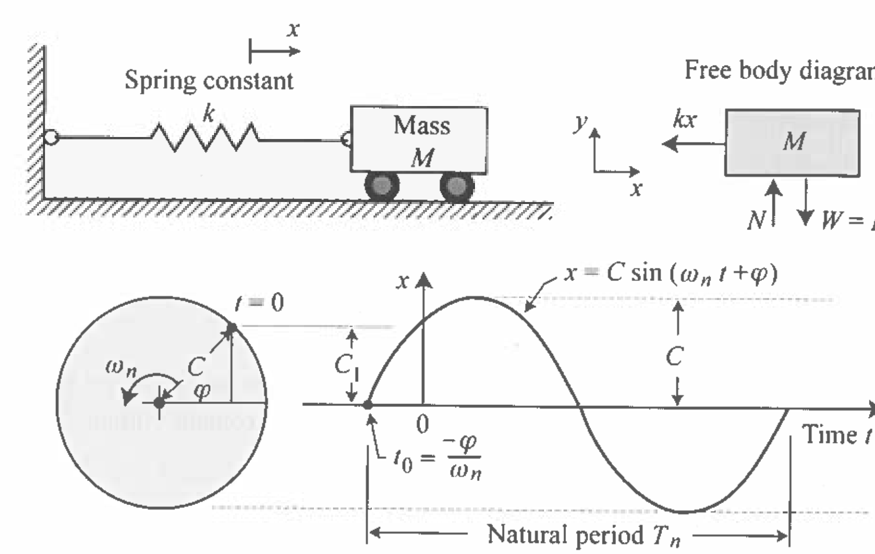

Figure 7.1 Oscillations of a spring–mass system

The short-hand notation uses dots for time derivatives, $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$, and the force balance from the free-body diagram is $\sum F_x = -kx = M\ddot{x}$, or

$$\ddot{x} + \frac{k}{M}x = 0. \quad (7.1)$$

The solution for the displacement x as a function of time t is $x = C\sin(\omega_n t + \varphi)$ where C is the amplitude of the motion, ω_n is the natural circular frequency and φ is the initial angle at $t = 0$. The velocity and acceleration are $V = \dot{x} = \omega_n C\cos(\omega_n t + \varphi)$ and $a = \ddot{x} = -\omega_n{}^2 C\sin(\omega_n t + \varphi)$. Substitution into Eq. (7.1) gives $-\omega_n{}^2 C\sin(\omega_n t + \varphi) + \frac{k}{M}C\sin(\omega_n t + \varphi) = 0$ from which we obtain the natural circular frequency ω_n in radians per second,

$$\omega_n = \sqrt{\frac{k}{M}}. \quad (7.2)$$

The corresponding natural frequency is $f_n = \frac{\omega_n}{2\pi}$ in cycles per second or hertz (1 Hz = 60 rpm), and the natural period of oscillations without friction is $T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$. Instead of using the initial angle φ, the equation of motion can be written as $x = C_1\cos(\omega_n t) + C_2\sin(\omega_n t)$. The initial displacement and velocity ($t = 0$) are $x_0 = C_1 = C\sin\varphi$ and $V_0 = \dot{x}_0 = \omega_n C_2 = \omega_n C\cos\varphi$, or $\varphi = \tan^{-1}(\omega_n x_0/V_0)$.

7.1.2 Flow Oscillations without Friction

The oscillations of a liquid without friction in the U-tube sketched in Figure 7.2 are now considered. Given the cross-sectional area A of the tube and the length L of the water column, the mass to be accelerated is ρAL. Euler's equation of motion considers that the pressure force $\rho g(z_2 - z_1)A$ equals the fluid mass times its acceleration:

$$pA = \rho g(z_2 - z_1)A = -\rho AL\frac{dV}{dt}.$$

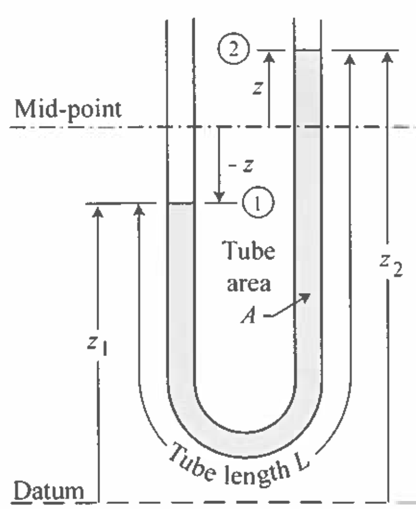

Figure 7.2 Oscillations without friction

Dividing by ρAL, and considering both $(z_2 - z_1) = 2z$ and $a = \ddot{z} = \frac{dV}{dt}$, we obtain the main equation $\ddot{z} + \frac{2g}{L}z = 0$.

This equation is simply Eq. (7.1) where $\omega_n^2 = \frac{k}{M} = \frac{2g}{L}$, with the natural period of oscillations,

$$T_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{L}{2g}}. \quad (7.3)$$

The fluid elevation z in a water column without friction varies with time t as

$$z = C_1\cos\sqrt{\frac{2g}{L}}t + C_2\sin\sqrt{\frac{2g}{L}}t. \quad (7.4)$$

The position z_0 at $t = 0$ defines $C_1 = z_0$ and the initial velocity V_0 gives

$$C_2 = V_0/\omega_n = V_0\sqrt{L/2g}.$$

Numerical solutions to the equation of motion are also possible (and often preferable). The acceleration $\ddot{z}$ is first solved and used to calculate the velocity from $V = V_0 + \ddot{z}\Delta t$. The displacement over this short time interval Δt is then simply obtained from $z = z_0 + V\Delta t$. The method becomes increasingly accurate as $\Delta t \to 0$. Example 7.1 shows detailed calculations for pipe flow oscillations without friction.

♦♦ Example 7.1: Pipe flow oscillations without friction

A frictionless fluid column 4.025-ft long has an initial upward velocity of $V_0 = 4$ ft/s at $z_0 = 1$ ft. Find: (1) the period; (2) the equation of motion; and (3) the maximum elevation z_{max} and velocity V_{max}.

Solution:

(1) The natural circular frequency is

$$\omega_n = \sqrt{\frac{2g}{L}} = \sqrt{\frac{2 \times 32.2}{4.025}} = 4 \text{ rad/s}$$

and the natural period of oscillations is

$$T_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = 1.57 \text{ s.}$$

(2) From the initial conditions $C_1 = z_0 = 1$ and $C_2 = V_0/\omega_n = 4/4 = 1$, we obtain the equation of motion $z = \cos(4t) + \sin(4t)$.

Alternatively, $C = \sqrt{C_1^2 + C_2^2} = \sqrt{2}$ and $\varphi = \tan^{-1}C_1/C_2 = \pi/4 = 45°$, and $z = \sqrt{2}\sin(4t + \pi/4)$.

(3) The maximum elevation z_{max} when $\sin(4t_m + \pi/4) = 1$ gives $z_{max} = C = \sqrt{2} = 1.41$ ft.

The time for z_{max} occurs when $\dot{z} = -4\sin(4t) + 4\cos(4t) = 0$ or $\tan(4t_{max}) = 1$ and $t_{max} = \frac{\pi}{16} = 0.196$ s.

The maximum velocity occurs when the acceleration $\ddot{z} = -16\cos(4t) - 16\sin(4t) = 0$, which is $\tan(4t_{Vmax}) = -1$, or the time where the vecocity is maximum is

$$t_{Vmax} = -\frac{\pi}{16}, \text{or } \frac{7\pi}{16} = 1.374 \text{ s,}$$

and the maximum velocity becomes

$$V_{max} = \dot{z}_{max} = -4\sin\left(\frac{-4\pi}{16}\right) + 4\cos\left(\frac{-4\pi}{16}\right) = 5.66 \text{ ft/s.}$$

Alternatively, we simply obtain the maximum velocity from $V_{max} = \omega_n C = 4\sqrt{2} = 5.66$ ft/s as shown in Fig. E-7.1.

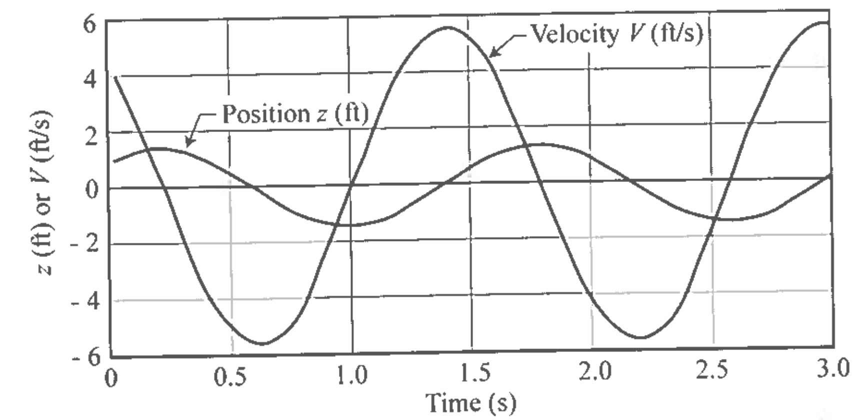

Fig. E-7.1 Pipe flow oscillations without friction

7.2 Oscillations for Laminar Flow

We first review damped spring-mass systems in Section 7.2.1 with applications to laminar flow in capillary tubes in Section 7.2.2.

7.2.1 Damped Spring–Mass Oscillations

Let us first consider the damped vibrations of the spring-mass system shown in Figure 7.3. The damping coefficient represents resistance to the motion. The damping

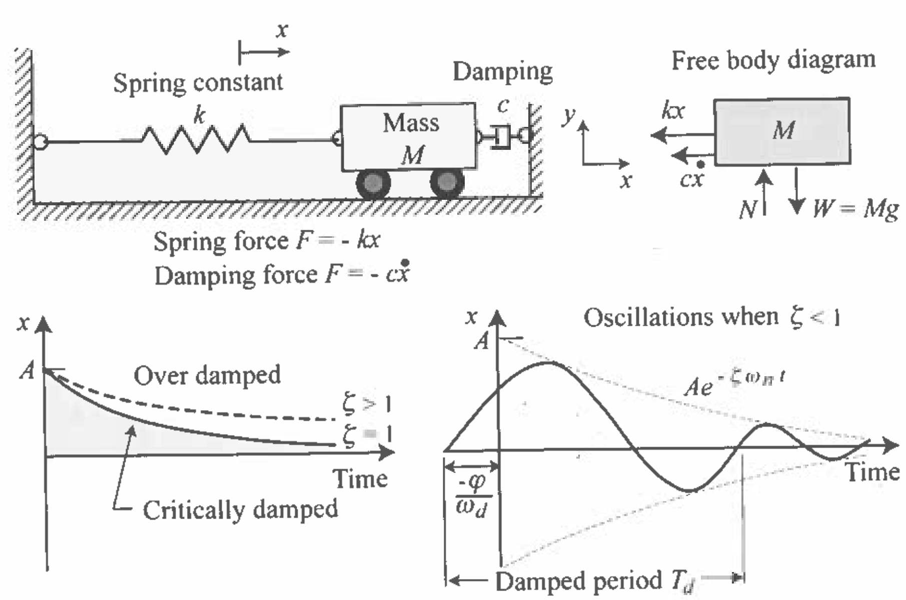

Figure 7.3 Oscillations of a damped spring–mass system

force is opposite to the velocity such that when the mass moves to the right ($\dot{x} > 0$), the force is applied to the left ($F = -c\dot{x}$).

The sum of forces in the x direction includes the damping force component of magnitude $-c\dot{x}$ in the direction opposite to the velocity $\dot{x}$, or $\sum F_x = -kx - c\dot{x} = M\ddot{x}$, or $\ddot{x} + \frac{c}{M}\dot{x} + \frac{k}{M}x = 0$.

The general form of this differential equation is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \tag{7.5}$$

with the dimensionless damping coefficient $\zeta = \frac{c}{2M\omega_n}$ and the natural circular frequency $\omega_n = \sqrt{\frac{k}{M}}$.

The general equations for the position and velocity are, respectively,

$$x = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \varphi), \tag{7.6}$$

$$\dot{x} = Ae^{-\zeta\omega_n t}\{[\omega_d\cos(\omega_d t + \varphi)] - [\zeta\omega_n\sin(\omega_d t + \varphi)]\}, \tag{7.7}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped circular frequency (the system only vibrates when $\zeta < 1$), with the phase angle φ, damped frequency $f_d = \frac{\omega_d}{2\pi}$ and damped period $T_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d}$.

The solution to the problem of damped oscillations with laminar flow is illustrated for two practical cases: (1) case A with initial displacement; and (2) case B with initial velocity.

Case A: initial displacement at $t = 0$, $x_0 = C$ and $V_0 = \dot{x}_0 = 0$.
From the displacement in Eq. (7.6) at $t = 0$, we obtain $A = C/\sin\varphi$.
From the velocity in Eq. (7.7) at $t = 0$, $\tan\varphi = \frac{\omega_d}{\zeta\omega_n}$, or $\varphi = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right)$.
The successive maximum/minimum positions are at times $t = 0, \frac{T_d}{2} = \frac{\pi}{\omega_d}, T_d = \frac{2\pi}{\omega_d}\ldots$
The successive maximim/minimum velocities are at times $t = \frac{T_d}{4} = \frac{\pi}{2\omega_d}, \frac{3T_d}{4} = \frac{3\pi}{2\omega_d}\ldots$
The graph of the position as a function of time is sketched in Figure 7.4.

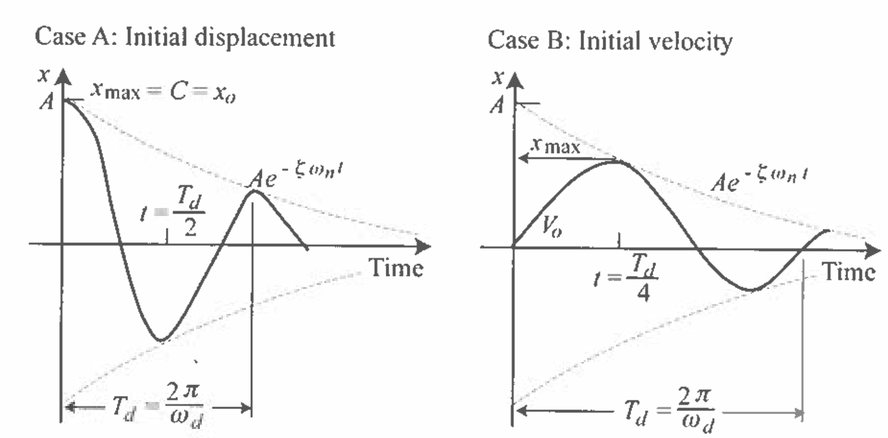

Figure 7.4 Position of a damped spring–mass for initial displacement and velocity

Case B: initial velocity at $t = 0$, $x_0 = 0$ and $V_0 = A\omega_d$.
From the position and velocity in Eqs. (7.6) and (7.7), we obtain $\varphi = 0$ and $A = V_0/\omega_d$.
The successive maximum/minimum positions are at times $t = \frac{T_d}{4} = \frac{\pi}{2\omega_d}, \frac{3T_d}{4} = \frac{3\pi}{2\omega_d} \ldots$
The successive maximum/minimum velocities are at times $t = 0, \frac{T_d}{2} = \frac{\pi}{\omega_d}, T_d = \frac{2\pi}{\omega_d} \ldots$
We are now ready to study the unsteady fluid motion in capillary tubes with viscous damping.

7.2.2 Damped Flow Oscillations in Capillary Tubes

This section is applicable to damped oscillations for laminar flow (Re < 2,000) in capillary tubes of constant area A. With reference to Figure 7.5, the equation of motion includes the friction losses h_f:

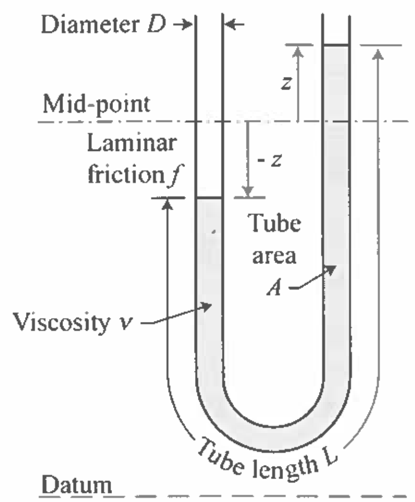

Figure 7.5 Oscillations with viscous friction

$$\rho g(z_2 - z_1)A + \rho g h_f A = -\rho A L \frac{dV}{dt},$$

where the friction loss are

$$h_f = \frac{fL}{D}\frac{V|V|}{2g}.$$

For laminar flow, the Darcy–Weisbach friction factor $f = \frac{64}{\mathrm{Re}} = \frac{64\upsilon}{VD}$ depends on the kinematic viscosity υ of the fluid. Therefore, the friction loss is given by

$$h_f = \frac{64\upsilon}{VD}\left(\frac{L}{D}\right)\frac{V|V|}{2g}.$$

Dividing the equation of motion by ρA with $z_2 - z_1 = 2z$ gives

$$L\frac{dV}{dt} + \frac{32\upsilon L V}{D^2} + 2gz = 0.$$

This is simplified further with $V = dz/dt = \dot{z}$ and $dV/dt = \ddot{z}$; thus,

$$\ddot{z} + \frac{32\upsilon}{D^2}\dot{z} + \frac{2g}{L}z = 0,$$

which is equivalent to Eq. (7.5) where $\omega_n = \sqrt{2g/L}$, and $2\zeta\omega_n = 32\upsilon/D^2$ gives $\zeta = 16\upsilon/\omega_n D^2$, and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$.

In summary, the displacement can be analytically defined from Eq. (7.6)

$$z = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \varphi) \tag{7.8}$$

and the velocity is given as

$$V = \dot{z} = Ae^{-\zeta\omega_n t}\{[\omega_d\cos(\omega_d t + \varphi)] - [\zeta\omega_n\sin(\omega_d t + \varphi)]\}, \tag{7.9}$$

where $\omega_n = \sqrt{2g/L}$, $\zeta = 16\upsilon/\omega_n D^2$ and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$.

Finally, A and φ depend on the initial displacement and velocity conditions. For instance, the boundary conditions for an initial displacement C without velocity is $A = C/\sin\varphi$, and $\varphi = \tan^{-1}(\omega_d/\zeta\omega_n)$. This may sound complicated without the application detailed in Example 7.2.

♦ **Example 7.2:** Laminar flow oscillations in a capillary tube
Consider oscillations in a 10-ft-long and 1.0-in.-diameter U-tube containing a fluid more viscous than water, $\upsilon = 1 \times 10^{-4}$ ft²/s. The initial head difference between both ends of the tube is 16 in. Can you find the equation for the position z as a function of time? Also find the maximum velocity and Reynolds number to double check that the flow is laminar.

Solution:
Step (1): The natural circular frequency is

$$\omega_n = \sqrt{\frac{2g}{L}} = \sqrt{\frac{2 \times 32.2}{10}} = 2.54 \text{ radians per second.}$$

The damping coefficient is

$$\zeta = \frac{16\,\upsilon}{\omega_n D^2} = \frac{16 \times 10^{-4} \times 12^2}{2.54} = 0.091,$$

with oscillations, because $\zeta < 1$.
The damped circular frequency is
$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 2.54\sqrt{1 - 0.091^2} = 2.53$ rad/s and $\zeta\omega_n = 0.091 \times 2.54 = 0.231$.
Step (2): As in case A, the initial conditions give the phase angle:

$$\varphi = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right) = \tan^{-1}\left(\frac{2.53}{0.231}\right) = 1.48 \text{ rad} = 85°.$$

The initial conditions are $C = 0.5 \times 16/12 = 0.667$ ft, and

$$A = \frac{C}{\sin\varphi} = \frac{0.667}{\sin 85°} = 0.669 \text{ ft.}$$

Step (3): From Eqs. (7.8) and (7.9), the height in ft and velocity in ft/s at time t in seconds are, respectively,

$$z = 0.669e^{-0.231t}\sin(2.53t + 1.48 \text{ rad})$$

and

$$V = 0.669e^{-0.231t}\{[2.53\cos(2.53t + 1.48 \text{ rad})] - [0.231\sin(2.53t + 1.48)]\}.$$

Step (4): The minimum velocity V_{min} at $t = T_d/4 = \pi/2\omega_d = \pi/(2 \times 2.53) = 0.62$ s gives $V_{min} = -1.46$ ft/s and

$$\mathrm{Re}_{min} = \frac{|V|D}{\upsilon} = \frac{1.46 \times 1}{12 \times 10^{-4}} = 1{,}220$$

and the flow is laminar because Re < 2,000.

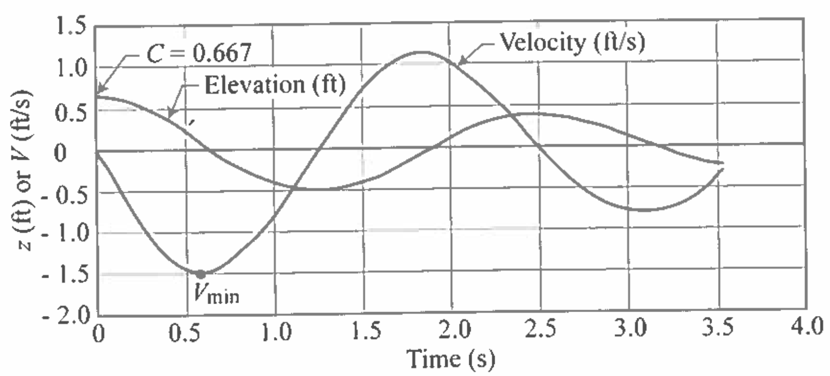

Fig. E-7.2 Pipe flow with viscous oscillations

7.3 Oscillations for Turbulent Flow

The case of oscillations for turbulent flows in large pipes is far more complicated because the equation of motion becomes nonlinear. The governing equation was derived in Section 7.2.2, except that the Darcy-Weisbach friction coefficient is now constant:

$$\rho g(z_2 - z_1)A + \rho g\left(\frac{fL}{D}\frac{V|V|}{2g}\right)A = -\rho AL\frac{dV}{dt}.$$

Dividing by ρAL and given $(z_2 - z_1) = 2z$, with $V = \dot{z}$ and $a = dV/dt = \ddot{z}$, we obtain

$$\ddot{z} + \frac{f}{2D}\dot{z}|\dot{z}| + \frac{2gz}{L} = 0. \tag{7.10}$$

This is a nonlinear differential equation because of the squared velocity term. The absolute value of the velocity term is needed to ensure that the resistance opposes the velocity in both flow directions. The equation can be integrated once with respect to t and the first integration is given here without derivation (see Rainville 1964, Streeter 1971):

$$\left(\frac{dz}{dt}\right)^2 = \frac{4gD^2}{f^2L}\left(1 + \frac{fz}{D}\right) + Ce^{\left(\frac{fz}{D}\right)}$$

The integration constant C is evaluated for minimum/maximum values, i.e. $z = z_m$ at $dz/dt = 0$ (where the subscript m represents the maximum or minimum),

$$C = -\frac{4gD^2}{f^2L}\left(1 + \frac{fz_m}{D}\right)e^{\left(-\frac{fz_m}{D}\right)},$$

and the velocity relationship becomes

$$V^2 = \left(\frac{dz}{dt}\right)^2 = \frac{4gD^2}{f^2L}\left\{\left(1 + \frac{fz}{D}\right) - \left(1 + \frac{fz_m}{D}\right)\exp\left[\frac{f(z - z_m)}{D}\right]\right\}. \tag{7.11}$$

This equation is useful to determine the successive peaks (high z_m^+ and low z_{m+1}^-) obtained when $V = \dot{z} = 0$, and the equation simplifies to

$$\left(1 + \frac{fz_m}{D}\right)\exp\left(-\frac{fz_m}{D}\right) = \left(1 + \frac{fz_{m+1}}{D}\right)\exp\left(-\frac{fz_{m+1}}{D}\right).$$

The main equation to be solved with $\phi = fz/D$ is

$$F(\phi) = (1 + \phi)e^{-\phi}. \tag{7.12}$$

With changes in flow direction, the successive values of ϕ are obtained by alternating the signs of ϕ and solving $F(\phi) = (1 + \phi)e^{-\phi} = (1 - \phi)e^{+\phi}$. This is plotted on Figure 7.6 with an illustrated example.

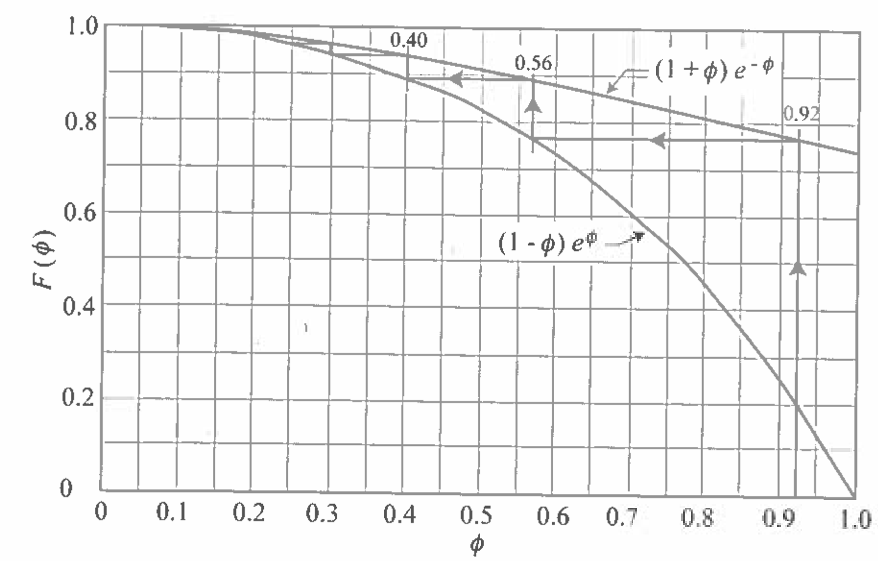

Figure 7.6 Plot of $F(\phi) = (1 + \phi)e^{-\phi}$ for turbulent pipe flow oscillations

The graphical procedure to determine successive peaks is illustrated in Figure 7.6. With an example starting at $\phi_1 = 0.92$, we obtain $F(\phi) = (1 + \phi)e^{-\phi} = 1.92e^{-0.92} = 0.77$, which corresponds to $\phi_2 = -0.56$ because $F(\phi) = (1 + \phi)e^{-\phi} = (1 - 0.56)e^{+0.56} = 0.77$.

To find the next minimum, we reverse the sign of the last value $\phi_2 = +0.56$ to find $F(0.56) = (1 + \phi)e^{-\phi} = 1.56e^{-0.56} = 0.89$, and the subsequent value is $\phi_3 = -0.40$ because $F(-0.40) = (1 + \phi)e^{-\phi} = (1 - 0.40)e^{+0.40} = 0.89$, and this results in $\phi_3 = 0.40$, and so on.

The successive peaks are therefore $\phi_1 = 0.92$, $\phi_2 = -0.56$ and $\phi_3 = 0.40$, etc. and the corresponding successive maximum/minimum elevations using f and D are $z_1 = \phi_1 D/f$, $z_2 = \phi_2 D/f$, etc.

The maximum value of V is found by equating $dV^2/dt = 0$ from Eq. (7.11) at the position z':

$$\frac{dV^2}{dt} = 0 = \frac{f}{D} - \left(1 + \frac{f z_m}{D}\right)\left\{\exp\left[\frac{f(z' - z_{max})}{D}\right]\right\}\frac{f}{D},$$

which gives

$$z' = z_{max} - \frac{D}{f}\ln\left(1 + \frac{f z_{max}}{D}\right).$$

The result is substituted back into Eq. (7.11) for the maximum velocity:

$$V_{max}^2 = \frac{4gD^2}{f^2 L}\left[\left(\frac{f z_{max}}{D}\right) - \ln\left(1 + \frac{f z_{max}}{D}\right)\right]. \tag{7.13}$$

This procedure looks awfully complicated, and Example 7.3 should be very helpful.

Example 7.3: Turbulent flow oscillations in a large pipe

A 1,000-ft-long U-tube consists of 2.0-ft-diameter pipe with $f = 0.03$. The initial water-level difference between both ends is $z_1 = 20$ ft. Find the successive minimum and maximum elevations.

Solution:

Step (1): The initial value is $\phi_1 = \frac{f z_1}{D} = \frac{0.03 \times 20}{2} = 0.3$ and $F(0.3) = (1+\phi)e^{-\phi} = 1.3\,e^{-0.3} = 0.963$.

This corresponds to $\phi_2 = -0.25$ because $F(\phi) = (1+\phi)e^{-\phi} = (1 - 0.25)\,e^{+0.25} = 0.963$.

Step (2): The sign of $\phi_2 = 0.25$ is reversed to give $F(0.25) = (1+\phi)e^{-\phi} = 1.25\,e^{-0.25} = 0.973$.

The third peak is $\phi_3 = -0.215$ because $F(-0.215) = (1 - 0.215)\,e^{+0.215} = 0.973$, and $\phi_3 = 0.215$.

Step (3): The corresponding sequence of maximum/minimum elevations for $\phi_1 = 0.3$, $\phi_2 = -0.25$ and $\phi_3 = 0.215$ is $z_1 = 20$ ft, $z_2 = \phi_2 D/f = -0.25 \times 2/0.03 = -16.7$ ft, and $z_3 = \phi_3 D/f = +0.215 \times 2/0.03 = +14.3$ ft.

Step (4): The maximum velocity from Eq. (7.13) corresponds to the first maximum $\phi_1 = 0.3$, and

$$V_{max}^2 = \frac{4gD^2}{f^2 L}\left[\left(\frac{f z_{max}}{D}\right) - \ln\left(1 + \frac{f z_{max}}{D}\right)\right] = \frac{4 \times 32.2 \times 2^2}{0.03^2 \times 1{,}000}[(0.3) - \ln(1 + 0.3)]$$

$$= 21.5\ \text{ft}^2/\text{s}^2,$$

or

$V_{max} = -4.64$ ft/s; this occurs when

$$z' = z_{max} - \frac{D}{f}\ln\left(1 + \frac{f z_{max}}{D}\right) = 20 - \frac{2}{0.03}\ln(1 + 0.3) = 2.5\ \text{ft}.$$

The basic equation of motion (Eq. (7.10)) can also be solved numerically at very short time increments to show the displacement and the velocity as a function of time. Figure E-7.3 shows a comparison of the numerical method and the successive maximum/minimum elevation and velocity values.

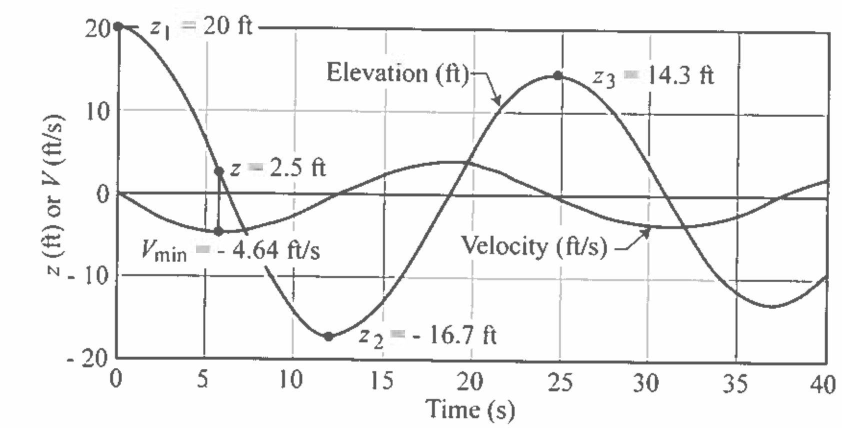

Fig. E-7.3 Turbulent pipe flow oscillations

7.4 Oscillations between Reservoirs

Two reservoirs are connected by a pipeline of length L and diameter D in Figure 7.7. We study the flow oscillations given the reservoir surface areas A_1 and A_2 with water elevations z_1 and z_2, respectively. For the analysis, the equilibrium position from the volumetric relationship $zA = z_1 A_1 = z_2 A_2$, gives $z_1 = zA/A_1$ and $z_2 = zA/A_2$. Also, given $H = z_1 + z_2$, we find $z_1 = HA_2/(A_1 + A_2)$ and $z_2 = HA_1/(A_1 + A_2)$. Minor losses are considered by using the equivalent friction factor $f_e = f + KD/L$ with the pipe friction factor f and minor-loss coefficient K.

The equation of motion when neglecting the momentum flux ($\rho QV << pA$) is

$$-\gamma(z_2 + z_1)A - \frac{\gamma A f_e L}{2gD}\frac{dz}{dt}\left|\frac{dz}{dt}\right| = \frac{\gamma A L}{g}\frac{d^2 z}{dt^2},$$

which gives

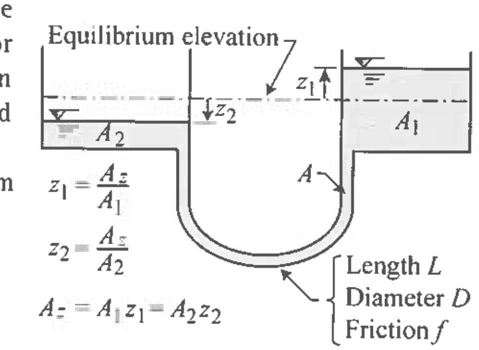

Figure 7.7 Oscillations between reservoirs

$$\frac{d^2z}{dt^2} + \frac{f_e}{2D}\frac{dz}{dt}\left|\frac{dz}{dt}\right| + \frac{gA}{L}\left(\frac{1}{A_2} + \frac{1}{A_1}\right)z = 0.$$

We are already very familiar with this formulation and the solution is the same as in Section 7.3 after replacing f with f_e, using $\phi = f_e\frac{z}{D}$, and replacing $2g/L$ by $gA[(1/A_1) + (1/A_2)]/L$. Notice that oscillations can be independent of the original elevation z_{10}, because $F(\phi) = (1+\phi)e^{-\phi} \to 0$ when $\phi > 5$. The application in Example 7.4 considers the oscillations between two large tanks.

♦ Example 7.4: Oscillations between large tanks

A valve is opened in a pipe connecting two water tanks of surface areas $A_1 = 200\ \text{ft}^2$ and $A_2 = 300\ \text{ft}^2$. The initial head difference between the two tanks is 66.7 ft. The 2,000-ft pipe has a 3-ft diameter with a friction factor $f = 0.024$ and minor losses $3.5\ V^2/2g$. Find the high and low water levels in tank 1.

Solution:

Step (1): The initial head of tank 1 is $z_{10} = HA_2/(A_1 + A_2) = 66.7 \times 300/(200 + 300) =$ 40 ft and the initial level in tank 2 is $z_{20} = \frac{z_{10}A_1}{A_2} = 40 \times \frac{200}{300} = 26.7$ ft below the reference elevation.

Step (2): The equivalent friction factor is $f_e = f + KD/L = 0.024 + (3.5 \times 3/2{,}000) =$ 0.02925.

The initial high level in the first tank $z_{10} = 40$ ft gives

$$z_{m0} = \frac{z_{10}A_1}{A} = \frac{40 \times 200 \times 4}{\pi \times 3^2} = 1{,}132 \text{ ft.}$$

The corresponding ϕ is

$$\phi_0 = \frac{f_e z_{m0}}{D} = \frac{0.029 \times 1{,}132}{3} = 11.0;$$

note that $\phi_0 > 5$ here, and $F(\phi_0) = (1+\phi_0)e^{-\phi_0} = (1+11)e^{-11} = 0.0002$.

The first oscillation starts when $\phi_0 < 5$, or after

$$z_{10} = \frac{z_{m0}A}{A_1} = \frac{A\phi_0 D}{A_1 f_e} = \frac{\pi \times 3^2 \times 5 \times 3}{4 \times 200 \times 0.029} = 18.1 \text{ ft.}$$

Step (3): The first minimum will happen when $\phi_1 = -1$, which corresponds to

$$z_{m1} = \frac{\phi_1 D}{f_e} = \frac{-1 \times 3}{0.02925} = -102.6 \text{ ft,}$$

and thus

$$z_{11} = \frac{z_{m1}A}{A_1} = \frac{-102.6 \times \pi \times 3^2}{200 \times 4} = -3.62 \text{ ft.}$$

Step (4): For the next maximum, the absolute value of $\phi_1 = +1$ gives $F(\phi_1) = (1+\phi_1)e^{-\phi_1} = (1+1)e^{-1} = 0.736$, which corresponds to $\phi_2 = 0.593$ given that $F(\phi_2) = (1-\phi_2)e^{\phi_2} = (1-0.593)e^{+0.593} = 0.736$. This second peak is at

$$z_{m2} = \frac{\phi_2 D}{f_e} = \frac{0.593 \times 3}{0.02925} = 60.9 \text{ ft,}$$

which is at elevation

$$z_{12} = \frac{z_{m2}A}{A_1} = \frac{60.9 \times \pi \times 3^2}{200 \times 4} = 2.15 \text{ ft.}$$

Step (5): The maximin/minimum sequence in the first tank is: $z_{10} = 40$ ft, $z_{11} = -3.62$ ft and $z_{12} = 2.15$ ft, etc.

Step (6): The numerical methods become increasingly accurate as $\Delta t \to 0$. The numerical scheme consists of three parts:

(a) Eq. 7.10 is solved for the acceleration $\ddot{z}$ given the initial values of elevation and velocity.

(b) The velocity is then calculated from $V = V_0 + \ddot{z}\Delta t$.

(c) The displacement is then simply obtained from $z = z_0 + V\Delta t$.

This maximum/minimum calculation sequence is compared with numerical calculations in Fig. E-7.4.

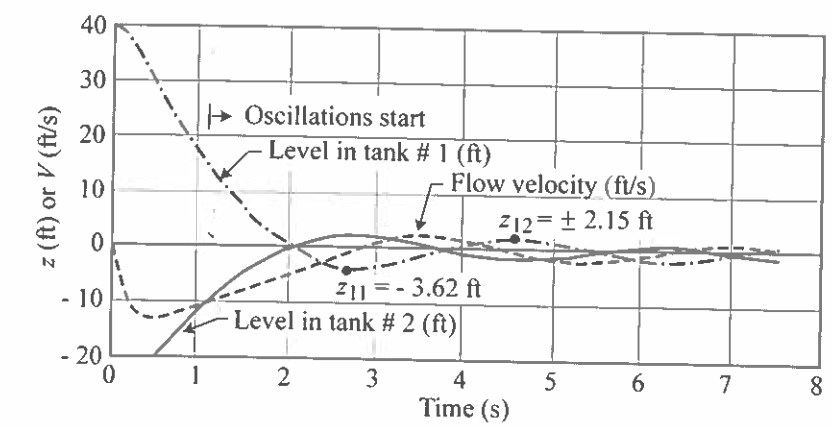

Fig. E-7.4 Turbulent flow in a pipe connecting two reservoirs

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Additional Resources

Additional information on transients and flow oscillations in pipes can be found in Streeter (1971), Wylie and Streeter (1978), Naudascher and Rockwell (1994), Ghidaoui et al. (2005) and Chaudhry (2014). Numerical methods which successively solve Eq. (7.10) for acceleration, velocity and position at very short time intervals are quite popular and effective nowadays on fast computers. Alternatively, analytical solutions for flow oscillations with turbulent friction losses are now becoming possible with the use of Lambert functions and elliptic integrals readily available in Matlab (Guo et al. 2017).

EXERCISES

These exercises review the essential concepts from this chapter.

1. What is the source of elasticity in pipe flow oscillations?
2. What is the effect of friction on the frequency of oscillations?
3. When do oscillations start in laminar flow?
4. When do oscillations start in turbulent flow?
5. True or false?
   (a) The pressure at the reference elevation (midpoint) in the pipe remains constant during the oscillation.
   (b) The period of oscillations in pipes without friction only depends on the pipe length.
   (c) Friction increases the frequency of oscillations.
   (d) Friction decreases the period of oscillations.

PROBLEMS

1. ♦♦ A 10-m-long, 5-mm-diameter U-shaped plastic tube is holding water at 20 °C. There is a 1-m head difference between both ends when the pressure is suddenly released at $t = 0$. Determine the following: (a) the natural circular frequency of the oscillations; (b) the damping factor; (c) the circular frequency of the damped oscillations; (d) the period of the damped oscillations; (e) the lowest water level; (f) the maximum flow velocity; and (g) whether or not the flow is laminar.
2. ♦♦ A 25-m-long, 25-cm-diameter U-shaped pipe has an 8-m difference between both ends as the system is released from rest. If $f = 0.04$, calculate the successive maxima during the oscillations and find the maximum velocity in the pipe. Compare the calculations with $f = 0.08$.

8 Steady Uniform Flow

As opposed to pressurized flow in closed conduits, open-channel flows convey water by gravity in man-made channels and natural waterways. The cross-sectional area of open channels varies with discharge as described in Section 8.1. Section 8.2 examines resistance to flow, the normal depth is considered in Section 8.3 and shear stress in Section 8.4.

8.1 Open-Channel Geometry

The cross section of a channel is measured perpendicular to the main flow direction. Figure 8.1 depicts the geometric elements of a typical cross section. The main parameters are: flow depth y, surface width W, wetted perimeter P, cross-sectional area A, averaged depth $h = A/W$ and hydraulic radius $R_h = A/P$.

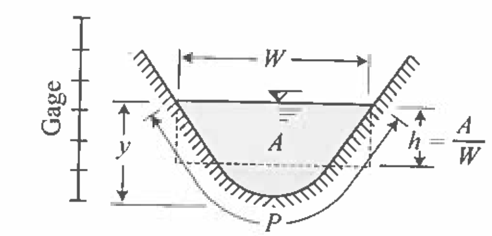

Figure 8.1 Cross-sectional geometry

The geometry of open-channel cross sections is summarized in Figure 8.2.

For the compound sections sketched in Figure 8.3a, the hydraulic radius is calculated from the sums of partial areas and partial wetted perimeters. Typical river cross-section profiles in Figure 8.3b indicate the substrate material and floodplain vegetation types in terms of deciduous and coniferous trees, shrubs and grasses. The bankfull elevation is usually important because floodplains can extend laterally over long distances. The vegetation on the floodplain also increases roughness while the main channel will tend to limit vegetation growth.

Three typical calculation examples are presented for: (1) a circular section for sewers and culverts in Example 8.1; (2) a trapezoidal canal in Example 8.2; and a compound section in Example 8.3.

Example 8.1: Circular cross section

Define the hydraulic geometry of a circular open-channel cross section shown in Fig. E-8.1.